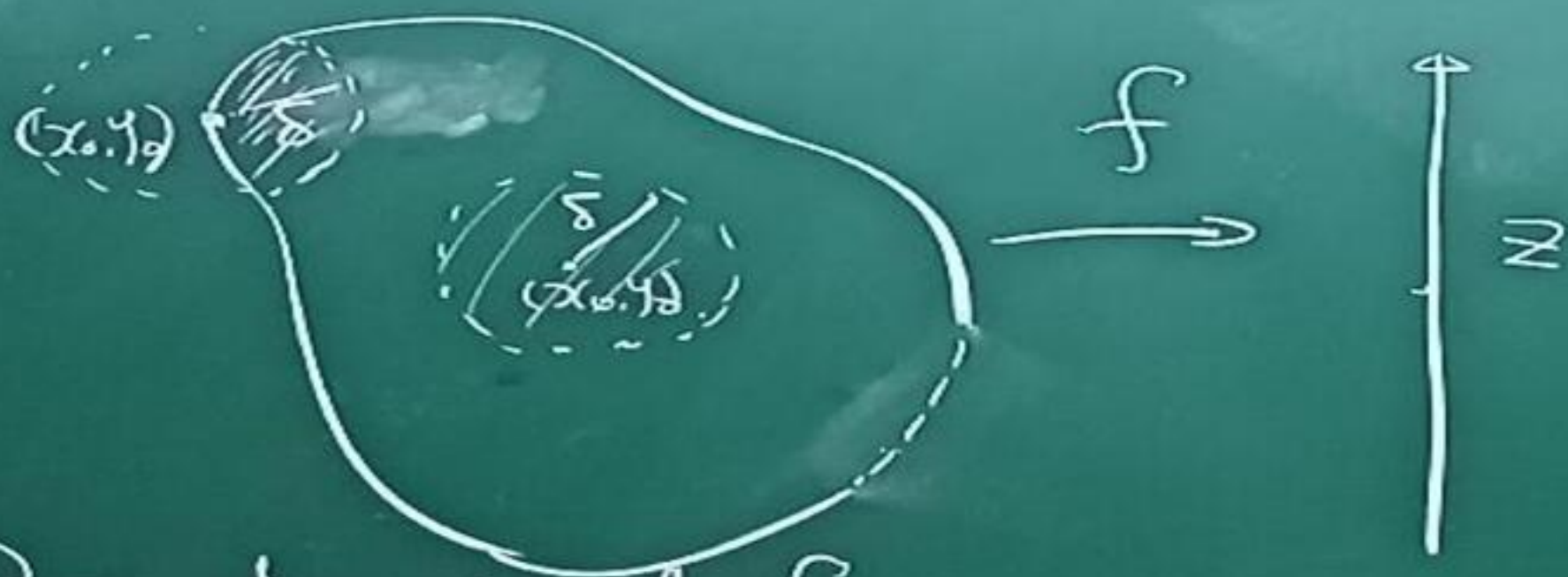


Limit and Continuity in higher dimension

2D case



$D_f = \text{domain of } f$

$$f: D_f \longrightarrow \mathbb{R}$$

$$(x, y) \longmapsto z = f(x, y)$$

Def $\lim_{(x,y) \rightarrow (x_0,y_0)} f(x,y) = L$

if for every $\varepsilon > 0$
there exists a corresponding
 $\delta > 0$, such that

$$\begin{aligned} & \text{"} 0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta \implies |f(x,y) - L| < \varepsilon \text{"} \\ & + ((x,y) \in D_f) \quad (*) \end{aligned}$$

Remark: If (*) is changed to

$$\text{"} \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta \implies |f(x,y) - L| < \varepsilon \text{"} \quad (**)$$

then $\begin{cases} \lim_{(x,y) \rightarrow (x_0,y_0)} f(x,y) = L \\ f(x_0,y_0) = L \end{cases}$

Def f is cont. at (x_0, y_0)

if $\lim_{(x, y) \rightarrow (x_0, y_0)} f(x, y) = f(x_0, y_0)$

That is, for every $\varepsilon > 0$,
there exists a corresponding
 $\delta > 0$, such that

$$\underbrace{0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta}_{(\sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta)} \Rightarrow |f(x, y) - f(x_0, y_0)| < \varepsilon$$

$$(\sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta)$$

Limit Rules: See Section 14.2

Therorem 1

$$\text{Eg 1 } \lim_{(x,y) \rightarrow (0,1)} \frac{x - xy + 3}{x + y^2} = ?$$

$$\text{Ans} = \frac{0 - 0 \cdot 1 + 3}{0 + 1^2} = 3$$

Here we have used the facts that x , xy , y^2 are continuous.

$$\text{Eg 2 } \lim_{(x,y) \rightarrow (0,0)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}} = ?$$

$$= \lim_{(x,y) \rightarrow (0,0)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}} \cdot \frac{\sqrt{x} + \sqrt{y}}{\sqrt{x} + \sqrt{y}}$$

$$= \lim_{(x,y) \rightarrow (0,0)} x(\sqrt{x} + \sqrt{y}) = 0$$

$$\text{Dom } D_f = \{x \geq 0, y \geq 0, x \neq y\}$$

Ex 3. $\lim_{\substack{(x,y) \rightarrow (0,0) \\ r \rightarrow 0}} \frac{4xy^2}{x^2+y^2} = ?$

" $0 < r < \delta \Rightarrow |f(x,y) - ?| < \epsilon$ "

Ans: $x = r \cos \theta, y = r \sin \theta$.

$= \lim_{r \rightarrow 0} \frac{4r^3 \cos \theta \sin^2 \theta}{r^2} = 0$

Rm

" $0 < \sqrt{x^2+y^2} < \delta \Rightarrow |f(x,y) - L| < \epsilon$ "

\iff

" $0 < r < \delta \Rightarrow |f(r \cos \theta, r \sin \theta) - L| < \epsilon$ "

i.e. $\lim_{(x,y) \rightarrow (0,0)} f(x,y) = \lim_{r \rightarrow 0} f(r \cos \theta, r \sin \theta)$

$$\text{Eg 4 } \lim_{(x,y) \rightarrow (0,0)} \frac{y}{x} = ?$$

Ans: $x = r \cos \theta$, $y = r \sin \theta$

$$\lim_{r \rightarrow 0} \frac{r \sin \theta}{r \cos \theta} \neq L \quad (\text{indep of } \theta)$$

$$= \lim_{r \rightarrow 0} \frac{\sin \theta}{\cos \theta} \text{ does not exist.}$$

or Two Path Theorem (See below)

$$\lim_{(x,y) \rightarrow (0,0), x=y} \frac{y}{x} = 1$$

$$\lim_{(x,y) \rightarrow (0,0), x=-y} \frac{y}{x} = -1$$

$\Rightarrow \lim_{(x,y) \rightarrow (0,0)}$
does not exist

Two Path Theorem

If $f(x, y)$ have different limits (one dimensional limits) along two Paths passing through (x_0, y_0)

as $(x, y) \rightarrow (x_0, y_0)$

Then $\lim_{(x, y) \rightarrow (x_0, y_0)} f(x, y)$

does not exist

Eg 5. $\lim_{(x,y) \rightarrow (0,0)} \frac{2xy}{x^2+y^2}$

See Fig 14.14 page 819

Sol. Let $(x,y) \rightarrow (0,0)$

along $y=mx$, $m \in \mathbb{R}$ (*)

$$\lim_{(*)} \frac{2xy}{x^2+y^2} = \lim_{x \rightarrow 0} \frac{2m}{1+m^2} = \frac{2m}{1+m^2}$$

different $m \in \mathbb{R}$ give different one-dimensional limit.

Two Path Thm $\Rightarrow \lim_{(x,y) \rightarrow (0,0)}$ does not exist

Eg 6. $\lim_{(x,y) \rightarrow (0,0)} \frac{2x^2y}{x^4+y^2} = ?$

See Fig 14.15 page 820

Sol. Let $y = mx$, $x \rightarrow 0$

$$\lim_{x \rightarrow 0} \frac{2x^2 mx}{x^4 + m^2 x^2} = \begin{cases} 0 & \text{(line "y=0") } m=0 \\ 0 & m \neq 0 \\ 0 & m = \infty \end{cases}$$

However,

if we let $y = kx^2$, $x \rightarrow 0$

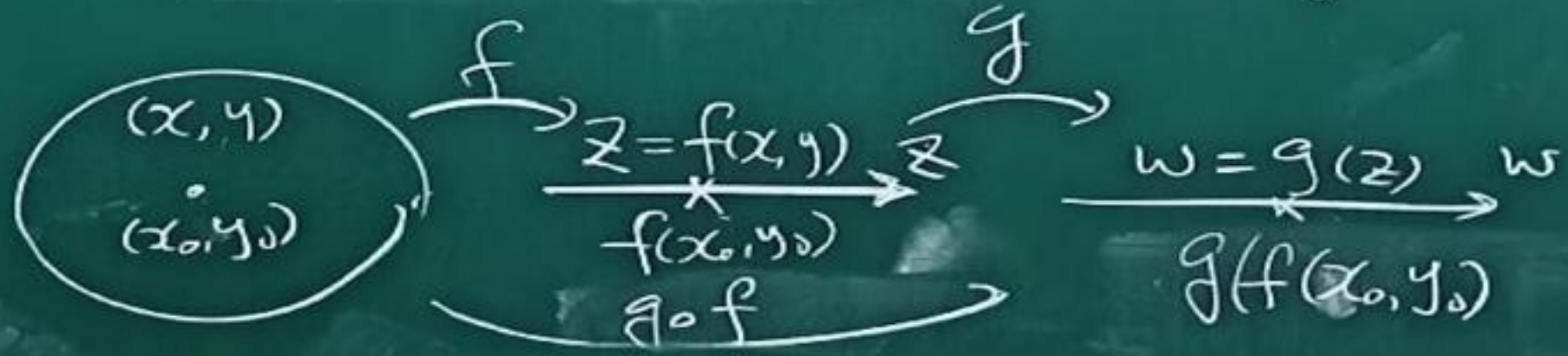
$$\lim_{x \rightarrow 0} \frac{2kx^4}{x^4(1+k^2)} = \frac{2k}{1+k^2} \quad \left(\begin{array}{l} \text{different } k \\ \text{different lim} \end{array} \right)$$

\therefore Two Path Thm $\Rightarrow \lim_{(x,y) \rightarrow (0,0)}$ does not exist

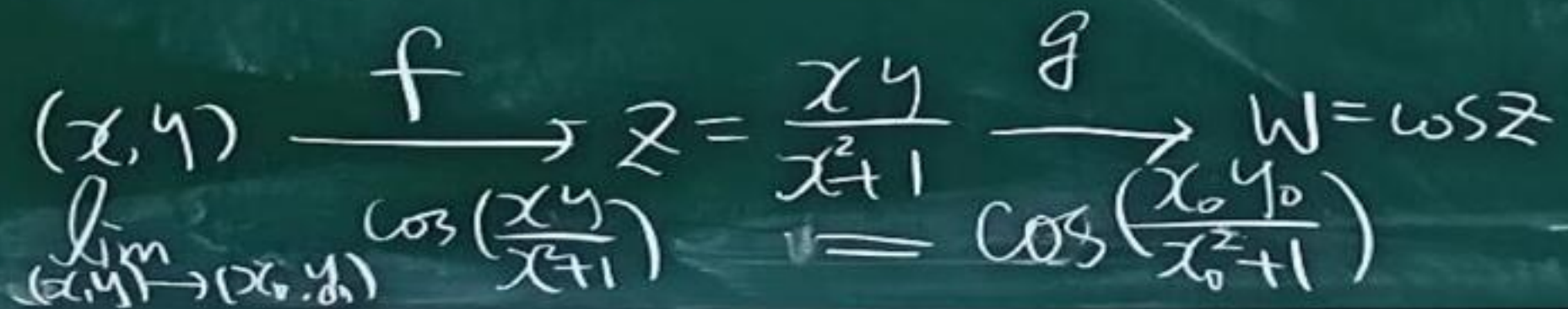
Remark If

$$\begin{cases} f \text{ is cont. at } (x_0, y_0) \\ g \text{ is cont. at } z_0 = f(x_0, y_0) \end{cases}$$

Then $g \circ f$ is cont. at (x_0, y_0)
($g \circ f(x, y) \stackrel{\text{def}}{=} g(f(x, y))$)



Ex: $\cos\left(\frac{xy}{x^2+1}\right)$ is continuous on \mathbb{R}^2



Partial derivatives

$$\text{Def: } \frac{\partial f}{\partial x}(x_0, y_0) = \lim_{x \rightarrow x_0} \frac{f(x, y_0) - f(x_0, y_0)}{x - x_0}$$

$$\frac{\partial f}{\partial y}(x_0, y_0) = \lim_{y \rightarrow y_0} \frac{f(x_0, y) - f(x_0, y_0)}{y - y_0}$$

$$\text{Notation: } \frac{\partial f}{\partial x} = f_x = \partial_x f = \partial_1 f$$

$$\frac{\partial f}{\partial y} = f_y = \partial_y f = \partial_2 f$$

$$\text{Ex 1. } f(x, y) = x^2 + 3xy + y - 1$$

$$\partial_x f = 2x + 3y + 0$$

$$\partial_y f = 0 + 3x + 1$$

$$\text{Eg 2 } f(x, y, z) = x \sin(y + 3z)$$

$$\frac{\partial f}{\partial x} \underset{y, z \sim \text{constant}}{=} \sin(y + 3z)$$

$$\frac{\partial f}{\partial y} \underset{x, z \sim \text{const}}{=} x \cos(y + 3z)$$

$$\frac{\partial f}{\partial z} \underset{x, y \sim \text{const}}{=} x \underbrace{\cos(y + 3z)}_{\sin'} \cdot \underbrace{3}_{\frac{\partial}{\partial z}(y + 3z)}$$

Ex 3 Find $\frac{\partial z}{\partial x}$ at $(0, 1, 1)$

if $z(x, y)$ is implicitly defined by $yz + \ln z = x + y$

Ans. $yz(x, y) + \ln z(x, y) = x + y$

$$\partial_x \Rightarrow y \cdot z_x + \frac{z_x}{z} = 1 + 0$$

$$\Rightarrow \frac{\partial z}{\partial x}(x, y) = \frac{1}{y + \frac{1}{z}}$$

Check $(0, 1, 1)$ is on this surface
(Yes: $1 \cdot 1 + \ln 1 = 0 + 1$)

$$\therefore \frac{\partial z}{\partial x} \Big|_{(0, 1, 1)} = \frac{1}{1 + \frac{1}{1}} = \frac{1}{2}$$

Higher order Partial derivatives

$$z = f(x, y)$$

$$\frac{\partial^2 f}{\partial x^2} = \partial_x^2 f = \partial_x(\partial_x f) = f_{xx}$$

$$\frac{\partial^2 f}{\partial y^2} = \partial_y^2 f = \partial_y(\partial_y f) = f_{yy}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \partial_x(\partial_y f) = f_{yx}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \partial_y(\partial_x f) = f_{xy}$$

Similarly $\partial_y \partial_x^2 f = f_{xxy}$, etc.

Ex 4: $f(x, y) = x^2 + y^2$

$$\partial_x f = 2x, \quad \partial_y f = 2y$$

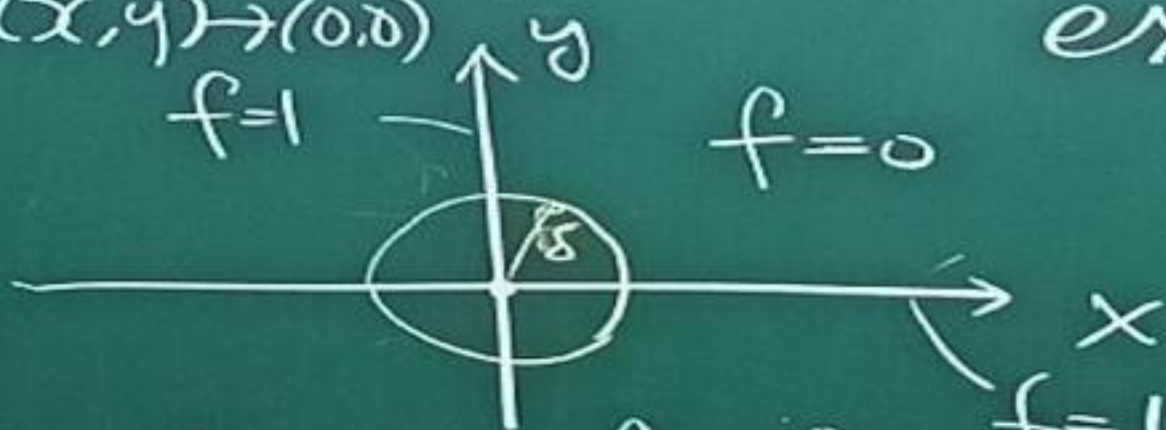
$$\partial_x^2 f = \partial_x(2x) = 2, \quad \partial_y^2 f = 2$$

$$\partial_x \partial_y f = \partial_x(2y) = 0, \quad \partial_y(\partial_x f) = \partial_y(2x) = 0$$

Ex 5 $\frac{\partial f}{\partial x}(x_0, y_0)$ and $\frac{\partial f}{\partial y}(x_0, y_0)$ exist $\not\Rightarrow$ f is cont. at (x_0, y_0)

$$f(x, y) = \begin{cases} 0 & xy \neq 0 \\ 1 & xy = 0 \end{cases}$$

(i) $\lim_{(x, y) \rightarrow (0, 0)} f(x, y)$ does not exist.



$$\begin{aligned} \text{(ii)} \quad \frac{\partial f}{\partial x}(0, 0) &= \lim_{x \rightarrow 0} \frac{f(x, 0) - f(0, 0)}{x - 0} \\ &= \lim_{x \rightarrow 0} \frac{1 - 1}{x - 0} = 0 \end{aligned}$$

$$\text{(iii)} \quad \frac{\partial f}{\partial y}(0, 0) = \lim_{y \rightarrow 0} \frac{f(0, y) - f(0, 0)}{y - 0} = 0$$

Def $z = f(x, y)$ is differentiable at (x_0, y_0) if

(i) $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ exist.

(ii) $f(x, y) = L(x, y) + \varepsilon_1 \cdot (x - x_0) + \varepsilon_2 \cdot (y - y_0)$
(or $= L(x, y) + \underline{\varepsilon} \cdot \sqrt{(x - x_0)^2 + (y - y_0)^2}$)

where

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

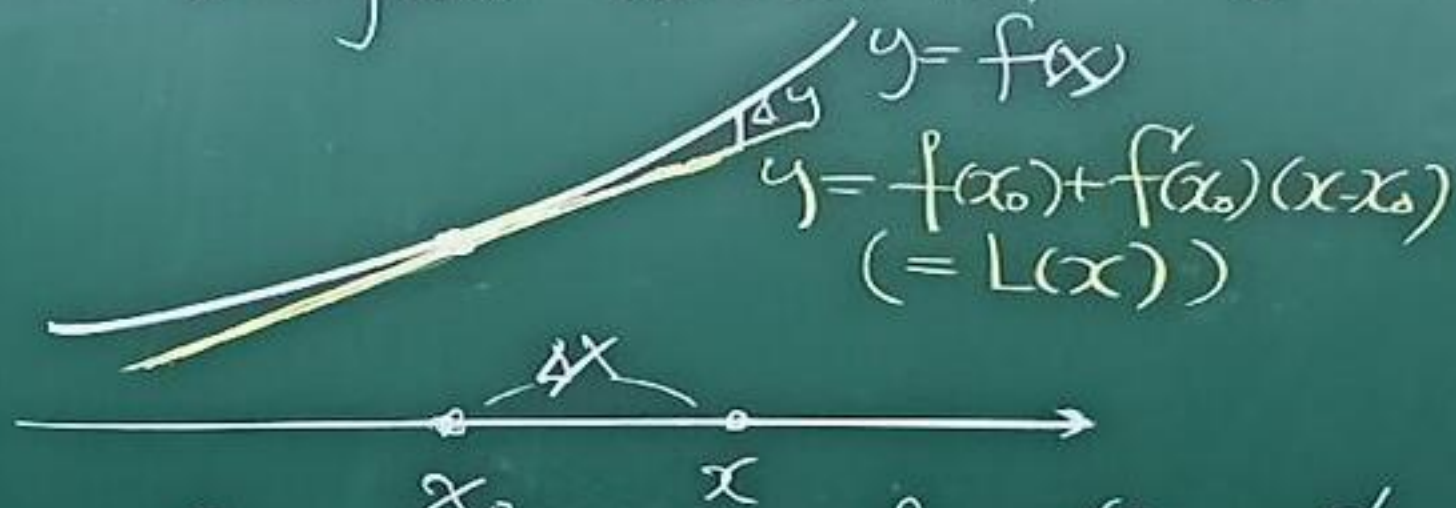
where $\lim_{(x, y) \rightarrow (x_0, y_0)} \varepsilon_1, \varepsilon_2, \underline{\varepsilon} = 0$

i.e. $f(x, y)$ and $L(x, y)$ are

tangent at $(x_0, y_0, f(x_0, y_0))$

i.e. $\lim_{(x, y) \rightarrow (x_0, y_0)} \frac{f(x, y) - L(x, y)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} = 0 \quad (= \lim_{(x, y) \rightarrow (x_0, y_0)} \underline{\varepsilon})$

Remark differentiability and tangent line in 1D



$$\lim_{x \rightarrow x_0} \frac{f(x) - L(x)}{x - x_0} = \lim_{x \rightarrow x_0} \frac{f(x) - (f(x_0) + f'(x_0)(x - x_0))}{x - x_0}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} \quad \parallel \quad 0$$

Rm Two curves $\begin{cases} y = f(x) \\ y = g(x) \end{cases}$ are tangent at (x_0, y_0)

if (i) $f(x_0) = g(x_0) = y_0$

(ii) $\lim_{x \rightarrow x_0} \frac{f(x) - g(x)}{x - x_0} = 0$ ($|f(x) - g(x)|$ is smaller than $|x - x_0|$)

Rm (homework)

$$\varepsilon_1(x-x_0) + \varepsilon_2(y-y_0) = \varepsilon \cdot \sqrt{(x-x_0)^2 + (y-y_0)^2}$$

Eg 6. Are $\begin{cases} y=f_1(x)=e^x \\ y=f_2(x)=1+x \end{cases}$ tangent at $(0,1)$?

Sol. (i) $f_1(0)=1, f_2(0)=1$

(ii) $\lim_{x \rightarrow 0} \frac{f_1(x) - f_2(x)}{x - 0} = \lim_{x \rightarrow 0} \frac{e^x - 1 - x}{x} = 0$ (Yes)

Eg 7. Are $\begin{cases} z=f_1(x,y)=x^2+y^2 \\ z=f_2(x,y)=0 \end{cases}$ tangent at $(0,0,0)$?

Sol. (i) $f_1(0,0)=0, f_2(0,0)=0$

(ii) $\lim_{(x,y) \rightarrow (0,0)} \frac{f_1(x,y) - f_2(x,y)}{\sqrt{x^2+y^2}} = \lim_{(x,y) \rightarrow (0,0)} \frac{x^2+y^2}{\sqrt{x^2+y^2}} = 0$ (Yes)