$$T_{f,a}(x) \stackrel{\text{def}}{=\!=} \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

(1) For what values of x does it converge? i.e. $R = ?$

(2) When it converges, does it equal $f(x)$?

Eg: $f(x) = \begin{cases} 0 & x = 0 \\ e^{-\frac{1}{x^2}} & x \neq 0 \end{cases}$

$T_{f,a}(x) = 0 \neq f(x)$ for $x \neq 0$

In the following example

$f^{(k)}(0) = 0$ for all k

$\Rightarrow T_{f,0}(x) \equiv 0$ but $f(x) \neq 0$
$(x \neq 0)$

<u>Eg5</u> $f(x) = \begin{cases} e^{\frac{-1}{x^2}}, & x \neq 0 \\ 0, & x = 0, \end{cases}$ $T_{f,0}(x) = ?$

<u>Ans</u>: $f(0) = 0$

$$f'(0) = \lim_{h\to 0} \frac{e^{\frac{-1}{h^2}} - 0}{h - 0} \quad \left(\begin{array}{l} \underline{\text{Not}} \text{ always} \\ = \lim_{h\to 0} f'(h) \end{array} \right)$$

$$\left(= \text{“}\frac{0}{0}\text{”}\right) = \lim_{h\to 0} \frac{\left(\frac{1}{h}\right)}{e^{\frac{1}{h^2}}} \quad \left(\pm \text{“}\frac{\infty}{\infty}\text{”}\right)$$

<u>l'Hôpital</u> $\lim_{h\to 0} \frac{-h^{-2}}{-2h^{-3} e^{\frac{1}{h^2}}} = \lim_{h\to 0} \frac{h}{2e^{\frac{1}{h^2}}} = 0$
$(**)$

$$f''(0) = \lim_{h \to 0} \frac{f'(h) - f'(0)}{h - 0}$$

$$\overset{(**)}{=} \lim_{h \to 0} \frac{2h^{-3} e^{-\frac{1}{h^2}}}{h - 0}$$

$$= \lim_{h \to 0} \frac{2 e^{-\frac{1}{h^2}}}{h^4}$$

$$= \lim_{h \to 0} \frac{2h^{-4}}{e^{\frac{1}{h^2}}}$$

$\underset{\text{l'Hôpital}}{=}$ (homework)

$$= 0$$

In fact, $f^{(k)}(0) = 0$ for all $k \in \mathbb{N}$

Thm Taylor's Thm

If $f, f', f'', \cdots$

all exist on $|x-a|<\delta$

Then, for any $n \in \mathbb{N}$

$$f(x) = P_n(x) + R_n(x) \quad \cdots (*)$$

on $|x-a|<\delta$, where

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

for some c between a and x

Since $\overline{T}_{f,a}(x) = \lim_{n\to\infty} P_n(x)$

$\therefore$ $\overline{T}_{f,a}(x) = f(x) \iff \lim_{n\to\infty} R_n(x) = 0$

Corollary: If $|f^{(n+1)}(c)| \leq M$

for all c between a and x

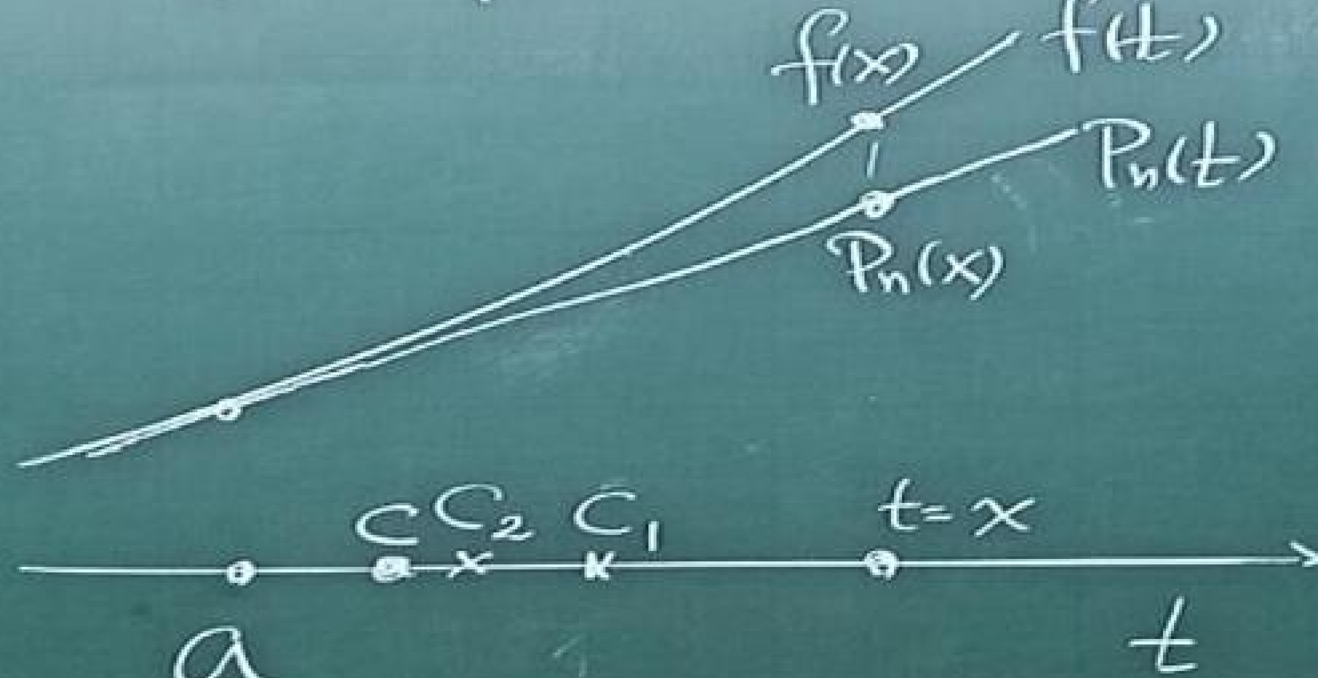
and all $n \in \mathbb{N}$

Then $|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1}$

$\Rightarrow \lim_{n\to\infty} R_n(x) = 0$

$\Rightarrow \overline{T}_{f,a}(x) = f(x)$

Proof of Taylor's Thm



We now write (for fixed x)

$$f(x) = P_n(x) + K(x-a)^{n+1}$$

$$\left(\text{i.e. } K = \frac{f(x) - P_n(x)}{(x-a)^{n+1}}\right)$$

We want to show $K = \dfrac{f^{(n+1)}(c)}{(n+1)!}$

Define $F(t) = f(t) - \left(P_n(t) + K(t-a)^{n+1}\right)$

$$F(a) = f(a) - (P_n(a) + 0) = 0$$

$$F(x) = 0$$

$\xRightarrow{\text{M.V.T.}}$ $F'(c_1)=0$ for some c_1 between a, x

$F'(a)=f'(a)-(P_n'(a)+0)=0$

$\because P_n'(a)=\frac{f'(a)}{1}+\frac{2f''(a)}{2!}(a-a)+\frac{3f'''(a)}{3!}(a-a)^2+\cdots$

$\xRightarrow[\text{for } F']{\text{MVT}}$ $F''(c_2)=0$ for some c_2 between a and c_1

$\vdots$

$F^{(n+1)}(c)=0$ for some c between a and c_n

$\|$

$f^{(n+1)}(c)-(0+K\cdot n!)=0$

$\Rightarrow K=\frac{f^{(n+1)}(c)}{(n+1)!}$ ——— $\square$

From the proof, it is easy to see that $c\ (=c_{n+1})$ depends on c

Eg 1 $f(x) = e^x$

$$\overline{T}_{f,a}(x) = \sum_{k=0}^{\infty} \frac{e^a}{k!}(x-a)^k$$

$$\xRightarrow{\text{Ratio Test}} \rho = 0 \Rightarrow R = \infty$$

$\therefore \overline{T}_{f,a}(x)$ converges for all $x \in \mathbb{R}$

Moreover $R_n(x) = \dfrac{e^c}{(n+1)!}(x-a)^{n+1}$

c is between a and x

$$a < x \Rightarrow c < x$$
$$x < a \Rightarrow c < a$$

$$\therefore e^c \leq \underbrace{\max(e^a, e^x)}_{\text{independent of } n}$$

$$\therefore \lim_{n \to \infty} R_n(x) = 0 \Rightarrow \underset{x \in \mathbb{R}}{\overline{T}_{f,a}(x) = e^x}$$

Eg 2. $f(x) = \sin x$, $a = 0$

$$T_{\sin x, 0} = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

From Ratio Test $\Rightarrow R = \infty$

$\therefore T_{\sin x, 0}$ converges for $x \in \mathbb{R}$

Secondly $R_n(x) = \frac{\sin^{(n+1)}(c)}{(n+1)!} x^{n+1}$

$$\left|\sin^{(n+1)}(c)\right| = \left|\left(\frac{d^{n+1}}{dx^{n+1}} \sin x\right)_{x=c}\right| \leq 1$$

$$\therefore |R_n| \leq \frac{|x|^{n+1}}{(n+1)!} \xrightarrow[n \to \infty]{} 0$$

$$T_{\sin x, 0}(x) = \sin x, \ \forall x \in \mathbb{R}$$

Rm(1). Similarly

$$\cos x = T_{\cos x, 0}(x)$$
$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

for any $x \in \mathbb{R}$

(2) $$T_{\sin x, a}(x) = \sum_{k=0}^{\infty} \frac{\sin^{(k)}(a)}{k!}(x-a)^k$$

$$\lim_{n\to\infty} |R_n| = \lim_{n\to\infty} \left| \frac{\sin^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} \right|$$
$$= 0 \quad \forall\, x \in \mathbb{R}$$

$\Longrightarrow T_{\sin x, a}(x)$ converges $\forall\, x \in \mathbb{R}$

and $= \sin x$

Similarly for $T_{\cos x, a}(x)$

Thm A: If $f(x)$ has a power series representation $\left(f(x)=\sum_{k=0}^{\infty} a_k (x-a)^k\right)$ on $|x-a|<R$, $R>0$

$$\left(\begin{array}{l} \Longrightarrow f^{(k)}(a) = k!\, a_k \quad k\in\mathbb{N} \\ \text{(Term by Term diff)} \qquad \text{(derivations)} \\ \Longrightarrow T_{f,a}(x) = \sum_{k=0}^{\infty} \frac{k!\, a_k}{k!}(x-a)^k = f(x) \\ \quad \text{on } |x-a|<R \end{array}\right)$$

conclusion:

$$\Longrightarrow \underline{f(x) = T_{f,a}(x) \text{ on } |x-a|<R}$$

Method 2 for Eg 4:

$$\because \frac{1}{1-x} = 1+x+x^2+\cdots \ (R=1>0) \Rightarrow T_{\frac{1}{1-x},0}(x) = 1+x+x^2+\cdots$$

Eg3. Find first few terms of $T_{f,0}(x)$ where $f(x) = e^x \cos x$

<u>Sol</u>. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$

both holds for all $x \in \mathbb{R}$

From Multiplication of Power Series

$(e^x \cdot \cos x)$ has Power Series Representation

and $= 1 + x + \left(\frac{1}{2!} - \frac{1}{2!}\right)x^2 + \frac{x^3}{3!} + \left(\frac{1}{4!} - \frac{1}{2!}\frac{1}{2!}\right)x^4 + \cdots$

$\overset{\text{Thm A}}{\Longrightarrow} = T_{e^x \cos x, 0}(x)$

Binomial Series

$f(x) = (1+x)^m$. $m \in \mathbb{R}$

$T_{f,0}(x) = ? \stackrel{?}{=} f(x)$

(1) If $m \in \mathbb{N}$. $f(x)$ = polynomial

$\Rightarrow T_{f,0}(x) = f(x)$

(2) If $m \notin \mathbb{N}$

$f^{(k)}(x) = m(m-1)\cdots(m-k+1)(1+x)^{m-k}$

$f^{(k)}(0) = m(m-1)\cdots(m-k+1)$

$\Rightarrow f(x) = P_n(x) + R_n(x)$

where $P_n(x) = \sum_{k=0}^{n} \binom{m}{k} x^k$

$R_n(x) = \binom{m}{n+1}(1+c_{n+1})^{m-n-1} x^{n+1}$

$\therefore\ T_{f,0}(x) = \sum_{k=0}^{\infty} \binom{m}{k} x^k$

where $\binom{m}{k} = \dfrac{m(m-1)\cdots(m-k+1)}{k!}$

(i) Does $T_{f,0}(x)$ converge?

Ratio Test $\Rightarrow \rho = |x|$

(m fixed, $k \to \infty$)

$\therefore\ T_{f,0}(x)$ converges on $|x|<1$, diverges on $|x|>1$

(ii) $T_{f,0}(x) \overset{?}{=} f(x)$ on $|x|<1$?

$\lim R_n(x) = 0$?

Ans: Not clear if $-1<x<0$

It can be shown indirectly (and not easily) that

$$\lim_{n\to\infty} R_n(x) = 0 \quad \text{on } |x|<1$$

(Section 10.10. problem 58)

$$\therefore \ (1+x)^m = \sum_{k=0}^{\infty} \binom{m}{k} x^k$$

for all $m \in \mathbb{R}$, $|x|<1$

Important! (注目!)

Known Taylor Series (외워!)

* $\dfrac{1}{1\pm x} = 1 \mp x + x^2 \mp x^3 + \cdots, \quad |x|<1$
* $e^{\pm x} = 1 \pm x + \dfrac{x^2}{2!} \pm \dfrac{x^3}{3!} + \cdots \quad x \in \mathbb{R}$
* $\cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \cdots \quad x \in \mathbb{R}$
* $\sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \cdots \quad x \in \mathbb{R}$
* $\ln(1\pm x) = \pm x - \dfrac{x^2}{2} \pm \quad -1 < x \le 1$
* $\tan^{-1} x = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \cdots \quad |x| \le 1$
* $(1\pm x)^m = 1 \pm mx + \dfrac{m(m-1)}{2!}x^2 \pm \dfrac{m(m-1)(m-2)}{3!}x^3 + \cdots \quad |x|<1$
* $f(x) = \begin{cases} e^{-\frac{1}{x^2}}, & x \ne 0 \\ 0 & x = 0 \end{cases}, \quad T_{f,0}(x) = 0 \ne f(x) \quad x \ne 0$

Eg 1 $T_{\sin^{-1},0}(x) = ?$

<u>Sol</u> $(\sin^{-1}x)' = \dfrac{1}{\sqrt{1-x^2}} \quad |x|<1$

$\sin^{-1}x - \sin^{-1}0 = \int_0^x (\sin^{-1} t)' \, dt$

(Fundamental Thm of Calc)

$= \int_0^x (1-t^2)^{-\frac{1}{2}} dt$ (Binomial, $m = -\frac{1}{2}$)

$= \int_0^x \left(1 - \frac{1}{2}(-t^2) + \frac{(-\frac{1}{2})(-\frac{3}{2})}{2!}(-t^2)^2 + \cdots \right) dt$

$= x + \dfrac{x^3}{6} + \dfrac{x^5}{40} + \cdots, \quad |x|<1$

$\overset{(\text{Thm A})}{=} T_{\sin^{-1},0}(x)$

Eg 2 $\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots,\ |x|<1$

$$\frac{\pi}{4} = \tan^{-1}1 \overset{?}{=} 1 - \frac{1}{3} + \frac{1}{5} - \cdots\ ?$$

<u>Sol</u> $\frac{1}{1+t^2} = 1 - t^2 + \cdots + (-1)^n t^{2n}$

(<u>not</u> Taylor's Thm) $+ \frac{(-1)^{n+1} t^{2n+2}}{1+t^2}\ \forall t \in \mathbb{R}$

$\left(\because 1 + x + \cdots + x^n = \frac{1 - x^{n+1}}{1-x},\ x = -t^2\right)$

$$\Rightarrow \tan^{-1}x - \tan^{-1}0 = \int_0^x \frac{1}{1+t^2}\,dt$$

$$= \int_0^x \left(1 - t^2 + t^4 - \cdots + (-1)^n t^{2n} + \frac{(-1)^{n+1} t^{2n+2}}{1+t^2}\right) dt$$

$$= x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + \frac{(-1)^n x^{2n+1}}{2n+1} + \tilde{R}_n(x)$$

Here $\tilde{R}_n(x) = \int_0^x \frac{(-1)^{n+1} t^{2n+2}}{1+t^2}\, dt$

If $x = 1$.

$|\tilde{R}_n(1)| = \int_0^1 \frac{t^{2n+2}}{1+t^2}\, dt$

$\leq \int_0^1 \frac{t^{2n+2}}{1+0}\, dt = \frac{1}{2n+3}$

(Similarly for $x = -1$)

$\Rightarrow \lim_{n \to \infty} \tilde{R}_n(\pm 1) = 0$

$\therefore \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \cdots$

Leibnitz formula for π

Similarly $\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$

Remark: If $|t|<1$

$$\frac{1}{\underset{大}{1}+\underset{小}{t^2}} = 1 - t^2 + t^4 - \cdots\cdots$$

If $|t|>1$

$$\frac{1}{\underset{小}{1}+\underset{大}{t^2}} = \frac{1}{t^2}\left(\frac{1}{\underset{大}{1}+\underset{小}{\frac{1}{t^2}}}\right)$$

$$= \frac{1}{t^2}\left(1 - \frac{1}{t^2} + \frac{1}{t^4} - \cdots\cdots\right);\ |t|>1$$

Applications:

(I) Approximation and error estimate

(II) Altenative method for "$\lim \frac{0}{0}$"

(*)(III) Find $f(x)$ from $T_{f,a}(x)$

Eg3 (Application I)

Find app. value of $\int_0^{\frac{1}{2}} \sin t^2 \, dt$ and estimate the error.

Ans: $\int_0^{\frac{1}{2}} \sin t^2 \, dt$

$$= \int_0^{\frac{1}{2}} \left(t^2 - \frac{t^6}{3!} + \frac{t^{10}}{5!} \cdots\cdots\right) dt$$

$$= \frac{t^3}{3} - \frac{t^7}{7\cdot 3!} + \frac{t^{11}}{11\cdot 5!} \cdots\cdots \Bigg|_0^{\frac{1}{2}}$$

(Alternating Series

$$= \underbrace{\frac{1}{3\cdot 2^3} - \frac{1}{7\cdot 3!\cdot 2^7} + \frac{1}{11\cdot 5!\cdot 2^{11}}}_{\text{Approximation}} - \cdots$$

$$|E| \leq \frac{\left(\frac{1}{2}\right)^{15}}{15\cdot 7!}$$ (error estimate of Alternating Series)

Eg7 (App II)

$$\lim_{x\to 0} \frac{\sin x - \tan x}{x^3} = ?$$

Sol: Method 1: l'Hôpital.

Method 2.

$$f(x) = \frac{\sin x - \frac{\sin x}{\cos x}}{x^3}$$

$$= \frac{\frac{1}{2}\sin 2x - \sin x}{x^3 \cos x}$$

$$= \frac{\frac{1}{2}\left(\cancel{2x} - \frac{(2x)^3}{3!} + \cdots\right) - \left(\cancel{x} - \frac{x^3}{3!} + \cdots\right)}{x^3\left(1 - \frac{x^2}{2!} + \cdots\right)}$$

$$= \frac{\cancel{x^3}\left(\frac{-4}{3!} + \frac{1}{3!} + \cdots\right)}{\cancel{x^3}\left(1 + \cdots\right)}$$

$x \to 0 \Rightarrow$ Ans $= \frac{-1}{2}$