$$T_{f,a}(x) \stackrel{\text{def}}{=} \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

Questions:

Q1 : For what values of x does it converge? i.e. $R = ?$

Ans: ratio and root test

Q2 : When it converges, does it equal $f(x)$?

Eg: $f(x) = \begin{cases} 0 & x = 0 \\ e^{-\frac{1}{x^2}} & x \neq 0 \end{cases}$

$T_{f,a}(x) = 0 \neq f(x)$ for $x \neq 0$

(A counter example for Q2)

In the following example

$f^{(k)}(0) = 0$ for all k

$\Rightarrow T_{f,0}(x) \equiv 0$ but $f(x) \neq 0$
$(x \neq 0)$

<u>Eg5</u> $f(x) = \begin{cases} e^{\frac{-1}{x^2}}, & x \neq 0 \\ 0, & x = 0, \end{cases}$ $T_{f,0}(x) = ?$

<u>Ans</u>: $f(0) = 0$

$$f'(0) = \lim_{h\to 0} \frac{e^{\frac{-1}{h^2}} - 0}{h - 0} \quad \left(\begin{array}{c} \underline{\text{Not}} \text{ always} \\ = \lim_{h\to 0} f'(h) \end{array} \right)$$

$$\left(= \text{``}\frac{0}{0}\text{''}\right) = \lim_{h\to 0} \frac{\left(\frac{1}{h}\right)}{e^{\frac{1}{h^2}}} \quad \left(\pm \text{``}\frac{\infty}{\infty}\text{''}\right)$$

<u>l'Hôpital</u>
$$\lim_{h\to 0} \frac{-h^{-2}}{-2h^{-3} e^{\frac{1}{h^2}}} = \lim_{h\to 0} \frac{h}{2e^{\frac{1}{h^2}}} = 0$$
$(**)$

$$f''(0) = \lim_{h \to 0} \frac{f'(h) - f'(0)}{h - 0}$$

$$\overset{(**)}{=} \lim_{h \to 0} \frac{2h^{-3} e^{-\frac{1}{h^2}}}{h - 0}$$

$$= \lim_{h \to 0} \frac{2 e^{-\frac{1}{h^2}}}{h^4}$$

$$= \lim_{h \to 0} \frac{2h^{-4}}{e^{\frac{1}{h^2}}}$$

$$\underset{\text{l'Hôpital}}{=} \text{(homework)}$$

$$= 0$$

In fact, $f^{(k)}(0) = 0$ for all $k \in \mathbb{N}$

Thm Taylor's Thm

If $f, f', f'', \ldots$

all exist on $|x-a|<\delta$

Then, for any $n \in \mathbb{N}$

$$f(x) = P_n(x) + R_n(x) \quad \cdots (*)$$

on $|x-a|<\delta$, where

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

for some c between a and x

Since $T_{f,a}(x) = \lim_{n\to\infty} P_n(x)$

$\therefore$ $T_{f,a}(x) = f(x) \Leftrightarrow \lim_{n\to\infty} R_n(x) = 0$

(Answer of Q2)

Corollary: If $|f^{(n+1)}(c)| \leq M$

for all c between a and x

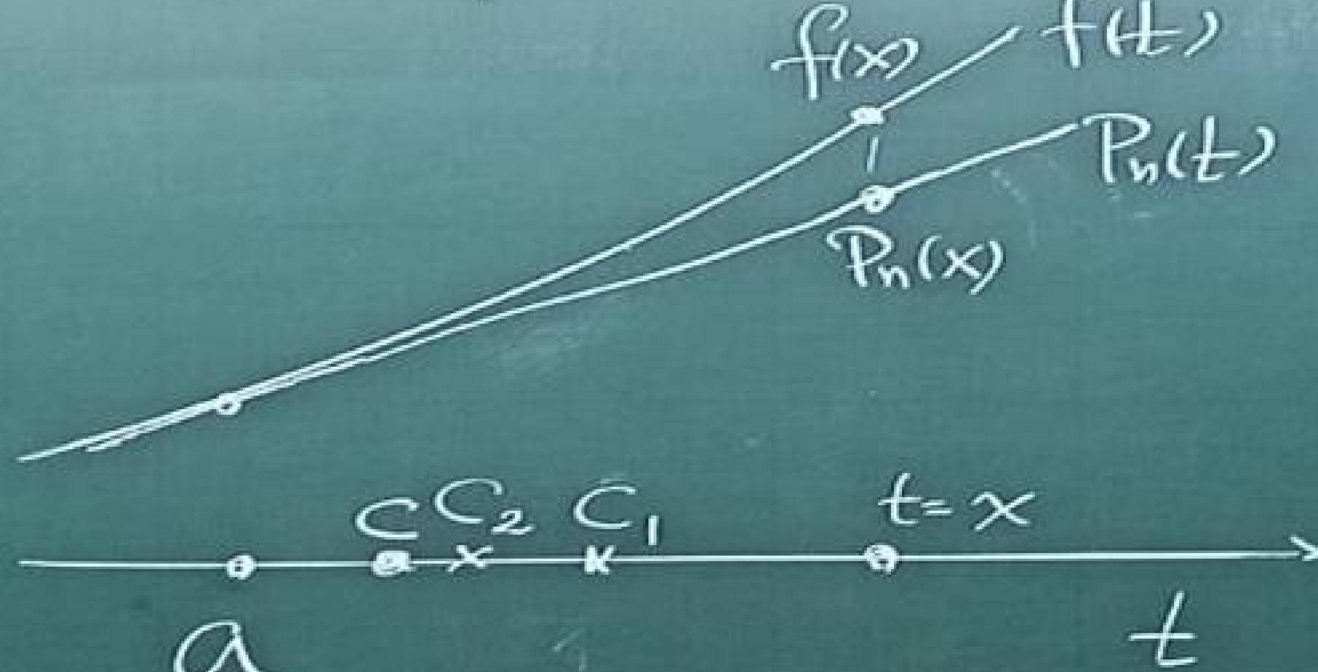
and all $n \in \mathbb{N}$

Then $|R_n(x)| \leq \dfrac{M}{(n+1)!}|x-a|^{n+1}$

$\Rightarrow \lim_{n\to\infty} R_n(x) = 0$

$\Rightarrow T_{f,a}(x) = f(x)$

Proof of Taylor's Thm



We now write (for fixed x)

$$f(x) = P_n(x) + K(x-a)^{n+1}$$

(i.e. $K = \dfrac{f(x) - P_n(x)}{(x-a)^{n+1}}$)

We want to show $K = \dfrac{f^{(n+1)}(c)}{(n+1)!}$

Define $F(t) = f(t) - (P_n(t) + K(t-a)^{n+1})$

$$F(a) = f(a) - (P_n(a) + 0) = 0$$

$$F(x) = 0$$

$\xRightarrow{\text{M.V.T.}}$ $F'(C_1)=0$ for some C_1 between a, x

$F'(a) = f'(a) - (P_n'(a) + 0) = 0$

$\because\ P_n'(a) = \frac{f'(a)}{1} + \frac{2f''(a)}{2!}(a-a) + \frac{3f'''(a)}{3!}(a-a)^2 + \cdots$

$\xRightarrow[\text{for } F']{\text{MVT}}$ $F''(C_2) = 0$ for some C_2 between a and C_1

$\vdots$

$F^{(n+1)}(C) = 0$ for some C between a and C_n

$\parallel$

$f^{(n+1)}(C) - (0 + K(n+1)!) = 0$

$\Rightarrow K = \frac{f^{(n+1)}(C)}{(n+1)!}$ ——— $\square$

From the proof, it is easy to see that $C\ (= C_{n+1})$ depends on n

Eg 1 $f(x) = e^x$

$$T_{f,a}(x) = \sum_{k=0}^{\infty} \frac{e^a}{k!}(x-a)^k$$

$$\xrightarrow{\text{Ratio Test}} \rho = 0 \Rightarrow R = \infty$$

$\therefore T_{f,a}(x)$ converges for all $x \in \mathbb{R}$

Moreover $R_n(x) = \dfrac{e^c}{(n+1)!}(x-a)^{n+1}$

c is between a and x

$$a < x \Longrightarrow c < x$$

$$x < a \Longrightarrow c < a$$

$$\therefore e^c \leq \underbrace{\max(e^a, e^x)}_{\text{independent of } n}$$

$$\therefore \lim_{n \to \infty} R_n(x) = 0 \Rightarrow T_{f,a}(x) = e^x \quad x \in \mathbb{R}$$

Eg 2. $f(x) = \sin x,\ a = 0$

$$T_{\sin x, 0} = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

From Ratio Test $\Rightarrow R = \infty$

$\therefore T_{\sin x, 0}$ converges for $x \in \mathbb{R}$

Secondly $R_n(x) = \dfrac{\sin^{(n+1)}(c)}{(n+1)!} x^{n+1}$

$$\left|\sin^{(n+1)}(c)\right| = \left|\left(\frac{d^{n+1}}{dx^{n+1}} \sin x\right)_{x=c}\right| \leq 1$$

$$\therefore |R_n| \leq \frac{|x|^{n+1}}{(n+1)!} \xrightarrow[n \to \infty]{} 0$$

$$T_{\sin x, 0}(x) = \sin x,\ \forall x \in \mathbb{R}$$

Rm(1). Similarly

$$\cos x = T_{\cos x, 0}(x)$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

for any $x \in \mathbb{R}$

(2) $$T_{\sin x, a}(x) = \sum_{k=0}^{\infty} \frac{\sin^{(k)}(a)}{k!}(x-a)^k$$

$$\lim_{n\to\infty} |R_n| = \lim_{n\to\infty} \left| \frac{\sin^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} \right|$$

$$= 0 \quad \forall\, x \in \mathbb{R}$$

$\Longrightarrow T_{\sin x, a}(x)$ converges $\forall\, x \in \mathbb{R}$

and $= \sin x$

Similarly for $T_{\cos x, a}(x)$

(Another answer to Q2)

Thm A: If $f(x)$ has a power series representation $\left(f(x)=\sum_{k=0}^{\infty} a_k (x-a)^k\right)$ on $|x-a|<R$, $R>0$

$$\Longrightarrow f^{(k)}(a) = k!\,a_k \quad k\in\mathbb{N}$$

(Term by Term diff)

(derivations)

$$\Longrightarrow T_{f,a}(x) = \sum_{k=0}^{\infty} \frac{k!\,a_k}{k!}(x-a)^k = f(x)$$

on $|x-a|<R$

conclusion: $\Longrightarrow \underline{f(x) = T_{f,a}(x) \text{ on } |x-a|<R}$

Method 2 for Eg 4:

$\because \frac{1}{1-x} = 1+x+x^2+\cdots \ (R=1>0) \Rightarrow T_{\frac{1}{1-x},0}(x) = 1+x+x^2+\cdots$

Eg3. Find first few terms of $T_{f,0}(x)$ where $f(x) = e^x \cos x$

Sol. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$

both holds for all $x \in \mathbb{R}$

From Multiplication of Power Series

$(e^x \cdot \cos x)$ has Power Series Representation

and $= 1 + x + \left(\frac{1}{2!} - \frac{1}{2!}\right)x^2 + \frac{x^3}{3!} + \left(\frac{1}{4!} - \frac{1}{2!}\frac{1}{2!}\right)x^4 + \cdots$

$\xRightarrow{\text{Thm A}} = T_{e^x \cos x, 0}(x)$