

$$T_{f,a}(x) \stackrel{\text{def}}{=} \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x-a)^k$$

Questions:

Q1: For what values of x does it converge? i.e. $R = ?$

Ans: ratio and root test

Q2: When it converges, does it equal $f(x)$?

$$\text{Eg. } f(x) = \begin{cases} 0 & x=0 \\ e^{-\frac{1}{x^2}} & x \neq 0 \end{cases}$$

$$T_{f,a}(x) = 0 \neq f(x) \quad \text{for } x \neq 0$$

(A counter example for Q2)

In the following example

$$f^{(k)}(0) = 0 \text{ for all } k$$

$$\Rightarrow T_{f,0}(x) \equiv 0 \text{ but } f(x) \neq 0 \text{ (} x \neq 0 \text{)}$$

$$\text{Eg 5 } f(x) = \begin{cases} e^{-\frac{1}{x^2}}, & x \neq 0 \\ 0, & x = 0, \end{cases} \quad T_{f,0}(x) = ?$$

$$\text{Ans: } f(0) = 0 \quad \left(\begin{array}{l} \text{Not always} \\ = \lim_{h \rightarrow 0} f'(h) \end{array} \right)$$
$$f'(0) = \lim_{h \rightarrow 0} \frac{e^{-\frac{1}{h^2}} - 0}{h - 0}$$

$$\left(\frac{0}{0} \right) = \lim_{h \rightarrow 0} \frac{\left(\frac{1}{h} \right)}{e^{-\frac{1}{h^2}}} \quad \left(\pm \frac{\infty}{\infty} \right)$$

$$\underline{\underline{\text{L'Hopital}}} \quad \lim_{h \rightarrow 0} \frac{-h^{-2}}{-2h^{-3} e^{-\frac{1}{h^2}}} = \lim_{h \rightarrow 0} \frac{h}{2e^{-\frac{1}{h^2}}} = 0 \quad (**)$$

$$f''(0) = \lim_{h \rightarrow 0} \frac{f'(h) - f'(0)}{h - 0}$$

$$\underline{\underline{(**)}} \lim_{h \rightarrow 0} \frac{2h^{-3} e^{-\frac{1}{h^2}}}{h - 0}$$

$$= \lim_{h \rightarrow 0} \frac{2e^{-\frac{1}{h^2}}}{h^4}$$

$$= \lim_{h \rightarrow 0} \frac{2h^{-4}}{e^{-\frac{1}{h^2}}}$$

L'Hopital (homework)

$$= 0$$

In fact, $f^{(k)}(0) = 0$
for all $k \in \mathbb{N}$

Thm Taylor's Thm

If f, f', f'', \dots

all exist on $|x-a| < \delta$

Then, for any $n \in \mathbb{N}$

$$f(x) = P_n(x) + R_n(x) \dots (*)$$

on $|x-a| < \delta$, where

$$P_n(x) = \sum_{k=0}^n \frac{f^{(k)}(a)}{k!} (x-a)^k$$

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} (x-a)^{n+1}$$

for some c between a and x

Since $T_{f,a}(x) = \lim_{n \rightarrow \infty} P_n(x)$

$\therefore T_{f,a}(x) = f(x) \Leftrightarrow \lim_{n \rightarrow \infty} R_n(x) = 0$
(Answer of Q2)

Corollary: If $|f^{(n+1)}(c)| \leq M$

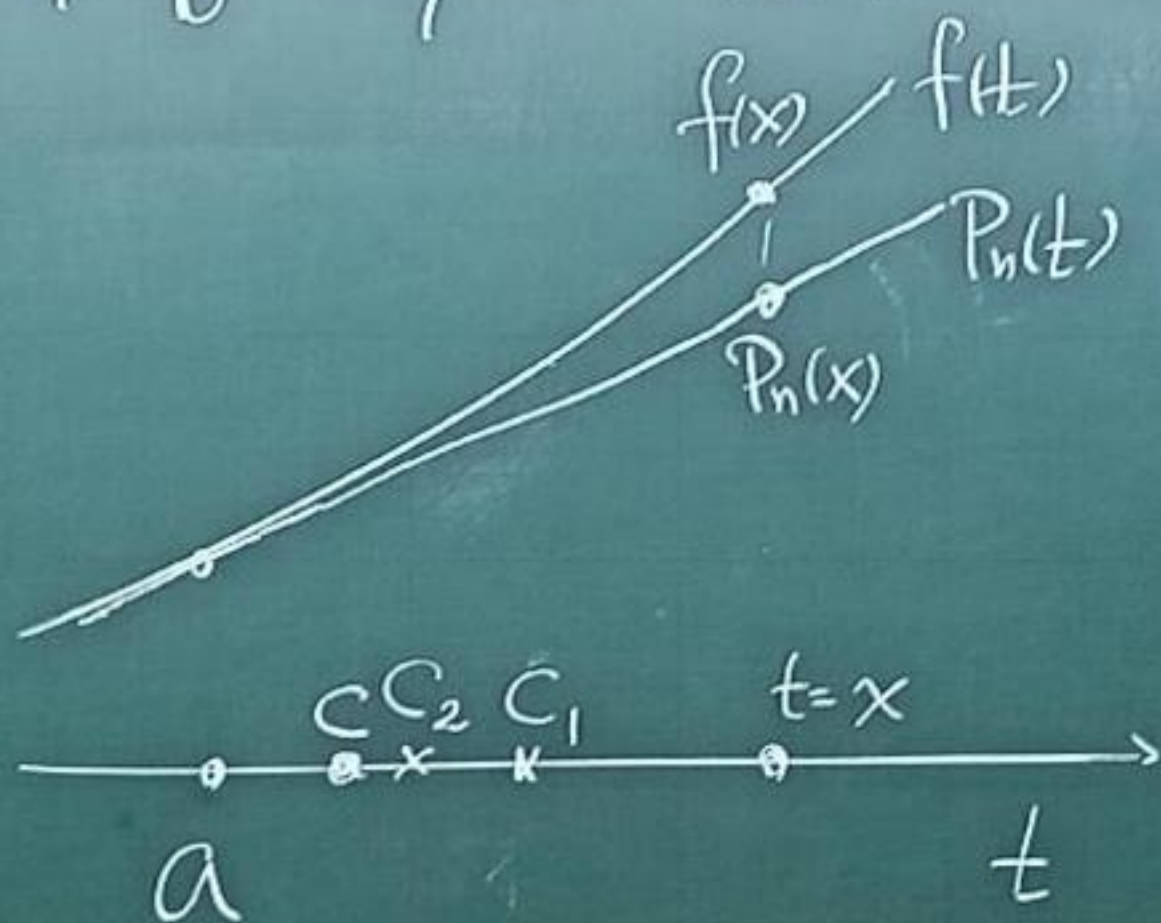
for all c between a and x
and all $n \in \mathbb{N}$.

Then $|R_n(x)| \leq \frac{M}{(n+1)!} |x-a|^{n+1}$

$\Rightarrow \lim_{n \rightarrow \infty} R_n(x) = 0$

$\Rightarrow T_{f,a}(x) = f(x)$

Proof of Taylor's Thm



We now write (for fixed x)

$$f(x) = P_n(x) + k(x-a)^{n+1}$$

(i.e. $k = \frac{f(x) - P_n(x)}{(x-a)^{n+1}}$)

We want to show $k = \frac{f^{(n+1)}(c)}{(n+1)!}$

Define $F(t) = f(t) - (P_n(t) + k(t-a)^{n+1})$

$$F(a) = f(a) - (P_n(a) + 0) = 0$$

$$F(x) = 0$$

M.V.T. $F'(c_1) = 0$ for some c_1 between a, x

$$F'(a) = f'(a) - (P'_n(a) + 0) = 0$$

$$\therefore P'_n(a) = \frac{f'(a)}{1} + \frac{2f''(a)}{2!}(a-a) + \frac{3f'''(a)}{3!}(a-a)^2$$

M.V.T.
for F' $F''(c_2) = 0$ for some c_2 between a and c_1

\vdots
 $F^{(n+1)}(c) = 0$ for some c between a and c_n

$$\begin{aligned} &= f^{(n+1)}(c) - (0 + K(n+1)!) = 0 \\ \Rightarrow K &= \frac{f^{(n+1)}(c)}{(n+1)!} \quad \square \end{aligned}$$

From the proof, it is easy to see that $c (= c_{n+1})$ depends on **n**

$$\text{Eg 1 } f(x) = e^x$$

$$T_{f,a}(x) = \sum_{k=0}^{\infty} \frac{e^a}{k!} (x-a)^k$$

Ratio Test

$$\implies \rho = 0 \implies R = \infty$$

$\therefore T_{f,a}(x)$ converges for all $x \in \mathbb{R}$

Moreover $R_n(x) = \frac{e^c}{(n+1)!} (x-a)^{n+1}$

c is between a and x

$$a < x \implies c < x$$

$$x < a \implies c < a$$

$$\therefore e^c \leq \max(e^a, e^x)$$

independent of n

$$\therefore \lim_{n \rightarrow \infty} R_n(x) = 0 \implies T_{f,a}(x) = e^x \quad x \in \mathbb{R}$$

$$\text{Ex 2. } f(x) = \sin x, a=0$$

$$T_{\sin x, 0} = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$$
$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

From Ratio Test $\Rightarrow R = \infty$

$\therefore T_{\sin x, 0}$ converges for $x \in \mathbb{R}$

$$\text{Secondly } R_n(x) = \frac{\sin^{(n+1)}(c)}{(n+1)!} x^{n+1}$$

$$\left| \sin^{(n+1)}(c) \right| = \left| \left(\frac{d^{n+1}}{dx^{n+1}} \sin x \right)_{x=c} \right| \leq 1$$

$$\therefore |R_n| \leq \frac{|x|^{n+1}}{(n+1)!} \xrightarrow{n \rightarrow \infty} 0$$

$$T_{\sin x, 0}(x) = \sin x, \forall x \in \mathbb{R}$$

Pn(1): Similarly

$$\cos x = T_{\cos x, 0}(x)$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

for any $x \in \mathbb{R}$

$$(2) T_{\sin x, a}(x) = \sum_{k=0}^{\infty} \frac{\sin^{(k)}(a)}{k!} (x-a)^k$$

$$\begin{aligned} \lim_{n \rightarrow \infty} |R_n| &= \lim_{n \rightarrow \infty} \left| \frac{\sin^{(n+1)}(c)}{(n+1)!} (x-a)^{n+1} \right| \\ &= 0 \quad \forall x \in \mathbb{R} \end{aligned}$$

$\Rightarrow T_{\sin x, a}(x)$ converges $\forall x \in \mathbb{R}$

and $= \sin x$

Similarly for $T_{\cos x, a}(x)$

(Another answer to Q2)

Thm A: If $f(x)$ has a power series representation $\left(f(x) = \sum_{k=0}^{\infty} a_k (x-a)^k\right)$ on $|x-a| < R, R > 0$

$$\Rightarrow f^{(k)}(a) = k! a_k \quad k \in \mathbb{N}$$

(Term by Term diff) (derivations)

$$\Rightarrow T_{f,a}(x) = \sum_{k=0}^{\infty} \frac{k! a_k}{k!} (x-a)^k = f(x)$$

on $|x-a| < R$

conclusion:

$$\Rightarrow f(x) = T_{f,a}(x) \text{ on } |x-a| < R$$

Method 2 for Eg 4:

$$\dots \frac{1}{1-x} = 1+x+x^2+\dots \quad (R=1>0) \Rightarrow T_{\frac{1}{1-x},0}(x) = 1+x+x^2+\dots$$

Ex 3. Find first few terms

of $T_{f,0}(x)$ where $f(x) = e^x \cos x$

Sol. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$$

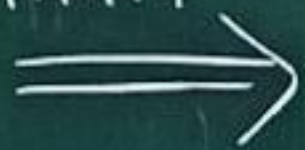
both holds for all $x \in \mathbb{R}$

From Multiplication of Power Series

$(e^x \cdot \cos x)$ has Power Series Representation

$$\begin{aligned} \text{and} &= 1 + x + \left(\frac{1}{2!} - \frac{1}{2!}\right)x^2 + \frac{x^3}{3!} \\ &+ \left(\frac{1}{4!} - \frac{1}{2!} \cdot \frac{1}{2!}\right)x^4 + \dots \end{aligned}$$

Thm A



$$= T_{e^x \cos x, 0}(x)$$