

Term by term differentiation

Thm 21 If  $f(x) = \sum_{n=0}^{\infty} C_n (x-a)^n$

converges (abs.) on  $|x-a| < R$

Then: (1):  $f', f'', f''', \dots$

all exist on  $|x-a| < R$

$$(2). f'(x) = \sum_{n=1}^{\infty} n C_n (x-a)^{n-1}$$

$$f''(x) = \sum_{n=2}^{\infty} n(n-1) C_n (x-a)^{n-2}$$

....

all converge on  $|x-a| < R$

$$\text{Ex 3: } f(x) = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \dots$$

$$= \frac{1}{1-x} \text{ on } |x| < 1$$

what is the power series representation of  $\frac{1}{(1-x)^2}$ ?

$$\underline{\text{Sol}} \quad \frac{1}{(1-x)^2} = f'(x)$$

$$= 1' + x' + (x^2)' + \dots + (x^n)' + \dots$$

$$= 1 + 2x + \dots + nx^{n-1} + \dots$$

$$= \sum_{n=1}^{\infty} nx^{n-1} \text{ converges on } |x| < 1$$

$$\underline{\text{Rm}} \quad \sum_{n=1}^{\infty} \frac{n}{2^{n-1}} = ? \quad \text{Ans.} = \left( \sum_{n=1}^{\infty} nx^{n-1} \right)_{x=\frac{1}{2}}$$

$$= \left( \frac{d}{dx} \sum_{n=0}^{\infty} x^n \right)_{x=\frac{1}{2}} = \frac{1}{(1-x)^2} \Big|_{x=\frac{1}{2}} = 4$$

$$\text{Ex 4. } \sum_{n=1}^{\infty} n^2 x^n = ?$$

$$\text{Sol } \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \text{--- } \textcircled{1}$$

$$\frac{d}{dx} \Rightarrow \frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1} \quad \text{--- } \textcircled{2}$$

$$\frac{d}{dx} \Rightarrow \frac{2}{(1-x)^3} = \sum_{n=2}^{\infty} n(n-1) x^{n-2} \quad \text{--- } \textcircled{3}$$

$$n^2 x^n = x^2 (n(n-1) x^{n-2}) + x (n x^{n-1})$$

$$\Rightarrow \sum_{n=1}^{\infty} n^2 x^n = \sum_{\substack{n=1 \\ (n=2)}}^{\infty} n(n-1) x^n + \sum_{n=1}^{\infty} n x^n$$

$$= x^2 \left( \frac{1}{1-x} \right)'' + x \left( \frac{1}{1-x} \right)'$$

$$= \frac{x+x^2}{(1-x)^3} \quad \text{valid on } |x| < 1$$

Remark Term by term differentiation  
may not be valid for other series

Eg 5:  $f(x) = \sum_{n=1}^{\infty} \frac{\sin(n!x)}{n^2}$  is not  
a power series

Since  $|a_n| \leq \frac{1}{n^2} \Rightarrow f(x)$  converges  
on  $x \in \mathbb{R}$

But  $\sum_{n=1}^{\infty} n a_n = \sum_{n=1}^{\infty} \frac{n!}{n^2} \cos(n!x)$

diverges for any  $x \in \mathbb{R}$ .

Thm 22 (term by term integration)

$$\text{If } f(x) = \sum_{n=0}^{\infty} C_n (x-a)^n$$

converges abs. on  $|x-a| < R$

$$\text{Then } \sum_{n=0}^{\infty} \frac{C_n}{n+1} (x-a)^{n+1} \text{ also}$$

converges on  $|x-a| < R$

$$\text{and } \int f(x) dx = \sum_{n=0}^{\infty} \frac{C_n}{n+1} (x-a)^{n+1} + C$$

In fact (Thm 22')

$$\begin{aligned} \int_a^x f(t) dt &= \sum_{n=0}^{\infty} \int_a^x C_n (t-a)^n dt \\ &= \sum_{n=0}^{\infty} \frac{C_n}{n+1} (x-a)^{n+1} \end{aligned}$$

Ex 6. Evaluate

$$f(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{2n-1}}{2n-1}$$

Sol. Radius of convergence:

Ratio test:  $f(x)$  converge if  $|x|^2 < 1$

Root  
 $\Rightarrow R = 1$

On  $|x| < 1$ ,  $f'(x) = 1 - x^2 + x^4 - \dots = \frac{1}{1+x^2}$

$$f(x) = \int_0^x f'(t) dt = \int_0^x \frac{1}{1+t^2} dt$$

$$= \tan^{-1} x$$

Note:  $\tan^{-1} x$  is defined for all  $x \in \mathbb{R}$   
but  $\neq \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{2n-1}}{2n-1}$  if  $|x| \geq 1$

$$\text{Ex 7: } \ln(1 \pm x) = ? \quad (|x| < 1)$$

Sol. Since  $\frac{d}{dx} \ln(1 \pm x) = \frac{1}{1 \pm x}$

We start with

$$\frac{1}{1 \pm x} = 1 \mp x + x^2 \mp x^3 + \dots, \quad |x| < 1$$

$$\Rightarrow \int_0^x \frac{1}{1 \pm t} dt = \int_0^x (1 \mp t + t^2 \mp t^3 + \dots) dt \quad |x| < 1$$

$$\stackrel{\text{Thm 22'}}{=} \int_0^x 1 dt \mp \int_0^x t dt + \int_0^x t^2 dt \mp \dots$$
$$\pm \ln|1 \pm t| \Big|_0^x = x \mp \frac{x^2}{2} + \frac{x^3}{3} \mp \frac{x^4}{4} + \dots$$

$$\Rightarrow \ln(1 \pm x) = \pm x - \frac{x^2}{2} \pm \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

on  $|x| < 1$

# Taylor Series

Question: For a given function  $f(x)$  and  $a \in \mathbb{R}$ , can we always find  $a_k \in \mathbb{R}$  and  $R > 0$

such that

$$(*) f(x) = \sum_{k=0}^{\infty} a_k (x-a)^k$$

on  $|x-a| < R$ ?

Ans: Not Necessarily.

(only for some  $f$ , not all  $f$ )

Remark If  $a_k \in \mathbb{R}$  and

$R > 0$  do exist, then

we must have  $a_k = \frac{f^{(k)}(a)}{k!}$  (\*)

from term by term diff. Thm.

That is, (\*) is the only candidate  
and it may or may not work!

## Question

If  $f^{(k)}(a)$  exist for all  $k=0, 1, 2, \dots$

Is it necessarily true that (\*) holds with (\*) for some  $R > 0$ ?

Ans: Not necessarily.

(Counter example below  
i.e.  $f(x) \neq \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x-a)^k$  for  $x \neq a$ )

Def. The Taylor Series  
generated by  $f$  at  $x=a$

$$T_{f,a}(x) \stackrel{\text{def}}{=} \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x-a)^k$$

(= Maclaurin Series if  $a=0$ )

Def. The Taylor Polynomial  
of degree  $n$  generated by

$$f \text{ at } x=a: P_{n,a}(x) = \sum_{k=0}^n \frac{f^{(k)}(a)}{k!} (x-a)^k$$

Ex 1:  $f$  is a polynomial.

$$f(x) = a_0 + a_1x + \dots + a_5x^5$$

Find  $P_{3,0}(x)$ ,  $P_{5,0}(x)$ ,  $P_{7,0}(x)$ ,  $T_{f,0}(x)$

Ans:  $f^{(k)}(0) = k! a_k$   $0 \leq k \leq 5$   
 $f^{(l)}(0) = 0$  for  $l > 5$

$$\Rightarrow P_{3,0}(x) = a_0 + a_1x + a_2x^2 + a_3x^3$$

$$P_{5,0}(x) = P_{7,0}(x) = T_{f,0}(x) = f(x)$$

R<sub>m</sub>  $P_{3,0}(x) \neq P_{3,1}(x)$ ,  $T_{f,0}(x) = T_{f,1}(x)$   
 $P_{5,0}(x) = P_{5,1}(x)$ ,  $P_{7,0}(x) = P_{7,1}(x)$

In general, if  $f(x)$  is a polynomial of degree  $n$ , then

$$P_{m,a}(x) = T_{f,a}(x) = f(x)$$

for all  $m \geq n$ .

$$\begin{aligned} \therefore f(x) &= a_0 + a_1x + \dots + a_nx^n = P_{n,0}(x) \\ &= b_0 + b_1(x-a) + \dots + b_n(x-a)^n = P_{n,a}(x) \end{aligned}$$

$$(b_k = \frac{f^{(k)}(a)}{k!})$$

$$P_{m,a}(x) = P_{n,a}(x) \text{ if } m > n.$$

$$\therefore P_{m,a}(x) = f(x) = T_{f,a}(x).$$

Eg 2  $f(x) = e^x$ ,  $T_{f,a}(x) = ?$

Ans:  $f^{(k)}(a) = e^a$

$\therefore T_{f,a}(x) = e^a \sum_{k=0}^{\infty} \frac{(x-a)^k}{k!} \left( \frac{x-a}{k!} \right)$

Remark: from ratio test

$\rho = 0 \Rightarrow R \text{ (for } \sum \frac{(x-a)^k}{k!} \text{)} = \infty$

$\therefore T_{f,a}(x)$  converges for any  $x \in \mathbb{R}$

(In fact,  $T_{f,a}(x) = e^x$  for any  $x \in \mathbb{R}$ )  
(later)

Eg 3:  $T_{\cos(x),0}(x)$

Sol  $\cos^{(n)}(0) = ?$

$n=0$ 4, 8, ...	$n=1$ 5, 9, ...	$n=2$ 6, 10, ...	$n=3$ 7, 11, ...
$\cos 0$	$-\sin 0$	$-\cos 0$	$\sin 0$
$\parallel$	$\parallel$	$\parallel$	$\parallel$
1	0	-1	0

$$\begin{aligned} &\Rightarrow T_{\cos(x),0}(x) \\ &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \\ &= \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} \quad \left( \frac{x^{2k}}{(2k)!} \right) \\ & (= \cos x \text{ for all } x \in \mathbb{R} \text{ (later)}) \end{aligned}$$

Similarly

$$\begin{aligned} T_{\sin(x), 0}(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \\ &= \sum_{k=1}^{\infty} (-1)^{k-1} \frac{x^{2k-1}}{(2k-1)!} \quad \left(\frac{x^k}{k!}\right) \end{aligned}$$

Ex 4  $T_{\frac{1}{1-x}, 0}(x) = ?$

Sol.  $f(x) = \frac{1}{1-x} = (1-x)^{-1}$

$$f'(x) = + (1-x)^{-2}, \quad f'(0) = 1$$

$$f''(x) = +2(1-x)^{-3}, \quad f''(0) = 2!$$

$$f^{(k)}(x) = k! (1-x)^{-k-1}, \quad f^{(k)}(0) = k!$$

$$\Rightarrow T_{\frac{1}{1-x}, 0}(x) = 1 + x + x^2 + x^3 + \dots$$