Term by term differentiation

Thm If $f(x)=\sum_{n=0}^{\infty} C_n (x-a)^n$

converges (abs.) on $|x-a|<R$

Then: (1): $f', f'', f''', \ldots$

all exist on $|x-a|<R$

(2). $f'(x)=\sum_{n=1}^{\infty} n C_n (x-a)^{n-1}$

$f''(x)=\sum_{n=2}^{\infty} n(n-1) C_n (x-a)^{n-2}$

$\ldots$

all converge on $|x-a|<R$

Eg 3: $f(x) = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots$

$= \frac{1}{1-x}$ on $|x| < 1$

what is the power series representation of $\frac{1}{(1-x)^2}$?

Sol $\frac{1}{(1-x)^2} = f'(x)$

$= 1' + x' + (x^2)' + \cdots + (x^n)' + \cdots$

$= 1 + 2x + \cdots + nx^{n-1} + \cdots$

$= \sum_{n=1}^{\infty} nx^{n-1}$ converges on $|x| < 1$

Rm $\sum_{n=1}^{\infty} \frac{n}{2^{n-1}} = ?$ Ans: $= \left(\sum_{n=1}^{\infty} nx^{n-1}\right)_{x=\frac{1}{2}}$

$= \left(\frac{d}{dx} \sum_{n=0}^{\infty} x^n\right)_{x=\frac{1}{2}} = \frac{1}{(1-x)^2}\Big|_{x=\frac{1}{2}} = 4$

Eg 4. $\displaystyle\sum_{n=1}^{\infty} n^2 x^n = ?$

Sol

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \text{—①} \quad (|x|<1)$$

$$\frac{d}{dx} \Rightarrow \frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1} \quad \text{—②}$$

$$\frac{d}{dx} \Rightarrow \frac{2}{(1-x)^2} = \sum_{\substack{n=2 \\ (n=1)}}^{\infty} n(n-1) x^{n-2} \quad \text{—③}$$

$$n^2 x^n = x^2\left(n(n-1)x^{n-2}\right) + x\left(n x^{n-1}\right)$$

$$\Rightarrow \sum_{n=1}^{\infty} n^2 x^n = \sum_{\substack{n=1 \\ (n=2)}}^{\infty} n(n-1) x^n + \sum_{n=1}^{\infty} n x^n$$

$$= x^2 \cdot \left(\frac{1}{1-x}\right)'' + x\left(\frac{1}{1-x}\right)'$$

$$= \frac{x+x^2}{(1-x)^3} \quad \text{valid on } |x|<1$$

<u>Remark</u> Term by term differentiation may not be valid for other series

Eg5: $f(x) = \sum_{n=1}^{\infty} \frac{\sin(n!x)}{n^2}$ is not a power series

Since $|a_n| \le \frac{1}{n^2} \Rightarrow f(x)$ converges on $x \in \mathbb{R}$

But $\sum_{n=1}^{\infty} n a_n = \sum_{n=1}^{\infty} \frac{n!}{n^2} \cos(n!x)$

diverges for any $x \in \mathbb{R}$.

Thm (term by term integration)

If $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ converges abs. on $|x-a|<R$

Then $\sum_{n=0}^{\infty} \frac{c_n}{n+1}(x-a)^{n+1}$ also converges on $|x-a|<R$

and $\int f(x)\,dx = \sum_{n=0}^{\infty} \frac{c_n}{n+1}(x-a)^{n+1} + C$

In fact

$$\int_a^x f(t)\,dt = \sum_{n=0}^{\infty} \int_a^x c_n (t-a)^n\,dt$$

$$= \sum_{n=0}^{\infty} \frac{c_n}{n+1}(x-a)^{n+1}$$

Eg6. Evaluate

$$F(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{2n-1}}{2n-1}$$

Sol. Radius of convergence:

Ratio / Root test: $F(x)$ converge if $|x|^2 < 1$

$\Rightarrow R = 1$

On $|x| < 1$, $F'(x) = 1 - x^2 + x^4 - \cdots = \frac{1}{1+x^2}$

$$F(x) = \int_0^x F'(t)\,dt = \int_0^x \frac{1}{1+t^2}\,dt$$

$$= \tan^{-1} x.$$

Note: $\tan^{-1} x$ is defined for all $x \in \mathbb{R}$

but $\neq \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{2n-1}}{2n-1}$ if $|x| \geq 1$

Eg: $\ln(1 \pm x)$. $|x| < 1$

$$\frac{1}{1 \pm x} = 1 \mp x + x^2 \mp x^3 + \cdots$$

$$\int_0^x \frac{1}{1 \pm t}\,dt = x \mp \frac{x^2}{2} + \frac{x^3}{3} \mp \frac{x^4}{4} + \cdots$$

$$\pm \ln|1 \pm t|\Big|_0^x \underset{|x|<1}{=\!=} \pm \ln(1 \pm t)\Big|_0^x = \pm \ln(1 \pm x)$$

$$\Rightarrow \ln(1 \pm x) = \begin{cases} x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \\ -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots \end{cases}$$

$|x| < 1$

Taylor Series:

Question: For a given function $f(x)$ and $a \in \mathbb{R}$, can we always find $a_k \in \mathbb{R}$ and $R > 0$ such that

$$(*) \quad f(x) = \sum_{k=0}^{\infty} a_k (x-a)^k$$

on $|x-a| < R$?

Ans: Not Necessarily.
(only for some f, not all f)

Remark If $a_k \in \mathbb{R}$ and $R > 0$ do exist, then we must have $a_k = \frac{f^{(k)}(a)}{k!}$ (✡) from term by term diff. Thm.

That is, (✡) is the only candidate and it may or may not work!

Question

If $f^{(k)}(a)$ exist for all $k = 0, 1, 2, \ldots$

Is it necessarily true that (*) holds with (**) for some $R > 0$?

Ans: Not necessarily.

(Counter example below: i.e. $f(x) \neq \sum_{k=0} \frac{f^{(k)}(a)}{k!}(x-a)^k$ for $x \neq a$)

Def: The Taylor Series generated by f at $x=a$

$$T_{f,a}(x) \overset{\text{def}}{=} \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

(= Mclaurin Series if $a=0$)

Def The Taylor Polynomial of degree n generated by f at $x=a$: $P_{n,a}(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$

Eg 1: f is a polynomial.

$$f(x) = a_0 + a_1 x + \cdots + a_5 x^5$$

Find $P_{3,0}(x)$, $P_{5,0}(x)$, $P_{7,0}(x)$, $T_{f,0}(x)$

Ans: $f^{(k)}(0) = k!\, a_k \quad 0 \le k \le 5$

$f^{(l)}(0) = 0 \quad$ for $l > 5$

$$\Rightarrow P_{3,0}(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

$$P_{5,0}(x) = P_{7,0}(x) = T_{f,0}(x) = f(x)$$

Rm $P_{3,0}(x) \neq P_{3,1}(x)$, $T_{f,0}(x) = T_{f,1}(x)$

$P_{5,0}(x) = P_{5,1}(x)$, $P_{7,0}(x) = P_{7,1}(x)$

In general, if $f(x)$ is a polynomial of degree n, then

$$P_{m,a}(x) = T_{f,a}(x) = f(x)$$

for all $m \geq n$.

$$\because f(x) = a_0 + a_1 x + \cdots + a_n x^n = P_{n,0}(x)$$

$$= b_0 + b_1(x-a) + \cdots + b_n(x-a)^n = P_{n,a}(x)$$

$$\left(b_k = \frac{f^{(k)}(a)}{k!}\right)$$

$$P_{m,a}(x) = P_{n,a}(x) \text{ if } m > n.$$

$$\therefore P_{m,a}(x) = f(x) = T_{f,a}(x).$$

Eg 2 $f(x) = e^x$. $T_{f,a}(x) = ?$

<u>Ans</u>: $f^{(k)}(a) = e^a$

$\therefore T_{f,a}(x) = e^a \sum_{k=0}^{\infty} \frac{(x-a)^k}{k!}$ (背!)

<u>Remark</u>: from ratio test

$\rho = 0 \Rightarrow R$ (for $\sum \frac{(x-a)^k}{k!}$) $= \infty$

$\therefore T_{f,a}(x)$ converges for any $x \in \mathbb{R}$

(In fact, $T_{f,a}(x) = e^x$ for any $x \in \mathbb{R}$ (later))

Eg 3: $T_{\cos(x),0}(x)$

Sol $\cos^{(n)}(0) = ?$

$n=0$	$n=1$	$n=2$	$n=3$
$4, 8, \cdots$	$5, 9, \cdots$	$6, 10, \cdots$	$7, 11, \cdots$
$\cos 0$	$-\sin 0$	$-\cos 0$	$\sin 0$
$\shortparallel$	$\shortparallel$	$\shortparallel$	$\shortparallel$
1	0	-1	0

$$\Rightarrow T_{\cos(x),0}(x)$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} \quad (背!)$$

$(= \cos x$ for all $x \in \mathbb{R}$ (later))

Similarly

$$T_{\sin(x),0}(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

$$= \sum_{k=1}^{\infty} (-1)^{k-1} \frac{x^{2k-1}}{(2k-1)!} \quad (背!)$$

Eg 4. $T_{\frac{1}{1-x},0}(x) = ?$

Sol: $f(x) = \frac{1}{1-x} = (1-x)^{-1}$

$f'(x) = +(1-x)^{-2}$, $f'(0) = 1$

$f''(x) = +2(1-x)^{-3}$ $\quad f''(0) = 2!$

$f^{(k)}(x) = k!(1-x)^{-k-1}$ $\quad f'''(0) = 3!$

$$\Rightarrow T_{\frac{1}{1-x},0}(x) = 1 + x + x^2 + x^3 + \cdots$$