**Def.** Power Series

$$\sum_{n=0}^{\infty} C_n(x-a)^n \underset{=}{\overset{\text{def}}{=}} C_0 + \sum_{n=1}^{\infty} C_n(x-a)^n$$

It is a funtion of $x$.

$a$ = center (fixed). $C_n$ = Coefficients

Question: When "$a$" and "$C_n$" are given, for what values of $x$ is the power series convergent?

**Eg 1.** $\displaystyle\sum_{n=0}^{\infty} \left(\frac{-1}{2}\right)^n (x-2)^n$. (Geometric Series)

$$r = \frac{-(x-2)}{2} \qquad (\text{div if } |r| \geq 1)$$

Power Series converges (abs) $\Leftrightarrow |r| < 1 \Leftrightarrow 0 < x < 4$

Eg 2 $\sum_{n=0}^{\infty} \frac{x^n}{n!}$.

$(0! \overset{\text{def}}{=} 1.\ x^0 \overset{\text{def}}{=} 1)$

<u>Sol</u>: Ratio test:

$|u_n| = \frac{|x|^n}{n!}$

$$\lim_{n\to\infty} \frac{|u_{n+1}|}{|u_n|} = \lim_{n\to\infty} \frac{|x|}{n} = 0$$

$\Rightarrow$ For any $x \in \mathbb{R}$

$\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges (absolutely)

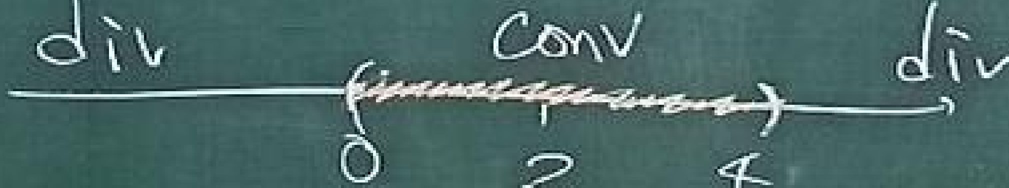
Eg3 $\sum_{n=0}^{\infty} n!\,x^n$

Ratio Test:

$$\lim_{n\to\infty}\frac{|u_{n+1}|}{|u_n|} = \lim_{n\to\infty} n|x| = \begin{cases} 0 & x=0 \\ \infty & x\neq 0 \end{cases}$$

It converges at $x=0$ only and diverges elsewhere.

Eg1




Eg2


Eg3

We now show that the region of convergence of a Power Series must be an interval centered at the "center of the Power Series". The radius of the interval is called the radius of convergence, usually can be found by Ratio Test or Root Test.

**Thm 18:** If $\sum_{n=0}^{\infty} a_n x^n$ $\left(\sum a_n (x-a)^n\right)$

converges at $x = c \neq 0$ $(x = c \neq a)$

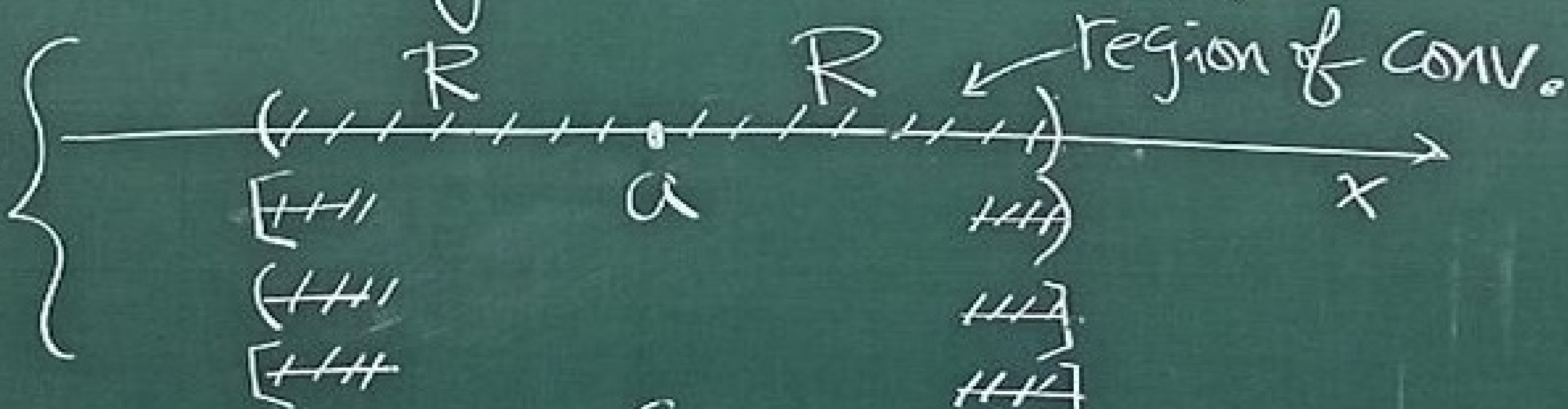
then it converges absolutely

for $|x| < |c|$ $\left(|x-a| < |c-a|\right)$

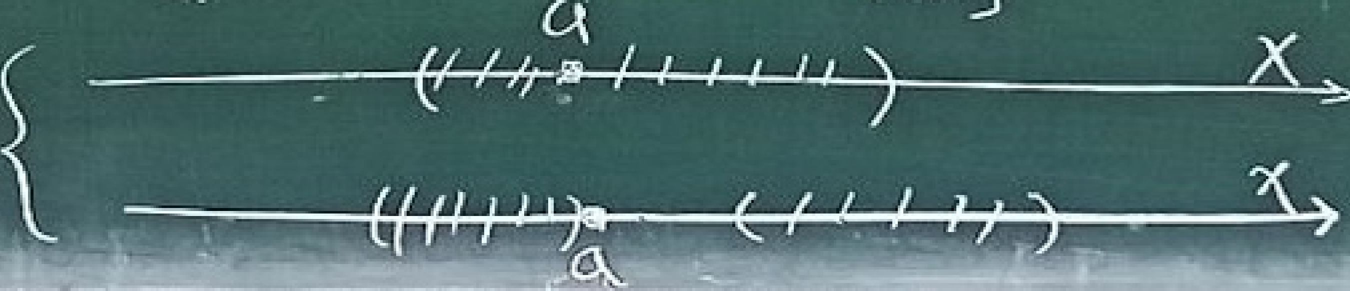
($\Rightarrow$ If it diverges at $x = d$,

then it diverges for $|x| > |d|$ $\left(|x-a| > |d-a|\right)$)

Possible




Impossible



Pf: If $\sum_{n=0}^{\infty} a_n c^n$ converges

$$\Longrightarrow \lim_{n\to\infty} a_n c^n = 0$$

$$\Longrightarrow |a_n c^n| < 1 \quad \forall n \geq N$$

$$\Longrightarrow |a_n| < \frac{1}{|c|^n} \quad \forall n \geq N$$

If $|x| < |c|$

$$\Rightarrow \sum_{n=N}^{\infty} |a_n x^n| \leq \sum_{n=N}^{\infty} \left(\frac{|x|}{|c|}\right)^n < \infty$$

$\therefore \sum_{n=0}^{\infty} a_n x^n$ converges absolutely on $|x| < |c|$

In Summary: Possible regions of convergence.

(1) It converges absolutely for all $x \in \mathbb{R}$.
i.e. Radius of conv. $R = \infty$

(2) Convergence only at $x = a$ $(R = 0)$

(3) $\exists\ 0 < R < \infty$

it $\begin{cases} \text{conv. (absolutely) on } |x| < R \\ \text{div. on } |x| > R \end{cases}$

$R$ = radius of conv. for $\sum a_n (x-a)^n$

In general $0 \leq R \leq \infty$

How to find $R$ for $\sum a_n(x-a)^n$?

Ans: If $\lim\limits_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \rho \quad 0 \le \rho \le \infty$

$$\Rightarrow \lim_{n\to\infty} \frac{|a_n(x-a)^{n+1}|}{|a_n(x-a)^n|} = \rho\,|x-a|$$

$$\Rightarrow \sum a_n(x-a)^n \begin{cases} \text{conv. abs.} & \text{if } |x-a| < \frac{1}{\rho} \\ \text{div.} & \text{if } |x-a| > \frac{1}{\rho} \end{cases}$$

i.e. $R = \frac{1}{\rho}$

Similarly, if $\lim\limits_{n\to\infty} |a_n|^{\frac{1}{n}} = \rho$

Then $R = \frac{1}{\rho}$

<u>Remark</u>: $R$ (radius of conv.) always exists for $\sum_{n=0}^{\infty} a_n(x-a)^n$

but $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$ or $\lim_{n\to\infty} |a_n|^{\frac{1}{n}}$

may or may not exist.

<u>Eg</u>. $\sum_{n=1}^{\infty} a_n x^n$

$= \left(\frac{x}{2}\right)^1 + \left(\frac{x}{4}\right)^2 + \left(\frac{x}{2}\right)^3 + \left(\frac{x}{4}\right)^4 + \cdots$

$\lim \frac{|a_{n+1}|}{|a_n|}$, $\lim |a_n|^{\frac{1}{n}}$ do not exist

<u>Rm</u>: $R = \left(\limsup_{n\to\infty} |a_n|^{\frac{1}{n}}\right)^{-1} = 2$

Eg 1 ⓐ $\sum_{n=0}^{\infty} x^n$ conv. on $(-1, 1)$
div. elsewhere

ⓑ $\sum_{n=0}^{\infty} \frac{x^n}{n}$ conv. on $[-1, 1)$
div. elsewhere

$a_n = \frac{1}{n} \xrightarrow[\text{Root}]{\text{Ratio}} \rho = 1 \Rightarrow R = 1$

$|x| < 1$. conv, $|x| > 1$. div

$x = 1$. p-series, $p = 1$

$x = -1$ Alternating Series test

ⓒ $\sum_{n=0}^{\infty} \frac{(-x)^n}{n}$ conv on $(-1, 1]$
div elsewhere.

ⓓ $\sum_{n=0}^{\infty} \frac{x^n}{n^2}$ conv on $[-1, 1]$
div elsewhere.

# Algebraic Manipulation of Power Series

If $A(x) = \sum_{n=0}^{\infty} a_n x^n$

$B(x) = \sum_{n=0}^{\infty} b_n x^n$

both conv on $|x| < R$.

Then $A(x) \pm B(x) = ?$

$A(x) \cdot B(x) = ?$

$A(x) / B(x) = ?$

Thm 19: If $A(x)=\sum_{n=0}^{\infty} a_n x^n$

$B(x)=\sum_{n=0}^{\infty} b_n x^n$

both conv (abs.) on $|x|<R$

and $C_n = a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0$

$\left(=\sum_{k=0}^{n} a_k b_{n-k}\right)$

Then $\sum_{n=0}^{\infty} C_n x^n$ conv. abs. and $= A(x)B(x)$

on $|x|<R$.

<u>Sol</u>. $A(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$

$B(x) = b_0 + b_1 x + b_2 x^2 + b_3 x^3 + \cdots$

$\Rightarrow A(x)\cdot B(x) = a_0 b_0 + \overset{(a_0b_1+a_1b_0}{\cancel{(a_1 b_0)}}\, x + (a_2 b_0 + a_1 b_1 + a_0 b_2) x^2 + \cdots$

Proof of Thm 19 requires this

Lemma. If $\sum_{n=0}^{\infty} a_n$ converges, then

$$\lim_{N\to\infty}\left(\sum_{n=N}^{\infty} a_n\right) = 0 \quad \text{---} \quad (*)$$

Pf $\displaystyle\sum_{n=0}^{N-1} a_n + \sum_{n=N}^{M} a_n = \sum_{n=0}^{M} a_n \quad (M>N)$

Take $\lim_{M\to\infty}(\quad)$

$$\Rightarrow \sum_{n=0}^{N-1} a_n + \sum_{n=N}^{\infty} a_n = \sum_{n=0}^{\infty} a_n$$

Take $\lim_{N\to\infty}(\quad)$

$$\Rightarrow \cancel{\sum_{n=0}^{\infty} a_n} + \lim_{N\to\infty}\left(\sum_{n=N}^{\infty} a_n\right) = \cancel{\sum_{n=0}^{\infty} a_n} \quad \#$$

Rm (1) $\int_1^{\infty} f(x)\,dx$ conv. $\Rightarrow \lim_{N\to\infty}\left(\int_N^{\infty} f(x)\,dx\right) = 0$

(2) $(*)$ is the same as $\lim_{N\to\infty}\left(\sum_{n=2N}^{\infty} a_n\right) = 0$
(used in next page)

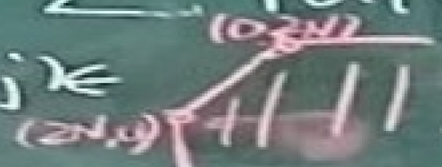
Pf (of Thm 19): Let $|x|<R$

$$\sum_{n=0}^{\infty} c_n x^n \text{ converges abs.}$$

$$\Leftrightarrow \lim_{N\to\infty} \sum_{n=2N}^{\infty} |c_n||x|^n = 0$$

($c_n = a_0 b_n + \dots + a_n b_0 \Rightarrow |c_n| <= |a_0 b_n| + \dots + |a_n b_0|$)

$$\sum_{n=2N}^{\infty} |c_n||x|^n \le \sum_{(i,j)\in} |a_i b_j x^{i+j}|$$

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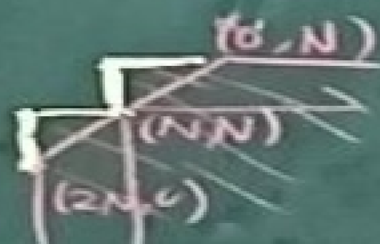
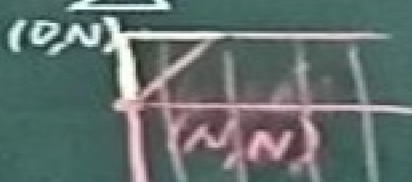
$$\le \sum_{(i,j)\in} |a_i b_j x^{i+j}|$$

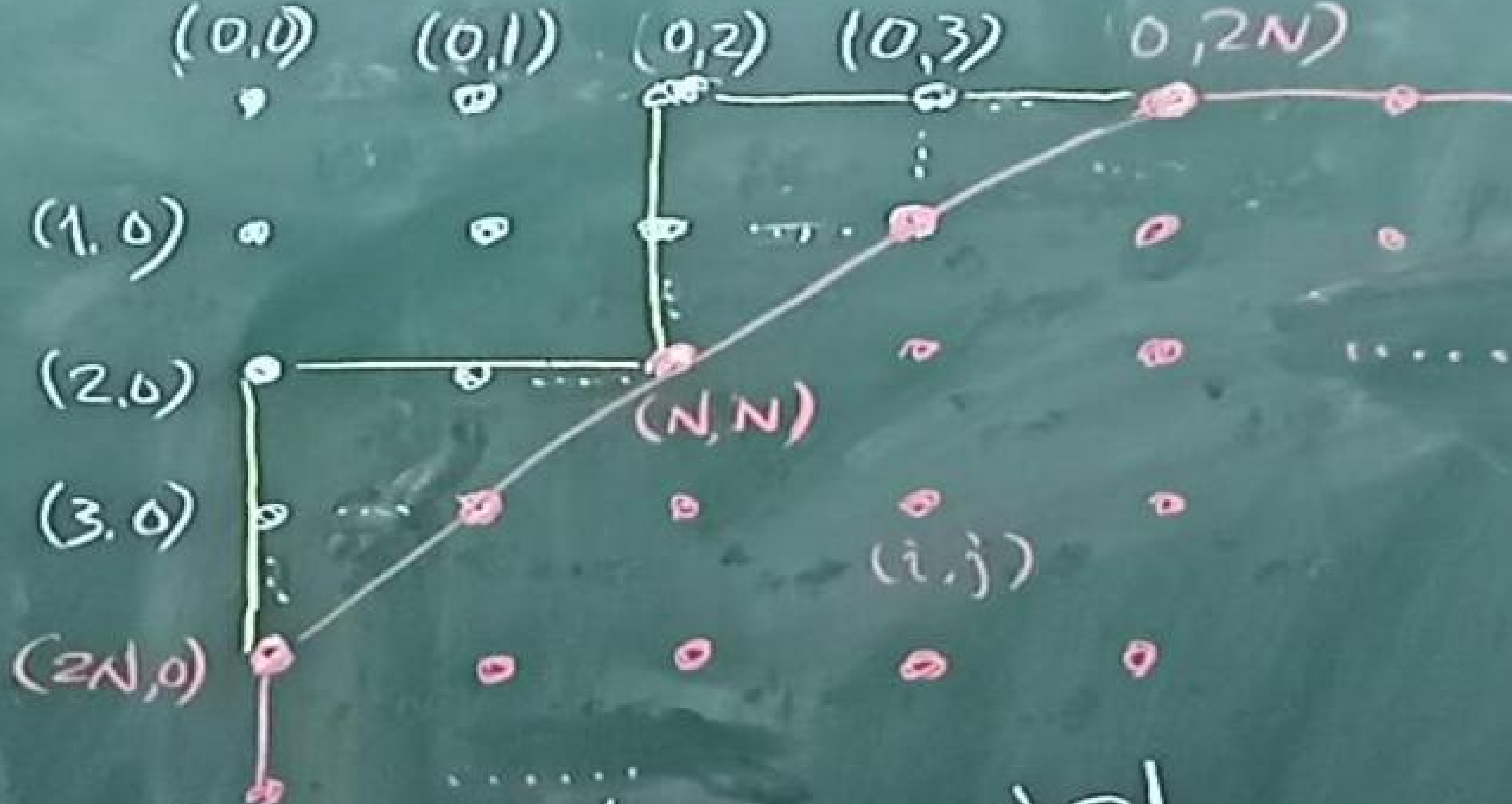



$$= \sum + \sum - \sum |a_i b_j x^{i+j}|$$



$$= \mathrm{I} + \mathrm{II} - \mathrm{III}$$




$$(i,j) \longleftrightarrow |a_i b_j x^{i+j}|$$

$$\mathrm{I} = \sum_{i=N}^{\infty} \sum_{j=0}^{\infty} |a_i b_j x^{i+j}| = \left(\sum_{i=N}^{\infty} |a_i x^i|\right)\left(\sum_{j=0}^{\infty} |b_j x^j|\right)$$

$$\mathrm{II} = \left(\sum_{i=0}^{\infty} |a_i x^i|\right)\left(\sum_{j=N}^{\infty} |b_j x^j|\right)$$

$$\mathrm{III} = \left(\sum_{i=N}^{\infty} |a_i x^i|\right)\left(\sum_{j=N}^{\infty} |b_j x^j|\right)$$

$\sum |a_n x^n|$, $\sum |b_n x^n|$ conv $\Longrightarrow \lim_{N\to\infty} \mathrm{I}, \mathrm{II}, \mathrm{III} = 0$

If $A(x)=\sum_{n=0}^{\infty} a_n x^n$

$B(x)=\sum_{n=0}^{\infty} b_n x^n$

both converge (abs.) on $|x|<R$

Can we find Power series Representation of $\frac{A(x)}{B(x)}$ if $b_0 \neq 0$?

Sol. If $C(x)=\frac{A(x)}{B(x)}=\sum_{n=0}^{\infty} c_n x^n$

$\Rightarrow A(x)=B(x)\cdot C(x)$

$\Rightarrow a_0=b_0c_0 \Rightarrow c_0=\frac{a_0}{b_0}$

$a_1=b_0c_1+b_1c_0 \Rightarrow c_1=\frac{1}{b_0}(\cdots)$

$a_2=b_0c_2+b_1c_1+b_2c_0 \Rightarrow c_2=\frac{1}{b_0}(\cdots)$

$\cdots\cdots$

Alternatively, we can find $C_n$ more efficiently by long division.

Eg 1. $A(x) = 1$

$$B(x) = 1 + x + \frac{x^2}{2} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

(In fact $B(x) = e^x$)

Find first few terms of $\frac{A(x)}{B(x)}$

$$
\begin{array}{r|l}
 & 1 \quad -1 \quad +\frac{1}{2} \quad -\frac{1}{6} + \cdots \\
\hline
1 + 1 + \frac{1}{2} + \frac{1}{6} + \cdots & 1 + 0 + 0 + 0 + \cdots \\
 & 1 + 1 + \frac{1}{2} + \frac{1}{6} + \cdots \\
\cline{2-2}
 & \quad -1 - \frac{1}{2} - \frac{1}{6} + \cdots \\
 & \quad -1 - 1 - \frac{1}{2} + \cdots \\
\cline{2-2}
 & \qquad \frac{1}{2} + \frac{1}{3} + \cdots \\
 & \qquad \frac{1}{2} + \frac{1}{2} \\
\cline{2-2}
 & \qquad\quad -\frac{1}{6} + \cdots
\end{array}
$$

Ans:

$$= 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \cdots$$

$(= e^{-x})$

<u>Rm</u> $B(x) = \sum_{n=0}^{\infty} b_n x^n$, $|x| < R$

can be defined for $z \in \mathbb{C}$, $|z| < R$

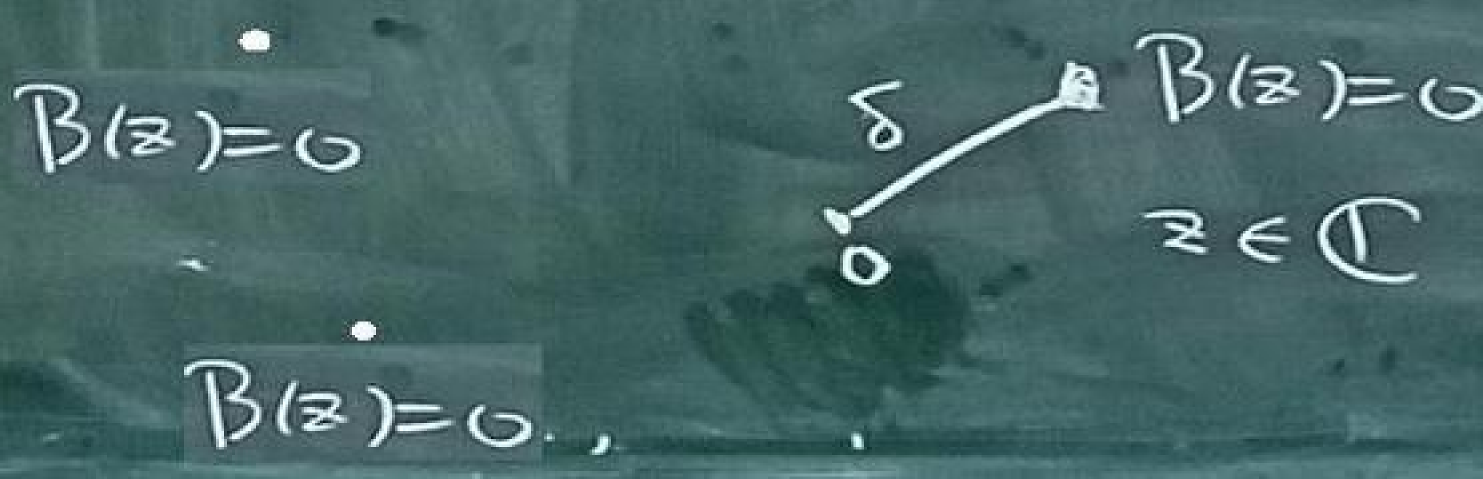
ie. $B(z) = \sum_{n=0}^{\infty} b_n z^n$, $|z| < R$

(and Thm 18 remains valid)

Moreover $\dfrac{1}{B(z)}$ can be defined

as a power series on $|z| < \delta$

where $\delta = \min\{|z|,\ B(z) = 0\}$



<u>Rm</u> If $A(x), B(x)$ both conv. (abs.) on $|x|<R$ and $b_0 = B(0) \neq 0$.

$\Rightarrow \exists\ \delta > 0$ such that $\frac{A(x)}{B(x)}$ computed above converges on $|x| < \delta$.

Eg2: $A(x) = 1$, $B(x) = 1 - x$ / $1 + x^2$

both converge on $|x| < \infty$

but $\frac{A(x)}{B(x)} = 1 + x + x^2 + x^3 + \cdots \quad (\delta = 1)$

$1 - x^2 + x^4 - x^6 + \cdots \quad (\delta = 1)$