Title: Last two pieces of puzzle for un-solvability of a system of two quadratic (in)equalities

Abstract:

Given two quadratic functions \(f(x) = x^T Ax + 2a^T x + a_0 \) and \(g(x) = x^T Bx + 2b^T x + b_0 ,\) each associated with either the strict inequality (<0); non-strict inequality (≤0); or the equality ($=0$), it is a fundamental question to ask whether or not the joint system has a solution? For homogeneous quadratic systems ($a=b=0,~a_0=b_0=0$), starting from Finsler's lemma in 1936 until Yuan's alternative lemma in 1990, all combinations of the un-solvability for $\{x\in \mathbb{R}^n \colon x^\top Ax \star 0\}\cap \{x\in \mathbb{R}^n
 \colon x^\top Bx \# 0\}\subset\{0\},$ where $\star$ and $\#$ can be any of $\{<,\le,=\},$ have been shown to possess either a positive definite or a positive semi-definite matrix pencil of A and $B.$ Extensions to non-homogeneous quadratic systems $\{x\in \mathbb{R}^n   \colon f(x) \star 0\}\cap \{x\in \mathbb{R}^n   \colon g(x) \# 0\}=\emptyset$ have been done for several cases already. There remains two challenging cases still open: the non-homogeneous Calabi Theorem which determines when $\{f(x)=0\}\cap\{ g(x)=0\}=\emptyset$; and the non-homogeneous (strict) Finsler lemma to determine whether $\{f(x)\le0\}\cap\{g(x)=0\}=\emptyset.$ The talk provides the answers to both.