CHAPTER 9, POLAR COORDINATES AND PARAMETRIC CURV ES

9.1 Conic section

A conic section with eccentricity e, is given by $e^2(x+p)^2 = (x-p)^2 + y^2$ which can be normalized to

$$\frac{(x - \frac{1+e^2}{1-e^2}p)^2}{\frac{4e^2p^2}{(1-e^2)^2}} + \frac{y^2}{\frac{4e^2p^2}{1-e^2}} = 1.$$

9.2 Polar Coordinates

Polar Coordinates. *A polar coordinate of a point P is given by (r, θ),where θ is a angle from the positive x-axis to the line OP,and r is the distance from O to P along the direction given by θ.*

Example. $(2, \frac{\pi}{3}), (-2, \frac{4\pi}{3}), (2, \frac{7\pi}{3}), (-2, -\frac{2\pi}{3})$.

Coordinate change. *The relation between the rectangular coordinate and the polar coordinate is given by*

$$x = r\cos\theta, \quad y = r\sin\theta,$$

$$r^2 = x^2 + y^2, \quad \tan\theta = \frac{y}{x}.$$

Polar coordinate equation. $r = f(\theta)$ or $F(r, \theta) = c$.

Example.

(1) $r = a$,
(2) $r = 2\sin\theta$,
(3) $x^2 + y^2 = 2y$,
(4) $r = 2 + 2\cos\theta, r = a = b\cos\theta$,
(5) $r = 2\cos 2\theta, r = a\cos n\theta$.

Symmetriy in polar coordinate.

(1) $F(r, \theta) = F(r, -\theta)$ implies symmetry with respect to x-axis.
(2) $F(r, \theta) = F(r, \pi - \theta)$ implies symmetry with respect to y-axis.
(3) $F(r, \theta) = F(-r, \theta)$ implies symmetry with respect to origin.

Example.

(1) $r^2 = -4\sin 2\theta$,
(2) $r = 1 + \sin\theta, r^2 = 4\sin\theta$.

9.3 Area Computation in Polar Coordinate

$$A = \frac{1}{2}\int_{\alpha}^{\beta} r^2 d\theta.$$

Example.

(1) $r = 3 + 2\cos\theta, 0 \le \theta \le 2\pi,$
(2) The area enclosed by inner and outer loop of $r = 1 + 2\cos\theta), 0 \le \theta \le 2\pi,$
(3) Area outside of $r = 1$ but inside of $r = 3 + 2\cos\theta), 0 \le \theta \le 2\pi.$

9.4 Parametric Curves

$$x = f(t), y = g(t).$$

Example.

(1) $x = \cos t, y = \sin t, 0 \le t \le 2\pi,$
(2) $x = \frac{1-t^2}{1+t^2}, y = \frac{2t}{1+t^2}, -\infty < t < \infty,$
(3) $x = t - 1, y = 2t^2 - 4t + 1, 0 \le t \le 2,$
(4) $x = \cos at, y = \sin bt, -\infty < t < \infty,$
(5) Cycloid $x = a(t - \sin t), y = a(1 - \cos t, 0 \le t \le 2\pi.$

Tangent line.

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}, \quad \frac{d^2y}{dx^2} = \frac{d}{dt}(\frac{dy}{dx}))/\frac{dx}{dt}$$

Example.

(1) $x = a(t - \sin t), y - a(1 - \cos t),$
(2) $x^3 = 2y^6 - 5y^4 + 9y.$

Polar curve as Parametric Curve.

$$x = f(\theta)\cos\theta, y = f(\theta)\sin\theta$$

$$\frac{dy}{dx} = \frac{r'\sin\theta + r\cos\theta}{r'\cos\theta - r\sin\theta}$$

Let $\psi = \alpha - \theta$, then $\cot\psi = \frac{r'}{r}$.

Example. $r = e^{\theta}, \psi = \frac{\pi}{4}$, at $(e^{\frac{\pi}{2}}, \frac{\pi}{2}))$ tangent line is $x + y = e^{\frac{\pi}{2}}$.

Integral Computation with Parametric Curves

(1) $A = \int_a^b y dx = \int_{\alpha,\beta}^{\beta,\alpha} g(t) f'(t) dt$,
(2) $V_x = \int_a^b \pi y^2 dx = \int_{\alpha,\beta}^{\beta,\alpha} \pi (g(t))^2 f'(t) dt$,
(3) $V_y = \int_a^b 2\pi x y dx = \int_{\alpha,\beta}^{\beta,\alpha} 2\pi f(x) g(t) f'(t) dt$,
(4) $L = \int_\alpha^\beta \sqrt{(f'(t))^2 + (g'(t))^2} dt$,
(5) $S_{x.y} = \int_\alpha^\beta 2\pi |g(t)| [|f(t)|] \sqrt{(f'(t))^2 + (g'(t))^2} dt$.

Example.

(1) $x = a \cos t, y = a \sin t$, area, volume, surface area.
(2) $x = a(t - \sin t), y = a(1 - \cos t)$ area under one arc.

Parametric Polar Curves.

$$x(t) = r(t) \cos \theta(t), y(t) = r(t) \sin \theta(t), ds = \sqrt{(r'(t))^2 + r(t)^2 \theta'(t)^2} dt.$$

Example. $r = 1 + \cos \theta$, perimeter and surface area.

Conic Section in Polar Coordintes

$$e(p - r \cos \theta) = r,$$
$$r = \frac{ep}{1 + e \cos \theta}.$$

Example. $r = \frac{16}{5 - 3 \cos \theta}$.