CHAPTER 8, DIFFERENTIAL EQUATIONS

8.1 Simple Differential Equations and Model

Initial value problem.

$$\frac{dy}{dx} = F(x, y), \qquad y(x_0) = y_0$$

Theorem. *Suppose that F satisfied the Lipshitz condition,*

$$|F(x, y) - F(x', y')| \le C\sqrt{(x - x')^2 + (y - y')^2},$$

for some fixed C, then the initial value problem has unique local solutuon.

In case one variable is missing.

$$\text{missiing } \ y \qquad \frac{dy}{dx} = g(x), \quad y(x) = \int g(x)dx + C$$

$$\text{missing } \quad x \qquad \frac{dy}{dx} = h(y), \quad \int \frac{dy}{h(y)} = x + C$$

Example. $y' = y^2, y(0) = 2$, then $-\frac{1}{y} = x + c$, since $y(0) = 2, c = -\frac{1}{2}$, $y = \frac{2}{1-2x}$

Natural growth equations.

$$\frac{dx}{dt} = kx, \quad \text{then} \quad x(t) = x_0 e^{kt}.$$

Example 3 page 579. $P(0) = 6 \times 10^9, P'(0) = 212,000 \times 365.25$,
(a) $k = ?$, (b) $P(50) = ?$, (c) $P(?) = 60 \times 10^9$.

Half life.

$$\frac{dN}{dt} = -kN,$$

then

$$N(t) = N_0 e^{-kt}, \frac{1}{2}N_0 = N_0 e^{-k\tau}, \tau = \frac{\ln 2}{k}.$$

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Torricelle's Law. page 583

$$v = \sqrt{2gy}, A(y)dy = dV = -avdt = -a\sqrt{2gy}dt,$$

so

$$y' = -k\frac{\sqrt{y}}{A(y)}, k = a\sqrt{2g}$$

Example.

(1) Example 8 page 584 $A(y) = A, y(0) = 10, y(1) = 5, y(?) = 0,$
(2) Example 9 page 584 $A(y) = \pi(4^2 - y^2), \pi(4^2 - y^2)y' = -\pi(\frac{1}{24})^2\sqrt{64y},$

$$32y^{1/2} - \frac{2}{5}y^{5/2} = -\frac{1}{72}t + C.$$

8.2 Slope Field and Euler Method

8.3 Separable Equations and Applications

$$y' = g(x)h(y), f(y)y' = g(x), \int f(y)dy = \int g(x)dx.$$

Example.

(1) $y' = -6xy, (a)y(0) = 7, (b)y(0) = -4,$
(2) $y' = 6x(y-1)^{2/3},$
(3) $y' = \frac{4-2x}{3y^2-5}, y(1) = 3.$

Heating and Cooling.

$$\frac{du}{dt} = -k(u - A), -\ln|u - A| = kt + C$$

Example 4 pag 602 $u(0) = 50. A = 375, u(75) = 125, u(?) = 150$

8.4 Linear Equations and Applications

$$y' + P(x)y = Q(x)$$

Example. $x^3y' + xy = 2x^3 + 1$

Integration factor.

$$I(x) = \int P(x)dx,$$

$$e^{I(x)}y' + e^{I(x)}P(x)y = e^{I(x)}Q(x),$$

$$e^{I(x)}y = \int e^{I(x)}Q(x)dx$$

Example.

(1) $y' - y = \frac{11}{8}e^{-x/3}, y(0) = 0,$
(2) $(x^2 + 1)y' + 3xy = 6x.$

In initial value problem, we can let

$$I(x) = \int_{x_0}^{x} P(t)dt,$$

then

$$y(x) = e^{-I(x)} \int_{x_0}^{x} e^{I(t)}Q(t)dt + y_0.$$

Example. $x^2y' + xy = \sin x, y(1) = y_1.$

Mixed Problem. page 612

$$x' = r_i c_i - r_0 \frac{x(t)}{V(t)}$$

Example.

(1) Example 5 page 613 $V = 480, r_i = r_0 = 350, c(0) = 5c, c(?) = 2c,$

$$x(t) = cV + 4cVe^{-\frac{r}{V}t}.$$

(2) Example 6 page 614 $x(0) = 90, V(0) = 90, c_i = 2, r_i = 4, r_0 = 3, V(t_1) = 120, x(t_1) =?$

Motion with resistance.

$$\frac{dv}{dt} = -g - kv$$

Example. $k = 0.16, v(0) = 160$

(a) $y_M =?$, (b) $y(t_1) = y_M$, (c) $v(t_2) =?$ Exerciae 32, 34, 36 page 619

$$v' = -kv,$$

$$v' = -kv^2,$$

$$v' = -kv^{3/2}.$$

8. Population Model

$$\frac{dP}{dt} = \beta(t)P - \delta(t)P,$$

where $\beta(t0$ is the birth rate and $\delta(t)$ is the death rate.

$\beta(t) \equiv \beta_0$.

$$\frac{dP}{dt} = (\beta_0 - \delta_0)P,$$

is the natural growth equation.

$$P(t) = P(t_0)e^{(\beta_0 - \delta_0)(t-t_0)}.$$

$\beta(t) = \beta P9t), \delta(t) \equiv 0.$.

$$\frac{dP}{dt} = \beta P^2.$$

$$P(t) = \frac{P(0)}{1 - \beta P(0)t}.$$

Example (1) page 620, $\beta = 0.0005, P(0) = 100$,

$$P(t) = \frac{2000}{20 - t},$$

$P(t) \to \infty$ as $t \to 20^-$.

Bounde Population, Logistic equation $\beta(t) = \beta_0 - \beta_1 P(t), \delta(t) = \delta.$.

$$\frac{dP}{dt} = (\beta_0 - \beta_1 P - \delta)P = kP(M - P),$$

where $k = \beta_1, M = \frac{\beta_0 - \delta}{\beta_1}$.

$$P(t) = \frac{MP(0)}{P(0) + (M - P(0))e^{-kMt}}.$$

Reamrk.

(1) *If $P(0) = M$ then $P(t) = M$.*
(2) *If $P(0) < M$ then $P(t) < M$ and $lim_{t\to\infty}P(t) = M$,*
(3) *If $P(0) > M$ then $P(t) > M$ and $lim_{t\to\infty}P(t) = M$.*

M is the limit population and sometimes called the carrying capacity of the envirment.

Example 2 page 621. $k = 0.0004, M = 150$, then

$$P(t) = \frac{150P(0)}{P(0) + (M - P(0))e^{-0.06t}}.$$

Example 3 page 623, $P(1885) = 50 \times (10)^6, P'(1885) = 0.75 \times (10)^6, P(1940) = 100 \times (10)^6, P'(1940) = (10)^6$. Then $k = (10)^{-4}, M = 200 \times (10)^6$ and

$$P(t) = \frac{(200 \times (10)^6)(100 \times (10)^6)}{100 \times (10)^6 + (100 \times (10)^6 e^{-0.02t}}.$$

Doomsday versus Extinction, $\beta(t) = kP`\delta(t) = \delta.$.

$$\frac{dP}{dt} = kP(P - M).$$

$$P9t) = \frac{MP(0)e^{-kmt}}{P(0)e^{-kMt} - (P(0) - M)}.$$

Example 7 page 625, $k = 0.0004, M = 150$ (a) $P(0) = 200$. (b) $P(0) = 100$.

8.6 Linear Second Order Equations

$$A(x)y'' + B(x)y' + C(x)y = F(x).$$

Homogeneous Equations.

$$A(x)y'' + B(x)y' + C(x)y = 0 \tag{H.}$$

Theorem. *If y_1, y_2 are solution of (H), then $y = c_1y_1 + c_2y_2$ is also a solution of (H)*

Remark. The correspondence $y \to (y(x_0), y'(x_0))$, which preserve the linear structure , by the uniquence of the solution of the initial value problem, is an isomorphism between the solution space of (H) and R^2. Two solutions y_1, y_2 of (H) is said to be independent if $y_1 \neq cy_2$ for any $c \in R$ as a function of x.

Example. $x^2y'' + 2xy' - 6y = 0, y_1 = x^2, y_2 = x^{-3}, y(1) = 10, y'(1) = 5.$

Theorem. *If y_1, y_2 are two indefendent solutions of (H), then any solution y of (H) can be wriiten as $y = c_1y_c + c_2y_2$ in an unique way.*

Constant Coefficients Equations.

$$ay'' + by' + cy = 0,$$

with characteristic polynomial

$$a\lambda^2 + b\lambda + c = 0 \tag{*}$$

(A) If $(*)$ has two distinct real roots λ_1, λ_2, then $e^{\lambda_1 x}, e^{\lambda_2 x}$ are independent solutions .

Example.

(1) $3y'' + 7y' + 2y = 0,$
(2) $5y'' - 2y = 0,$
(3) $y'' - 4y = 0$, $\cosh 2x, \sinh 2x$

(B) If $(*)$ has double real; roots λ, then $e^{\lambda x}, xe^{\lambda x}$ are independent solutions.

Example. $4y'' + 12y' + 9y = 0, y(0) = 4, y'(0) = -3.$

(C) If $(*)$ has complex conjugate roots $p \pm iq$, then bu Euler identity

$$e^{iu} = \cos u + i \sin u,$$

$e^{px}\cos qx, e^{px}\sin qx$ are independent solutions.

Example.

(1) $y'' + 4y = 0$,
(2) $9y'' + 6y' + 325y = 0, y(0) = 12, y'(0) = 325$.

Particular solution of the inhomogeneous equation.

$$ay'' + by' + cy = f(x).$$

(1) If $f(x) = a_0x^n + \cdots + a_n$, then if $c \neq 0$ $y_p(x) = b_0x^n + \cdots + b_n$, if $c = 0$ but $b \neq 0$ $y_p(x) = b_0x^{n+1} + \cdots + b_{n+1}$
(2) If $f(x) = \cos\omega x$ or $f(x) = \sin\omega x$ and $i\omega$ is not a root of (*), then $y_p(x) = \alpha\cos\omega x + \beta\sin\omega x$.
(3) If $f(x) = \cos\omega x$ or $f(x) = \sin\omega x$ and $i\omega$ is a root of (*), then $y_p(x) = \alpha x\cos\omega + \beta x\sin\omega$.
(4) If $f(x) = e^{\lambda x}$ and λ is not a root of (*), then $y_p(x) = ce^{\lambda x}$.
(5) If $f(x) = e^{\lambda x}$ and λ is a simple root of (*), then $y_p(x) = cxe^{\lambda x}$.
(6) If $f(x) = e^{\lambda x}$ and λ is a double root of (*), then $y_p(x) = cx^2e^{\lambda x}$.

Example.

(1) $y'' + 2y = x^2 + 1, y(0) = 2, y'(0) = 4$,
(2) $y''(x) + 4y = \sin 2x, y(0) = 1, y'(0) = 2$,
(3) $y'' - 4y = e^{2x}, y(0) = 3, y'(0) = 5$.

8.7 Mechanical Vibration

Force of the spring $F_S = -kx$
Force of the resistance $F_R = -cv$
External force $F_E = F(t)$
Total force $F_T = F_S + F_R + F_E$
Newton's law $F_T = ma$, so the mathematical model is $mx'' + cx' + kx = F(t)$.

Free Undamped Motion.

$$mx'' + kx = 0.$$

The general solution is

$$x(t) = A\cos\sqrt{\frac{k}{m}}t + B\sin\sqrt{\frac{k}{m}}t.$$

$\omega_0 = \sqrt{\frac{k}{m}}$ is the circular frequency. $C = \sqrt{A^2 + B^2}$ is the amptitute,

$$x(t) = C\cos(\omega_0 t - \alpha)$$

, α is the phase angle, $\cos\alpha = \frac{A}{C}, \sin\alpha = \frac{B}{C}$, $T = \frac{2\pi}{\omega_0}$ is the period and $v = \frac{1}{T}$ is the frequency.

Example.

(1) on page 642. $m = \frac{1}{2}kg, 100 = k2, k = 50, x(0) = \frac{1}{2}, x'(0) = 0$. $x'' + 100x = 0, x(t) = \frac{1}{2}\cos 10t$.

(2) on page 643 as in (1) $x'(0) = -10, A = \frac{1}{2}, B = -1, C = \sqrt{5}/2, , \cos\alpha = \frac{1}{\sqrt{5}}, \sin\alpha = -\frac{2}{\sqrt{5}}, T = \frac{2\pi}{10}, \alpha = 2\pi - \tan^{-1} 2$.

Free Damped Motion.

$$mx'' + cx' + kx = 0.$$

(A) Over Damped. $c^2 - 4km > 0, \, , p_1 = -\frac{c+\sqrt{c^2-4km}}{2m}$ $p_2 = -\frac{c-\sqrt{c^2-4km}}{2m}$

$$x(t) = c_1e^{-p_1t} + c_2e^{-p_2}.$$

(B) Critical Damped. $c^2 - 4km = 0$, $p = \frac{c}{2m}$

$$x(t) = c_1e^{-pt} + c_2te^{-pt}.$$

(C) Under Damped. $c^2 - 4km < 0$, $r = -\frac{c}{2m} \pm i\sqrt{\frac{k}{m} - (\frac{c}{2m})^2}, p = \frac{c}{2m}, \omega_1 = \sqrt{\omega_0^2 - p^2}$.

$$x(t) = e^{-pt}(A\cos\omega_1 t + B\sin\omega_1 t) = e^{-pt}C(\cos\omega_1 t - \alpha_1),$$

ω_1 is the citcular frequency, $T_1 = \frac{2\pi}{\omega_1}$ is the pseudo period, Ce^{-pt} is the time verying amptitute.

Example. (3) page 645. As in (2) $c = 6, x(0) = \frac{1}{2}, x'(0) = -10$,

$$x'' + 12x' + 100x = 0$$

$r = -6\pm 8i$, $x(t) = e^{-6t}(A\cos 8t + B\sin 8t)$, $x(0) = \frac{1}{2}, A = \frac{1}{2}, x'(0) = -10 = -6A + 8B, B = -\frac{7}{8}$, $C = \frac{\sqrt{65}}{8}$, $T_1 = \frac{2\pi}{8}, \alpha = 2\pi - \tan^{-1}\frac{7}{16}$.

Forced Oscillation. $F(t) \neq 0$,

$$mx'' + cx' + kx = F_0\cos\omega t$$

$$x_p(t) = A\cos\omega t + B\sin\omega t, x(t) = x_e(t) + x_p(t).$$

Example.(4) of page 647. $m = 1, c = 0, k = 9, F_0 = 80, \omega = 5$

$$x'' + 9x = 80\cos 5t, x(0) = x'(0) = 0.$$

(5) of page 648

$$m = 1, c = 2, k = 26, F_0 = 82, \omega = 4, x(0) = 6, x'(0) = 0.$$