CHAPTER 7, TECHNIQUE OF INTEGRATIONS

7.2 Integral Table , Simple Substitutions

Example.

(1) $\int \frac{1}{x}(1+\ln x)^5\,dx$,
(2) $\int \frac{x\,dx}{1+x^4}$,
(3) $\int \frac{x^2\,dx}{\sqrt{25-16x^2}}$.

7.3 Integration by Parts

$$\int f'(x)g(x)dx = f(x)g(x) - \int f(x)g'(x)dx.$$

Example.

(1) $\int \ln x\,dx$,
(2) $\int \sin^{-1} x\,dx$,
(3) $\int xe^x\,dx$,
(4) $\int x^3e^x\,dx$,
(5) $\int e^{2x}\sin 3x\,dx$,
(6) $\int \sec^n x\,dx$.

7.4 Trigonometric Intgrals

(1) $\int \sin^n x\,dx$,

$$\int \sin x\,dx = \cos x + C,$$

$$\int \sin^2 x\,dx = \int \frac{1-\cos 2x}{2}dx = \frac{x}{2} - \frac{\sin 2x}{4} + C,$$

$$\int \sin^{2m+1} x\,dx = \int (1-\cos^2 x)^m\,d\cos x,$$

$$\int \sin^{2m} x\,dx = \int (\frac{1-\cos 2x}{2})^m dx.$$

Reduction formulae, since

$$\int \sin^n x\,dx = -\cos x\sin^{n-1} x + \int \cos x(n-1)\sin^{n-2} x\cos x\,dx = -\cos x\sin^{n-1} x + \int (n-1)(1-\sin^2 x)\sin^n$$

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$$\int \sin^n x dx = -\frac{\cos x \sin^{n-1} x}{n} + \frac{n-1}{n} \int \sin^{n-2} x dx.$$

(2) $\int \cos^n x dx$,
(3) $\int \tan^n x dx$,

$$\int \tan x dx = -\ln|\cos x| + C,$$

$$\int \tan^2 x dx = \int \sec^2 x - 1 dx = \tan x - x + C.$$

Reduction formulae ,

$$\int \tan^n x dx = \int (\sec^2 x - 1) \tan^{n-2} x dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x dx.$$

(4) $\int \cot^n x dx$,
(5) $\int \sec^n x dx$,

$$\int \sec x dx = \int \sec x \frac{\tan x + \sec x}{\tan x + \sec x} dx = \ln|\sec x + \tan x| + C,$$

$$\int \sec^2 x dx = \tan x + C.$$

Reduction formulae , since

$$\int \sec^n x dx = \tan x \sec^{n-2} x - \int \tan x (n-2) \sec^{n-3} x \tan x \sec x dx,$$

$$\int \sec^n x dx = \frac{\tan x \sec^{n-2} x}{n-1} - \frac{n-2}{n-1} \int \sec^{n-2} x dx.$$

(6) $\int \csc^n x dx$,
(7) $\int \sin^n x \cos^m x dx$,
(8) $\int \sec^n x \tan^m x dx$.

Example.

(1) $\int \sin^3 x \cos^2 x dx$,
(2) $\int \sin^2 x \cos^2 x dx$,
(3) $\int \cos^4 x dx$,
(4) $\int_0^{\pi/2} \sec \frac{x}{2} dx$,
(5) $\int \tan^3 x \sec^3 x dx$,
(6) $\int \sec^6 2x dx$,
(7) $\int \tan^6 x dx$.

7.5 Ratioal Functions , Partial Fraction

Theorem. *(Partial Fraction) Suppose that the rational function* $\frac{P(x)}{Q(x)}$ *with ndegree* P *less than degree* Q *and* $Q(x) = C\Pi_{j=1}^{M}(x-a_j)^{m_j}\Pi_{k=1}^{N}(x^2+b_kx+c_k)^{n_k}$ *is the fractorization of* Q *into the product of irreducible factors. Then*

$$\frac{P(x)}{Q(x)} = \frac{1}{C}[\sum_{j=1}^{M}\sum_{i=1}^{m_j}\frac{\alpha_{j,i}}{(x-a_j)^i} + \sum_{k=1}^{N}\sum_{i=1}^{n_k}\frac{\beta_{k,i}x+\gamma_{k,i}}{(x^2+b_kx+c_k)^i}].$$

The theorem is a consequence of the next two lemmas.

Lemma. *Suppose that* $P9x), Q(x)$ *are two relatively prime polynomials (they don't have common factor of degree greater or equal one) and degree* $R(x)$ *is less than degree* $P(x)$ *plus degree* $Q(x)$*, then*

$$\frac{R(x)}{P(x)Q(x)} = \frac{S(x)}{P(x)} + \frac{T(x)}{Q(x)}$$

where degree $S(x)$ *and degree* $T(x)$ *are less than degree* $P(x)$*, degree* $Q(x)$ *respectively.*

Lemma. *Suppose that* $P(x) = p(x)^m$ *and degree* $Q(x)$ *is less than degree* $P(x)$*, then*

$$\frac{Q(x)}{P(x)} = \sum_{i=1}^{m}\frac{q_i(x)}{p(x)^i},$$

where degree $q_i(x)$ *is less then degree* $p(x)$*.*

Example.

(1) $\int \frac{x^3-1}{x^3+x}dx,$
(2) $\int \frac{x^3+x^2+x-1}{x^2+2x+2}dx,$
(3) $\int \frac{5}{(2x+1)(x-2)}dx,$
(4) $\int \frac{4x^-3x-4}{x^3+x^2-2x}dx,$
(5) $\int \frac{x^3-4x-1}{x(x-1)^3}dx,$
(6) $\int \frac{5x^3-3x^2+2x-1}{x^4+x^2}dx.$

7.6 Trigonometric Substitution

(1) $a^2-x^2, x = a\sin\theta,$
(2) $a^2+x^2, x = a\tan\theta,$
(3) $x^2-a^2, x = a\sec\theta.$

Example.

(1) $\int \frac{x^3}{\sqrt{1-x^2}}dx,$
(2) $\int \sqrt{a^2-u^2}du,$
(3) $\int \frac{dx}{(4x^2+9)^2},$
(4) $\int \frac{\sqrt{x^2-25}}{x}dx,$
(5) $\int \frac{dx}{\sqrt{x^2-1}}.$

7.7 Integral Involving Quadratic Polumials

Example.

(1) $\int \frac{dx}{9x^2+6x+5}$,
(2) $\int \frac{dx}{\sqrt{9+16x-4x^2}}$,
(3) $\int \frac{2x+3}{9x^2+6x+5}dx$,
(4) $\int \frac{2+6x}{(3+2x-x^2)^2}dx$,
(5) $\int \frac{u^1}{\sqrt{a^2-u^2}}du$.

Integral of rational function in trigonometric functions. *let* $t = \tan\frac{x}{2}$, *then* $\sin x = \frac{2t}{1+t^2}, \cos x = \frac{1-t^2}{1+t^2}, dx = \frac{2dt}{1+t^2}$.

Example. $\int \frac{2+\sin x}{3+\cos x}dx$

7.8 Improper Integrals

Either f is not bounded or I is not a bounded closed interval.

Example.

(1) a). $\int_1^\infty \frac{dx}{x^2}$, b) $\int_{-\infty}^1 \frac{dx}{\sqrt{1-x}}$,
(2) $\int_{-\infty}^\infty \frac{dx}{1+x^2}$,
(3) a) $\int_0^1 \frac{dx}{\sqrt{x}}$, b) $\int 1^2\frac{dx}{(x-2)^2}$,
(4) $\int_0^2 \frac{dx}{(2x-1)^{2/3}}$.

Special function and improper integrals.

Gamma function.

$$\Gamma(t) = \int_0^\infty x^{t-1}e^{-x}dx,$$

$\Gamma(1) = 1 = \Gamma(2)$, *and* $\Gamma(t+1) = t\Gamma(t)$, *hence* $\Gamma(n+1) = n!$, $\Gamma(\frac{1}{2}) = \frac{1}{2}\int_0^\infty e^{-x^2}dx$.

Escape velocity. $W_\infty = \int_R^\infty \frac{GMm}{r^2}dr = \frac{GMm}{R}$, $\frac{1}{2}mv^2 = \frac{GMm}{R}$, *hence* $v = \sqrt{\frac{2GM}{R}}$