# CHAPTER 6, APPLICATION OF INTEGRAL

## 6.1 Riemann Sum Approximations

$$\mathcal{F} \approx \sum_i F(x_i^*) \triangle x_i$$

where $F(x)$ us a continuous function on $[a, b]$, then $\mathcal{F} = \int_a^b F(x)dx$.

**Example.**

(1) Example 1 on p.416 $c(t) = 50 - t, t \in [0, 30]$,
(2) Example 2 p.417. $\rho(x) = 15 + 2x, x \in [0, 20]$,
(3) Example 3 p.417 $Q = \lim_{n\to\infty} 2x_i e^{-x_i^2}, x_i = 1 + \frac{i}{n}$,
(4) Example 4 p.418 $v(t) = t^2 - 11t + 24, t \in [0, 10]$, distance ? net distance ?
(5) Example 5 p.420 $\triangle F_i = 2\pi x_i v(x_i) \triangle x_i$, $v(x) = 1$,
(6) Example 6 p.420 $v(x) = \frac{P}{2\nu L}(r^2 - x^2)$,
(7) Example 7 p.421$\triangle A_i = Fc(t_i)\triangle t_i$, $F = \frac{A}{\int_0^T c(t)dt}$.

## 6.2 Volume by Method of Cross Sections

$$V = \int_a^b A(x)dx.$$

**Example.** $V = \int_0^h (\frac{b}{h}(h - y))^2 dy$.

**Solid of Revolution.**

$$V = \int_a^b \pi(f(x))^2 dx$$

.

**Example.**

(1) $y^2 = x, x \in [0, 2]$,
(2) Volume of the unit ball.
(3) Volume of the circular cone with $(r, h)$ is $\frac{\pi}{3} r^2 h$,
(4) $y = x^3, x = y^2$ about $x$-axis and $y$-axis.
(5) About $x = -1$,
(6) $x^2 + y^2 \leq 1, 0 \leq z \leq y$,
(7) Ex 47.

## 6.3 Volume by Method of Cylindrical Shells

$$V = 2\pi \int_a^b x f(x) dx.$$

**Example.**

(1) $y = 3x^2 - x^3, [0, 3]$,
(2) $y = \sqrt{b^2 - x^2}, [a, b]$

$$V = 2\pi \int_a^b x(f(x) - g(x)) dx.$$

**Example.**

(1) $y = x^3, y^2 = x$,
(2) As in (1) by $x = -1$.

## 6.4 Arc Length and Surface Area of Revolution

**Arc Lemght.**

$$s = \int_a^b \sqrt{1 + f'(x)^2} dx = \int_c^d \sqrt{1 + g'(y)^2} dy.$$

$$s = \int ds, \quad where \quad ds = \sqrt{1 + f'(x)^2} dx = \sqrt{1 + g'(y)^2} dy.$$

**Example.**

(1) $y = x^{3/2}, [0, 5]$ ,
(2) $y = \frac{1}{2} \sin \pi x, [0, 36]$,
(3) $x = \frac{1}{6} y^3 + \frac{1}{2y}, [1, 2]$

**Surface Area of Revolution.**

*If* $y = mx, [0, b]$ *then* $A = \pi m \sqrt{1 + m^2} b^2$.
*If* $0 < a < b$, *then* $A_{a,b} = \pi m \sqrt{1 + m^2}(b^2 - a^2)$.
*For general curve* $\triangle A_i \approx 2\pi f(x_i^*) \sqrt{1 + f'(x_i^{**})} \triangle x_i$. *Hence*

$$A = \int_a^b 2\pi f(x) ds.$$

**Example.**

(1) $y = x^3, [0, 2]$ about $x$-axis,
(2) $y = x^2, [0, \sqrt{2}]$ about $y$-axis.

## 6.5 Force and Work

Constant force, $W = F \cdot d$
$\triangle W_i \approx F(x_i^*)\triangle x_i$, $W = \int_a^b F(x)dx$

**Elastic Spring.** $F(x) = kx$.

Example (2) p.458 Natural length 1ft, 0.5 ft 10lb, $k = 20$, $W = \int_0^1 20xdx = 10$

**Work against gravity.** $F = \frac{k}{r^2}$.

Example (3) p.459 $s = 1000lb, R = 4000ml$, $k = 16 \times (10)^9$, $W \int_{4000}^{5000} \frac{k}{r2}dr = 16 \times (10)^9(\frac{1}{4000} - \frac{1}{5000})$.

**Work done Filling a Tank.** $\triangle F_i \approx \rho A(y_i^*)\triangle y_i$, $W = \int_a^b \rho y A(y)dy$.

Example (4) p.461 b=750, h=500, $\rho$ =120, 160/h, 12/d, 330/y, T=20.

**Work Emptying a Tank.** $W = \int_a^b \rho(h-y)A(y)dy$

Example (5) p.462 $r = 3, l = 10, \rho = 40, h = 5+3$.

**Force Exerted by Liquid.** $F = \int_a^b \rho(c-y)w(y)dy$.

Example (6) $r = 4, \rho = 75, c = 4$.

## 6.6 Centroid of Plane Region and Curve

$$M = \sum m_i, M_y = \sum m_i x_i, M_x = \sum m_i y_i, \overline{x} = M_y/M, \overline{y} = M_x/M.$$

**Moment and Centroid of Plane Region.** $m_i = f(x_i^*)\triangle x_i, y_i = \frac{f(x_i^*)}{2}$,

$$\overline{x} = \frac{\int_a^b xf(x)dx}{\int_a^b f(x)dx}, \quad \overline{y} = \frac{\int_a^b \frac{f(x)^2}{2}dx}{\int_a^b f(x)dx}$$

.

$$\overline{x} = \frac{\int_a^b x(f(x)-g(x))dx}{\int_a^b (f(x)-g(x))dx}, \quad \overline{y} = \frac{\int_a^b \frac{f(x)^2-g(x)^2}{2}dx}{\int_a^b (f(x)-g(x))dx}$$

(1) Example (1) p.479 $y = \sqrt{a^2-x^2}, [-a, a]$,
(2) Example (2) p.471 $[(0,0), (0,1), (1,0)]$.

**Theorem.** *(First Pappus theorem)*

$$V_x = 2\pi\overline{y}A, \quad V_y = 2\pi\overline{x}A$$

**Example.**

(1) (4) (1) about $x$-axis, $\overline{y} = \frac{4a}{3\pi}$,
(2) (5) $(x-b)^2 + y^2 = a^2$, about $y$-axis.

**Moment and Centroid of Curve.**

$$\overline{x} = \frac{\int xds}{s}, \quad \overline{y} = \frac{\int yds}{s}.$$

**Example.**(6) p.473 As (1)

**Theorem.** *(Second Pappus theorem)*

$$S_x = 2\pi\overline{y}s, \quad S_y = 2\pi\overline{x}s$$

Example (6) p.473. The torus.

## 6.7 Natural Logarithm as Integral

**Natural Logarithm.**

(1) *For* $x > 0$, $\ln x = \int_1^x \frac{dt}{t}$,
(2) $(\ln x)' = \frac{1}{x}, (\ln x)'' = -\frac{1}{x^2}$.
(3) $\ln e = 1$.

**Graph of $y = \ln x$.**

**Laws of Logarithm.**

(1) $\ln xy = \ln x + \ln y$,
(2) $\ln 1 = 0$,
(3) $\ln \frac{1}{x} = -\ln x$,
(4) $\ln \frac{x}{y} = \ln x - \ln y$,
(5) $\ln x^r = r \ln x, r \in \mathcal{Q}$.
(6) $\ln x : (0, \infty) \to (-\infty, \infty)$.

**Natural Exponential Function.**

(1) $\exp x = y$ *if and only if* $\ln y = x$,
(2) $\exp \ln y = y, \ln \exp x = x$,
(3) $D_x \exp x = \exp x$
(4) $e = \exp 1$, *then* $e^r = \exp r, r \in \mathbb{Q}$, *so define* $e^x = \exp x, x \in \mathbb{R}$, *hence* $\ln e^x = x \ln e$.

**Laws of Exponential.**

(1) $e^{x+y} = e^x e^y$,
(2) $e^{-x} = \frac{1}{e^x}$,
(3) $e^{rx} = (e^x)^r$,
(4) *For* $a > 0$ *define* $a^x = \exp(x \ln a)$, *then* $D_x a^x = \ln a a^x$,
(5) $(e^x)^y = \exp(y \ln(e^x)) = e^{xy}$.

**Example.**

(1) $D_x(3^{x^2})$,
(2) $\int \frac{10^{\sqrt{x}}}{\sqrt{x}} dx$,
(3) $D_x x^r = D_x \exp(r \ln x) = \frac{r}{x} \exp(r \ln x) = rx^{r-1}$.

**General Logarithm Function.** $\log_a x = y$ *iff* $a^y = x$, $\log_a x = \frac{\ln x}{\ln a}$.

## 6.8 Inverse Trigonometric Functions

**Inverse Trigonometric Functions.**

(1) $\sin^{-1} x : [-1, 1] \to [-\frac{\pi}{2}, \frac{\pi}{2}], D_x \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$,
(2) $\cos^{-1} x : [-1, 1] \to [0, \pi], D_x \cos^{-1} x = -\frac{1}{\sqrt{1-x^2}}$,
(3) $\tan^{-1} x : [-\infty, \infty] \to [-\frac{\pi}{2}, \frac{\pi}{2}], D_x \tan^{-1} x = \frac{1}{1+x^2}$,
(4) $\cot^{-1} x : [-\infty, \infty] \to [0, \pi], D_x \cot^{-1} x = -\frac{1}{1+x^2}$,
(5) $\sec^{-1} x : [1, \infty) \cup (-\infty, -1] \to [0, \frac{\pi}{2}) \cup (\frac{\pi}{2}), \pi], D_x \sec^{-1} x = \frac{1}{|x|\sqrt{x^2-1}}$,
(6) $\csc^{-1} x : [1, \infty) \cup (-\infty, -1] \to [-\frac{\pi}{2}, 0) \cup (0, \frac{\pi}{2}], D_x \csc^{-1} x = -\frac{1}{|x|\sqrt{x^2-1}}$

**Example.**

(1) Example (1) p.490. $\tan\theta = \frac{16t^2}{800}$,
(2) $D_x(\sin^{-1} x^2)$,
(3) $D_x(\sec^{-1} e^x)$,
(4) $\int_0^1 \frac{dx}{1+x^2}$,
(5) $\int \frac{1}{1+9x^2} dx$,
(6) $\int \frac{1}{\sqrt{4-x^2}} dx$,
(7) $\int_1^{\sqrt{2}} \frac{1}{x\sqrt{2x^2-1}} dx$.

## 6.9 Hyperbolic Functions

**Hyperbolic Functions.**

(1) $\sinh x = \frac{e^x - e^{-x}}{2}$,
(2) $\cosh x = \frac{e^x + e^{-x}}{2}$,
(3) $\tanh x = \frac{\sinh x}{\cosh x}$,
(4) $\coth x = \frac{\cosh x}{\sinh x}$,
(5) $sechx = \frac{1}{\cosh x}$,
(6) $cschx = \frac{1}{\sinh x}$.

**Laws of Hyperbolic Functions.**

(1) $\cosh^2 x - \sinh^2 x = 1$,
(2) $1 - \tanh x = sech^2 x$,
(3) $\coth^2 x - 1 = csch^2 x$,
(4) $\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$,
(5) $\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$,
(6) $\sinh 2x = 2 \sinh x \cosh x$,
(7) $\cosh 2x = \cosh^2 x + \sinh^2 x$,
(8) $\sinh^2 x = \frac{\cosh 2x - 1}{2}$,
(9) $\cosh^2 x = \frac{\cosh 2x + 1}{2}$,
(10) $D_x \sinh x = \cosh x$,
(11) $D_x \cosh x = \sinh x$,

(12) $D_x \tanh x = sech^2\ x$,
(13) $D_x \coth x = -csch^2\ x$,
(14) $D_x sechx = -\tanh x sech\ x$,
(15) $D_x cschx = -cothxcsch\ x$.

**Example.**
(1) $D_x f$, (a)$\cosh 2x$, (b) $\sinh^2 x$, (c)$x \tanh x$, (d) $sech\ x^2$. ,
(2) (a) $\int \cosh 3x dx$, (b) $\int \sinh x \cosh x dx$, (c) $\sinh^2 x dx$, (d) $\int \tanh^2 x dx$.

**Inverse Hyperbolic Functions.**
(1) $\sinh^{-1} x = \ln(x + \sqrt{x^2+1}) : (-\infty, \infty) \to (-\infty, \infty), D_x \sinh^{-1} x = \frac{1}{\sqrt{1+x^2}}$,
(2) $\cosh^{-1} x = \ln(x + \sqrt{x^2-1}) : [1, \infty) \to [0, \infty), D_x \cosh^{-1} x = \frac{1}{\sqrt{x^2-1}}$,
(3) $\tanh^{-1} x = \frac{1}{2} \ln \frac{1+x}{1-x} : (-1, 1) \to (-\infty, \infty), D_x \tanh^{-1} x = \frac{1}{1-x^2}$,
(4) $\coth^{-1} x = \frac{1}{2} \ln \frac{1+x}{1-x} : (-\infty, -1) \cup (1, \infty) \to (-\infty, 0) \cup (0, \infty), D_x \sinh^{-1} x = \frac{1}{1-x^2}$,
(5) $sech^{-1} x = \ln \frac{1+\sqrt{1-x^2}}{x} : (0, 1] \to [1, \infty), D_x sech^{-1} x = -\frac{1}{x\sqrt{1-x^2}}$,
(6) $csch^{-1} x = \ln \frac{1+\sqrt{x^2+1}}{x} : \mathbf{R} - \{0\} \to \mathbf{R} - \{0\}, D_x csch^{-1} x = -\frac{1}{|x|\sqrt{1-x^2}}$.

**Integral Formula.**
(1) $\int \frac{dx}{\sqrt{1+x^2}} = \sinh^{-1} x + C$,
(2) $\int \frac{dx}{\sqrt{x^2-1}} = \cosh^{-1} x + C$,
(3) $\int \frac{dx}{1-x^2} = \tanh^{-1} x + C, \quad |x| < 1$,
(4) $\int \frac{dx}{1-x^2} = \coth^{-1} x + C, \quad |x| > 1$,
(5) $\int \frac{dx}{x\sqrt{1-x^2}} = -sech^{-1}|x| + C$,
(6) $\int \frac{dx}{x\sqrt{1+x^2}} = -csch^{-1}|x| + C$.

**Example.**
(1) $\int \frac{dx}{\sqrt{4x^2+1}}$,
(2) $\int_1^{\frac{1}{2}} \frac{dx}{1-x^2}$,
(3) $\int_2^5 \frac{dx}{1-x^2}$.

**Hanging Cable.** Let $y = f(x)$ be a representation of thr cable, $T_0$ be the tension at the lowest point and $T$ be the tension anf $\rho$ be the density.

$$T \sin\theta = \rho s(x), T \cos\theta = T_0,$$

so

$$f'(x) = \tan\theta = \frac{\rho}{T_0} \int_0^x \sqrt{1 + (f'(t))^2} dt.$$

Take the derivative of both sides to get

$$f''(x) = \frac{\rho}{T_0} \sqrt{1 + (f'(x))^2}.$$

From

$$\int_0^x \frac{f''(x)}{\sqrt{1+(f'(x))^2}}dx = \int_0^x \frac{\rho}{T_0}dx,$$

we get

$$\sinh^{-1} f'(x) = \frac{\rho}{T_0}x + C.$$

Take $x = 0$ you get $C = 0$, so $f'(x) = \sinh \frac{\rho}{T_0}x$. Integrate it again, you have

$$f(x) = \frac{T_0}{\rho}\cosh \frac{\rho}{T_0}x + y_0$$

**Hyperboloc Area.**

$$A(t) = \frac{1}{2}(\sinh t \cosh t - \int_1^{\cosh t} \sqrt{x^2-1}dx.$$

Take the derivativeof both sides you get

$$A'(t) = \frac{1}{2}(\sinh^2 t + \cosh^2 t) - \sqrt{\cosh^2 t - 1}\sinh t = \frac{1}{2}.$$

Hence $A(t) = \frac{1}{2}t$.