CHAPTER 5, THE INTEGRAL

5.2 Antiderivative and Initial Value Problem

Differential Equations.

Example.

(1) $\frac{dT}{dt} = -k(T - A)$,
(2) $\frac{dP}{dt} = kP$,
(3) $\frac{dy}{dt} = -k\sqrt{y}$.

Antiderivative and Indefinit integral. *F is a antiderivative of f if $F' = f$. Indefinite integral $\int f dx = F(x) + C$is the general form of antiderivatives.*

Example.

(1) $f(x) = 3x^2$, $F(x) = x^3 + C$,
(2) $\int 1 dx = x + C$,
(3) $\int 2x dx = x^2 + C$,
(4) $\int x^3 dx = \frac{1}{4}x^4 + C$,
(5) $\int \cos x dx = \sin x + C$,
(6) $\int \sin 2x dx = -\frac{1}{2}\cos 2x + C$,
(7) $\int x^3 + 3\sqrt{x} - \frac{4}{x^2} dx$
(8) $\int (2\cos 3t + 5\sin 4t + 3e^{7t})dt$,
(9) $\int (x+5)^{10} dx$,
(10) $\int \frac{20}{(4-5x)^3} dx$.

Initial Value Problem. $\frac{dy}{dx} = f(x), y(0) = y_0$

Example.

(1) $\frac{dy}{dx} = 3x^2, y(0) = 2$,
(2) $\frac{dy}{dx} = 2x + 3, y(1) = 3$.

Rectilinear Motion.

(1) $x(t)$ *position,*
(2) $v(t)$ *velocity,*
(3) $a(t)$ *acceleration,*
(4) $x(0) = x_0, v(0) = v_0$.

Example. $a(t) = 12t, x(0) = 10, v(0) = 0$

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Constant acceleration Motion.

(1) $a(t) = a$,
(2) $v(t) = at + v_0$,
(3) $x(t) = \frac{1}{2}at^2 + v_0 t + x_0$.

Example.

(1) $a(t) = -20, x(0) = 0, x(t_1) = 160, v(t_1) = 0, v(0) = ?$
(2) $g = -32, y(0) = 0, y(20) = 0, v(t_1) = 0, y(t_1) = ?$.

5.3 Elementary Area Computation

Summation formula.

(1) $\sum_1^n k = \frac{n(n+1)}{2}$,
(2) $\sum_1^n k^2 = \frac{n(n+1)2n+1)}{6}$,
(3) $\sum_1^n k^3 = \frac{n^2(n+1)^2}{4}$.

Example.

(1) $f(x) = x^2, [0, 5]$,
(2) $\sum_1^{10}(7k^2 - 5k)$,
(3) $\lim_{n\to\infty} \frac{1+\cdots+n}{n^2}$,
(4) $f(x) = 100 - 3x^2, [1, 5]$.

5.4 Riemann Sum and Integral

Definition. *A partition $P = \{a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b\}$ is a list of real numbers with increasing order. When you add some real numbers to a partition and then list it in increasing order, you get a new partition, which is called a refinement.*

Riemann Sum. *f is a bounded function on $[a, b]$. $P = \{a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b\}$ is a partition of $[a, b]$ and $x_j^* \in [x_{j-1}, x_j], j = 1, 2, \cdots, n$ is a selection, and let $\triangle x_j = x_j - x_{j-1}$, $|P| = \max \triangle x_j$.*

$$R_P(f) = \sum_1^n f(x_j^*)\triangle x_j \tag{*}$$

is a Riemann sum of f on $[a, b]$, and

(1) *if $x_j^* = x_{j-1}$ (*) is the left sum $R_L(f)$,*
(2) *if $x_j^* = x_j$ (*) is the right sum $R_R(f)$,*
(3) *if $x_j^* = \frac{x_{j-1}+x_j}{2}$ (*) is the mid sum $R_M(f)$,*
(4) *$U_P = \sum_1^n f(x_j^*)\triangle x_j, f(x_j^*) = \mathbf{l.u.b.}_{x\in[x_{j-1},x_j]} f(x)$ is the upper sum ,*
(5) *$L_P = \sum_1^n f(x_j^*)\triangle x_j, f(x_j^*) = \mathbf{g.l.b.}_{x\in[x_{j-1},x_j]} f(x)$ is the lower sum .*

Remark. *If P' is a refinement of P, then $U_P(f) \geq U_{P'}(f)$ and $L_P(f) \leq L_{P'}(f)$.*

Riemann Integral. *The definite integral of f from a to b is the number*

$$I = \lim_{|P| \to 0} \sum_j f(x_j^*) \triangle x_j$$

provided the limit exist, in this case, we say that f is integrable on $[a, b]$ and write

$$I = \int_a^b f(x) dx$$

the definite integral of f on $[a, b]$.

Here $I = \lim_{|P| \to 0} \sum_j f(x_j^*) \triangle x_j$ means that for any $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that $|P| < \delta$ implies $|R_P(f) - I| < \epsilon$.

Theorem. *f is integrable on $[a, b]$ iff for any $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that $|P| < \delta$ implies $|U_P - L_P| < \epsilon$.*

Proof. Necessary: Assume f is integrable, given $\epsilon > 0$, there is $\delta > 0$ such that for any partition P with $|P| < \delta$, we have

$$|R_P(f) - \int_a^b f(x) dx| < \frac{\epsilon}{2}.$$

Which implies that $|U_P(f) - L_P(f)| < \epsilon$.

Sufficient : Since $U_P(f) \geq L'_P(f)|$ for all partitions P, P', $\{U_P(f)\}$is a set of real number bounded from below, hence has a g.l.b I_U , and similarly $\{L_P(f)\}$ has a l.u.b. I_L. From the assumption, given $\epsilon > 0$, there is a δ , any partition P with $|P| < \delta$, we have

$$I_U - I_L \leq U_P(f) - L_P(f) < \epsilon.$$

Since this inequality is true for all $\epsilon > 0$, we get $I_U = I_L$ and let us call this common value I. Now we show that the definit integral of f from a to b is I. Given $\epsilon > 0$, choose a δ such that $U_{\mathbf{P}}(f) - L_{\mathbf{P}}(f) < \epsilon$ whenever $|P| < \delta$. Then we have

$$R_P(f) - I \leq U_P(f) - I \leq U_P(f) - L_P(f)$$
$$R_P(f) - I \geq L_P(f) - I \geq L_P(f) - U_P(f)$$

hence

$$|R_P(f) - I| \leq U_P(f) - L_P(f) < \epsilon.$$

Theorem. *If f is monotone on $[a, b]$, then f is integrable on $[a, b]$.*

Theorem. *If f is continuous on $[a, b]$, then f is integrable on $[a, b]$.*

Uniform Continuous. *f is uniformly continuous on $[a, b]$ if for any $\epsilon > 0$ there is $\delta > 0$ such that for $x, y \in [a, b]$ and $|x - y| < \delta$ then $|f(x) - f(y)| < \epsilon$.*

Theorem. *If f is continuous on $[a, b]$ then f is uniformly continuous on $[a, b]$.*

Example.

(1) $\int_0^4 x^3 - 2x dx$,
(2) $\int_a^b x dx$,
(3) $\int_0^b e^x dx$.

5,4 5.6 Evaluation of integrals and Fundamental theorem of Calculus

Elementary Properties of Definite Integrals.

(1) *If f, g are integrable on $[a,b]$, the $\alpha f+\beta g$ is integrable on $[a,b]$ and*

$$\int_a^b(\alpha f+\beta g)(x)dx=\alpha\int_a^b f(x)dx+\beta\int_a^b g(x)dx.$$

(2) *If f is integrable on $[a,b]$ and on $[b,c]$, then f is integrable on $[a,c]$ and*

$$\int_a^c f(x)dx=\int_a^b f(x)dx+\int_b^c f(x)dx.$$

(3) *If f is integrable on $[a,b]$ and $f(x)\geq 0$ on $[a,b]$, then $\int_a^b f(x)dx\geq 0$.*

(4) *If f, g are integrable on $[a,b]$ and $f(x)\geq g(x)$ on $[a,b]$, then $\int_a^b f(x)dx\geq\int_a^b g(x)dx$.*

(5) *If f is integrable on $[a,b]$ and $m\leq f(x)\leq M$ on $[a,b]$, then*

$$m\leq\frac{1}{b-a}\int_a^b f(x)dx\leq M$$

.

(6) *If f is continuous on $[a,b]$, then $\frac{1}{b-a}\int_a^b f(x)dx=f(c)$ for some $c\in[a,b]$. (This is the mean value theorem of integral.)*

Theorem. *(Fundamental Theorem of Calculus) Suppose that f is continuous on $[a,b]$.*

(1) *Define $F(x)=\int_a^x f(t)dt$ for $x\in[a,b]$, then $F'(x)=f(x)$.*

(2) *If $G'=f$ on $[a,b]$, then $\int_a^b f(x)dx=G(b)-G(a)$.*

Example.

(1) $\int_0^b x^n dx$,

(2) $\int_a^b \cos x dx$,

(3) $\int_1^9(2x-x^{-\frac{1}{2}}-3)dx$, $\int_0^1(2x+3)^3dx$, $\int_0^{\frac{\pi}{2}}\sin 2x dx$, $\int_0^1 e^{2x}dx$,

(4) $\int_1^5\sqrt{3x+1}dx$,

(5) $P(0)=100, P'(t)=10+t+0.06t^2, P(10)=?$,

(6) $\lim_{n\to\infty}\sum\frac{2i}{n^2}$,

(7) $\int_{-1}^1 2|x|dx$,

(8) $\int_0^{2\pi}|\cos x-\sin x|dx$,

(9) $\sqrt{1+x}\leq\sqrt{1+\sqrt{x}}\leq 1.2+0.2x$ on $[0,1]$, estimate $\int_0^1\sqrt{1+\sqrt{x}}dx$.

(10) Average x^2 over $[0,2]$,

(11) $f(x)=\cos x, x\geq 0, -1-x^2, x<0, \int_{-1}^{\pi/2} fx)dx$,

(12) Find the area between the graph of x^3-x^2-6x and the x-axis,

(13) $\int_{-1}^2|x^3-x|dx$,

(14) $h'=?, h(x)=\int_0^{x^2} t^3\sin t dt$,

(15) $y'=\sec x$, $y(x)=y_0+\int_0^x\sec t dt$.

5.7 Integration by Substitution

Indefinite Integral. Suppose that $F' = f$ then $dF(g(x))/dx = f(g(x))g'(x)$, hence

$$\int f(g(x))g'(x)dx = F(g(x)) + C$$

Example.

(1) $\int (2x+1)^5 2dx$,
(2) $\int 2x\sqrt{1+x^2}dx$, $\int \frac{e^x}{\sqrt{1+e^x}}dx$,
(3) $\int x^2\sqrt{x^3+9}dx$,
(4) $\int \sin(3x+4)dx$,
(5) $\int 3x\cos x^2 dx$,
(6) $\int \sec^2 3x dx$,
(7) $\int 2\sin^3 x\cos x dx$,
(8) $\int 3\sqrt{x}e^{1+\sqrt{x^3}}dx$, $\int \frac{3\sqrt{x}}{1+\sqrt{x^3}}dx$

Definite Integral.

$$\int_a^b f(g(x))g'(x)dx = F(g(b)) - F(g(a)) = \int_{g(a)}^{g(b)} f(u)du.$$

Example.

(1) $\int_0^3 x^2\sqrt{x^3+9}dx$,
(2) $\int_3^5 \frac{xdx}{(30-x^2)^2}$,
(3) $\int_0^{\frac{\pi}{4}} \frac{\cos 2x}{1+\sin 2x}dx$.

5.8 Area of Plane Region

$$\int_a^b f(x) - g(x)dx.$$

Example.

(1) $y = x, x = 2, y = \frac{1}{x^2}$,
(2) $y = x, y = 6 - x^2$,
(3) $y = \frac{1}{2}x, y^2 = 8 - x$
(4) As (3) in y,
(5) $\int_0^1 \sqrt{1-x^2}dx = \frac{\pi}{4}$.

5.9 Numerical Integration

$$L_n = \sum_i y_{i-1} \triangle x_i,$$

$$R_n = \sum_i y_i \triangle x_i,$$

$$M_n = \sum_i y_{i-1/2} \triangle x_i,$$

Trapezoidal

$$T_n = \sum_i \frac{y_{i-1} + y_i}{2} \triangle x_i,$$

Simpson

$$S_n = \frac{1}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + \cdots + 4y_{n-1} + y_n)\frac{(b-a)}{2n}$$

, n is even.

Error Estimates.

(1)
(2) $ET_n = \frac{K_2(b-a)^3}{12n^2}$,
(3) $EM_n = \frac{K_2(b-a)^3}{24n^2}$,
(4) $ES_n = \frac{K_4(b-a)^5}{180n^4}$.

Remark.

(1) *When $K_2 = 0$ f is a linear function, then both ET_n and EM_n are* 0.
(2) *When $K_4 = 0$, f is a polynomial of degree three, then $ES_n = 0$.*