CHAPTER 4, ADDITIONAL APPLICATION OF THE DERIVATIVES

4.2 Increments, Defferential and Linear Approximation

Definition.

(1) $\triangle y = f(x + \triangle x) - f(x)$ *is the increment of* f *from* x *to* $x + \triangle x$,
(2) $df = f'(x)dx$ *is the differential of* f *at* x *if* f *is differentiable at* x,
(3) $f(x + \triangle x) \approx f(x) + f'(x)\triangle x$ *is the linear appeoximation of* f *near* x.

Example.

(1) $\sqrt{1+x}$ at $x = 0$,
(2) $122^{\frac{2}{3}}$ at $x = 125$,
(3) Example 3 p.228. $V(x) = \frac{\pi}{3}(30x^2 - x^3)$ at $x = 5, \triangle x = \pm\frac{1}{16}$.

Absolute and Relative Error.

Definition.

(1) *Absolute error* $= |f(x + \triangle x) - f(x) \approx f'(x)dx|$,
(2) *Relative error = Absolute error /Value.*

Example. As in (3) $\frac{dv}{v} = \frac{3(20-x)\triangle x}{(30-x)x}$

Error of Linear Approximation.

Example.

(1) x^3 at $x = 1, \triangle x = 0.1$,
(2) $\sqrt{1+x}$.

Differential. $df = f'(x)dx$

Example.

(1) $d(x^n) = nx^{n-1}dx$,
(2) $d(\sin x) = \cos x dx$,
(3) $d(e^x) = e^x dx$
(4) $y = 3x^2 - 2x^{\frac{3}{2}}, dy = (6x - 3x^{1/2})dx$,
(5) $u = \sin^2 t - \cos 2t, du = (2\cos 2t + 2\sin 2t)dt$,
(6) $w = ze^z, dw = (1+z)e^z dz$.

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4.3 Increasing and Decreasing Functions, Mean Value Theorem

Theorem. *(Mean Value Theorem [M.V.T]) Suppose that f is continuous on $[a,b]$ and differentiable on (a,b), then there is a $c \in (a,b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$, or $f(b)-f(a) = f'(c)(b-a)$*

Theorem. *(Rolle's Theorem) Suppose that f is continuous on $[a,b]$ and differentiable on (a,b) and $f(a) = f(b)$,then there is a $c \in (a,b)$ such that $f'(c) = 0$.*

Proof. Since f is continuous on $[a,b]$, Let x_M, x_m be a maximum, minimum point of f on $[a,b]$.

(1) If one of x_M, x_m is in (a,b), then we have a local extrmum point, so f' is zero there.
(2) Both x_M, x_m are boundary point, since $f(a) = f(b)$, which means that f is a constant function.

Proof of M.V.T.. Let $g(x) = f(a) + \frac{f(b)-f(a)}{b-a}(x-a)$ and let $h(x) = f(x) - g(x)$. Then $h(a) = 0 = h(b)$, so by Rolle's theorem, there is a $c \in (a,b)$ such that $h'(c) - 0$. Which implies that $f'(c) - \frac{f(b)-f(a)}{b-a} = 0$.

Example.

(1) $x^{\frac{1}{2}} - x^{\frac{3}{2}}$ on $[0,1]$,
(2) $1 - x^{\frac{2}{3}}$ on $[-1,1]$.

Corollary. *Suppose that f satisfies the condition in (M.V.T) and $f'(x) = 0$ on (a,b), then f is a constant function on $[a,b]$.*

Corollary. *Suppose that f, g satisfy the condition in (M.V.T) and $f'(x) = g'(x)$ on (a,b), then $f(x) = g(x) + K$ on $[a,b]$.*

Example. $f'(x) = 6e^{2x}$, and $f(0) = 7$

Corollary. *Suppose that f satisfies the condition in (M.V.T) and $f'(x) < (>) 0$ on (a,b), then f is a decreasing (increasing) function on $[a,b]$.*

Example.

(1) $x^2 - 4x + 5$,
(2) $e^x + x - 2 = 0$ has exactly one real root,
(3) $3x^4 - 4x^3 - 12x^2 + 5$,
(4) $\sin x < x$

4.4 The first derivative test and application

Theorem. *Suppose that f is continuous on I and differentiable on $I - \{c\}$*

(1) *If $f'(x) < 0$ for $x < c$ near c and $f'(x) > 0$ for $x > c$ near c, then $f(c)$ is a local minimum value.*

(2) *If $f'(x) > 0$ for $x < c$ near c and $f'(x) < 0$ for $x > c$ near c, then $f(c)$ is a local maximum value.*
(3) *If $f'(x) < 0$ for $x \neq c$ near c or $f'(x) > 0$ for $x \neq c$ near c, then $f(c)$ is neither a local minimum value nor a local maximum value.*

Example.

(1) $2x^3 - 3x^2 - 36x + 7$,
(2) $\frac{2\ln x}{x}$ on $(0, \infty)$,
(3) $x + \frac{4}{x}$ on $(0, \infty)$,
(4) Example 4 p.250 $V = 125, A = 8r^2 + 2\pi rh$,
(5) Example 5 p.251 $L = 4\csc\theta + 2\sec\theta$.

Theorem. *Suppose that f is continuous on I and differentiable on $I - \{c\}$*

(1) *If $f'(x) < 0$ for $x < c$ and $f'(x) > 0$ for $x > c$, then $f(c)$ is a global minimum value.*
(2) *If $f'(x) > 0$ for $x < c$ and $f'(x) < 0$ for $x > c$, then $f(c)$ is a global maximum value.*

4.5 Simple curve sketching

Key points.

(1) *Critical points.*
(2) *Intervals of increasing and decreasing.*
(3) $\lim_{x \to \pm\infty}$.

Example.

(1) $x^3 - 27x$,
(2) $8x^5 - 5x^4 - 20x^3$,
(3) $x^{\frac{2}{3}}[x^2 - 2x - 6]$,

Solution of equations.

(1) $x^3 - 3x + 1 = 0$,
(2) $x^3 - 3x + 2 = 0$,
(3) $x^3 - 3x + 3 = 0$.

4.6 Higher derivative test and Concavity

Definition.

(1) *Concave upward= band upward= Convex*

$$f(a) + \frac{f(b) - f(a)}{b - a}(c - a) \geq f(c) \qquad \text{(convex)}$$

for all $a, b, c \in I$ such that $a < c < b$.

(2) *Concave downward= band downward= Concave*

$$f(a) + \frac{f(b) - f(a)}{b - a}(c - a) \leq f(c) \qquad \text{(concave)}$$

for all $a, b, c \in I$ such that $a < c < b$.

Theorem. *f is convex on I iff $a, b, c \in I$ such that $a < c < b$, either of the following is true:*

(1) $\frac{f(b)-f(a)}{b-a} \geq \frac{f(c)-f(a)}{c-a}$,
(2) $\frac{f(b)-f(c)}{b-c} \geq \frac{f(b)-f(a)}{b-a}$,
(3) $\frac{f(b)-f(c)}{b-c} \geq \frac{f(c)-f(a)}{c-a}$,
(4) $f(a) \geq f(c) + \frac{f(b)-f(c)}{b-c}(a - c)$,
(5) $f(b) \geq f(a) + \frac{f(c)-f(a)}{c-a}(b - a)$.

Proof. Each of them are equivalent to

$$f(a)(b - c) + f(b)(c - a) \geq f(c)(b - a).$$

Theorem. *If f is convex on I, then f is continuous on I.*

Theorem. *If f is differentiable on I, then the following conditions are equivalent:*

(1) *f is convex on I,*
(2) *For all $a, x \in I$, we have $f(a) + f'(a)(x - a) \leq f(x)$,*
(3) *f' in nondecreasing on I.*

Proof.

(1) implies (3). For $a < x < b$, by (1) we have $\frac{f(b)-f(a)}{b-a} > \frac{f(x)-f(a)}{x-a}$ and $\frac{f(b)-f(x)}{b-x} > \frac{f(b)-f(a)}{b-a}$. Let $x \to b^-$ we get $f'(b) \geq \frac{f(b)-f(a)}{b-a}$ and let $x \to a^+$ we get $f'(a) \leq \frac{f(b)-f(a)}{b-a}$.

(3) implies (1). Suppose that for some $a < c < b$ such that $f(c) > f(a) + \frac{f(b)-f(a)}{b-a}(c-a)$, then we get $\frac{f(b)-f(a)}{b-a} > \frac{f(b)-f(c)}{b-c}$ and $\frac{f(c)-f(a)}{c-a} > \frac{f(b)-f(a)}{b-a}$. Now by M.V.T. there exist $d \in (a, c), e \in (c, b)$ such that

$$f'(d) = \frac{f(c) - f(a)}{c - a} > \frac{f(b) - f(a)}{b - a} > \frac{f(b) - f(c)}{b - c} = f'(e)$$

which is a contradiction to (3)

(2) implies (3). For $a < b$, by (2) we have $f'(a) \leq \frac{f(b)-f(a)}{b-a} \leq f'(b)$.

(3) implies (2). Assume that for some $a < b$ such that $f(b) < f(a) + f'(a)(b - a)$, that means $f'(a) > \frac{f(b)-f(a)}{b-a}$, then by M¿V¿T¿ there exists $c \in (a, b)$ such that $f'(a) > f'(c)$. Which is a contradiction to (3).

If f'' exists, then (3) is equivalent to $f''(x) \geq 0$, so we have the following theorem.

Theorem. *If f'' exists on I, then f is convex iff $f''(x) \geq 0$ on I.*

Theorem. *(Second derivative test)*

(1) *Suppose that $f" \geq 0$ on I and $f'(c) = o$, then $f(c)$ is the minimum value of f on I.*
(2) *Suppose that $f" \leq 0$ on I and $f'(c) = o$, then $f(c)$ is the maximum value of f on I.*

Example.

(1) $x^3 - 3x^2 + 3$,
(2) Example 4 p.269 $V = x^2 y = 500$ minimize A.

Inflection point. *c is an inflection point of f if f is concave upward on one side of c but concave downward on the other side of c.*

Example.

(1) $(2x^2 - 3x - 1)e^{-x}$,
(2) $8x^5 - 5x^4 - 20x^3$,
(3) $4x^{\frac{1}{3}} + x^{\frac{4}{3}}$,
(4) Example 8 p.276 $x^2 - xy + y^2 = 9$.

4.7 Curve sketching and Asympote

Vertical asymptote. *$x = a$ is a vertical asymptote of $y = f(x)$ if*

$$\lim_{x \to a^{\pm}} f(x) = \pm\infty.$$

Example. $\frac{1}{(x+2)^2}$, $\frac{x}{(x+2)^2}$.

Horizontal asymptote. *$y = L$ is a horizotal asymptote of $y = f(x)$ if*

$$\lim_{x \to \pm\infty} f(x) = L.$$

Example.

(1) $\frac{3x^3 - x}{2x^3 + 7x^2 - 4}$,
(2) $\sqrt{x + a} - \sqrt{x}$,
(3) $\frac{4e^{2x}}{(1+e^x)^2}$,
(4) $e^{-\frac{x}{5}} \sin 2x$,
(5) $\frac{x}{x-2}$,
(6) $\frac{x}{(x+2)^2}$,
(7) $\frac{2+x-x^2}{(x-1)^2}$.

Slant asymptote. *$y = mx + b$ is an slant asymptote of $y = f(x)$ if*

$$\lim_{x\to\pm\infty}[f(x) - (mx + b)] = 0.$$

Example. $\sqrt{x^2 - 1}$.

Key points of sketching.

(1) *$f'(x)$ for critical point, local maximum, local minmum, interval of increasing, decreasing.*
(2) *f"(x) for concave upward, downward, and point of inflection.*
(3) *Asymptotes.*
(4) *x-intercept, y-intercept.*
(5) *Symmetry. $f(-x) = f(x)$ even function symmetry with respect to y-axis. $f(-x) = -f(x)$ odd function, symmetry with respect to origin.*

Example. $\frac{x^2+x-1}{x-1}$.

4.9 Indeterminate forms, L'Hopital's Rule

Theorem. *Suppose that $f(a) = 0 = g(a)$ and f, g are differentiable near a. If*

$$\lim_{x\to a}\frac{f'(x)}{g'(x)} = L$$

then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = L.$$

Theorem. *(Cauchy's Generalize Mean Value Theorem) Suppose that f, g are contiuous on $[a, b]$ and differentiable on (a, b), then there is $c \in (a, b)$ such that*

$$f'(c)(g(b) - g(a)) = g'(c)(f(b) - f(a)).$$

Proof of L'Hopital's Rule. (1) $a \in \mathbb{R}$,

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f(x) - f(a)}{g(x) - g(a)} = \lim_{x\to a}\frac{f'(c)}{g'(c)} = L.$$

By the Cauchy's Generalize Mean Value Theorem we hav the second equality , and since c is between a, x, we get the last equality. (2) if $a = \pm\infty$ change variable $x = \frac{1}{y}$.

Example.

(1) $\lim_{x\to 0}\frac{e^x-1}{\sin 2x}$,
(2) $\lim_{x\to 1}\frac{1-x+\ln x}{1+\cos\pi x}$,
(3) $\lim_{x\to 0}\frac{\sin x}{x+x^2}$.

Theorem. *Suppose that* $\lim_{x\to a} f(x) = \infty = \lim_{x\to a} g(x)$ *and* f, g *are differentiable near* a*. If*

$$\lim_{x\to a} \frac{f'(x)}{g'(x)} = L$$

then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = L.$$

Example.

(1) $\lim_{x\to\infty} \frac{\ln 2x}{\ln x}$,
(2) $x^2 e^{-x}$.

Magnanitute of e^x and $\ln x$.

Theorem.

(1) $\lim_{x\to\infty} \frac{x^n}{e^x} = 0$,
(2) $\lim_{x\to\infty} \frac{\ln x}{x^{\frac{1}{m}}} = 0$.

4.10 More Indeterminate Forms

(1) $0 \cdot \infty$ $\lim_{x\to\infty} x \cdot \ln \frac{x-1}{x+1}$.
(2) $\infty - \infty$ $\lim_{x\to 0} \frac{1}{x} - \frac{1}{\sin x}$, $\lim_{x\to\infty}(\sqrt{x^2+3x} - x)$.
(3) 1^∞ $\lim_{x\to 0}(\cos x)^{\frac{1}{x}}$.
(4) 0^0 $\lim_{x\to 0^+} x^{\tan x}$, $\lim_{x\to\infty}(1+\frac{1}{x})^x = e$.