CHAPTER 3, THE DERIVATIVES

3.1 Derivative and Rates of Change

The Derivative. *The derivative of the function f is the function f' defined by*

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

for all a for which this limit exists.

The process from f to f' is call differentiation.

f is differentiable at a if $f'(a)$ exists. f is a differentiable function if f is differentiable at all x in the domain.

Example.

(1) $f(x) = \frac{x}{x+3}$ $f'(x) = \frac{3}{(x+3)^2}$,
(2) $f(x) = ax^2 + bx + c$ $f'(x) = 2ax + b$.

Differentiation notation. $y = f(x)$, $\frac{dy}{dx} = f'(x)$.

Example.

(1) $y = 3x^2 - 4x + 5$, $\frac{dy}{dx} = 6x - 4$,
(2) $z = 2t - 5t^2$, $\frac{dz}{dt} = 2 - 10t$.

Instantaneous Rates of Change. $Q = f(t)$, $\triangle Q = f(t + \triangle t) - f(t)$ *the increment of Q from t to $t + \triangle t$*

$\lim_{\triangle t \to 0} \frac{\triangle Q}{\triangle t} = f'(t)$ *is the instantaneous rates of change of Q with respect to t.*

Example.

(1) Example 3 p. 111 $V(t) = \frac{1}{6}(60 - t)^2$,$V'(15) =?$, $V'(45) =?$,
(2) $x(t)$ is the position at time t, $v(t) = \frac{dx}{dt}$ is the velocity at time t, $a(t) = \frac{dv}{dt}$ is the acceleration at time t,
(3) Example 5 p.113 $x(t) = 5t^2 + 100$, $v(t) = 10t$,
(4) The equation of vertical motion is $y = -\frac{1}{2}gt^2 + v_0 t + y_0$, Example 6 p. 114 $y(t) = -16t^2 + 96t$,
(5) Example 7 p.115 $A = x^2$, $x = 5t$, $\frac{dA}{dx} = 2x$, $\frac{dA}{dt} = 50t$.

Alternating notion of differentiability. *Suppose that f is differentiable at a let*

$$e(a,h) = \frac{f(a+h) - f(a)}{h} - f'(a),$$

then $\lim_{h\to 0} e(a,h) = 0$. *Now suppose that*

$$f(a+h) = f(a) + Ah + he(a,h) \tag{1}$$

such that

$$\lim_{h\to 0} e(a,h) = 0 \tag{2}$$

then f *is differentiable at* a *and* $f'(a) = A$. *Hence* f *is differentiable at* a *iff both (1) and (2) hold.*

Theorem. *If* f *is differentiable at* x , *then* f *is continuous at* x.

3.2 Basic Differentiation Rules

Theorem. *Let* $D_x f = f' = \frac{df}{dx}$ *then*

(1) $D_x c = 0$,
(2) $D_x x = 1$,
(3) $D_x x^n = nx^{n-1}$ *for* n *is a positive integer,*
(4) $D_x \frac{1}{x} = -\frac{1}{x^2}$,
(5) $D_x x^{\frac{1}{n}} = \frac{1}{n} x^{\frac{1}{n}-1}$ *for* n *is a positive integer.*

Example.

(1) $D_x x^7 = 7x^6$,
(2) $D_t t^{17} = 17t^{16}$,
(3) $D_z z^{100} = 100z^{99}$.

Derivative of linear combinations.

$$(af(x) + bg(x))' = af'(x) + bg'(x).$$

Example.

(1) $D_x(16x^6) = 96x^5$, $D_z(7z^3) = 21z^2$, $D_u(99u^{100}) = 9900u^{99}$,
(2) $D_x(36 + 26x + 7x^5 - 5x^9) = 26 + 35x^4 - 45x^8$,in general

$$D_x(a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0) = na_n x^{n-1} + \cdots + a_1,$$

(3) Find the tangent line to $y = 2x^3 - 7x^2 + 3x + 4$ at $(1, 2)$.

Differentiation of Product.

$$D_x(f \cdot g) = D_x f \cdot g + f \cdot D_x g.$$

Example. $[(1 - 4x^3)(3x^2 - 5x + 2)]' = -12x^2(3x^2 - 5x + 2) + (1 - 4x^3)(6x - 5)$.

Differentiation of Quotient.

$$D_x(\frac{1}{f(x)}) = -\frac{f'(x)}{(f(x))^2}$$

and

$$D_x\frac{f(x)}{g(x)} = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}.$$

Example.

(1) $D_x\frac{1}{x^2+1} = -\frac{2x}{(x^2+1)^2}$,
(2) $D_x x^{-n} = -nx^{-n-1}$,
(3) $D_x\frac{5x^4-6x+7}{2x^2}$,
(4) $D_t\frac{1-t^3}{1+t^4} = \frac{-3t^2(1+t^4)-(1-t^3)(4t^3}{(1+t^4)^2}$.

3.3 Chain Rule

Theorem. *Suppose that g is differentiable at x and f is differentiable at $g(x)$, then $H = f \circ g$ is differentiable at x and*

$$H'(x) = f'(g(x)) \cdot g'(x).$$

Suppsoe that w is a function of u and u is a function of x, then the rate of change of w with respect to x is $\frac{dw}{du} \cdot \frac{du}{dx}$.

Proof. Since g is differentiable at x,

$$g(x+h) = g(x) + g'(x)h + he_g,$$

and f is differentiable at$y$,

$$f(y+k) = f(y) + f'(y)k + ke_k \tag{*.}$$

Substitute $y = g(x)$ and $k = g'(x)h + he_g$ into (*) we get

$$f(g(x+h)) = f(g(x)) + f'(g(x))(g'(x)h + he_g) + (g'(x)h + he_g)e_k \tag{**.}$$

After collecting terms on the right hand side of (**) and let

$$e_{f\circ g} = f'(g(x))g'(x)e_g + (g'(x) + e_g)e_k,$$

we get

$$f(g(x+h)) = f(g(x)) + f'(g(x))g'(x)h + he_{f\circ g}.$$

Since

$$\lim_{h\to 0} e_{f\circ g} = 0,$$

the proof is complete.

Example.

(1) $D_x[(3x^2+5)^{17}]$,
(2) $D_x[\frac{1}{(2x^3-x+7)^2}]$,
(3) $D_z(\frac{z-1}{z+1})^5$,
(4) Example 6 p.135 Suppose that $\frac{dr}{dt}=0.2$, $\frac{dV}{dt}=?$ when $r=5$.
(5) Example 7 p.136 Suppose that $\frac{dM}{dt}=kS$, $M=\frac{4\pi}{3}r^3\rho, S=4\pi r^2$,$r(0)=0$, $r(20)=1$ then $r(?)=3$.

3.4 Derivatives of Algebraic Functions

$$D_x x^{\frac{n}{m}}=\frac{n}{m}x^{\frac{n}{m}-1}.$$

Generalized power rule.

$$D_x[f(x)]^n=n[f(x)]^{n-1}f'(x).$$

Example.

(1) a. $\sqrt{x}$, b. $x^{\frac{3}{2}}$, c. $t^{-\frac{2}{3}}$
(2) $\sqrt{4-x^2}$,
(3) $5x^{\frac{3}{2}}-2x^{\frac{2}{3}}$,
(4) $(3-5x)^7$,
(5) $\sqrt{2x^2-3x+5}$,
(6) $[5t+(3t-1)^{\frac{4}{3}}]^{10}$,
(7) $|x|=\sqrt{x^2}$,
(8) $x^{\frac{1}{3}}$,
(9) $x\sqrt{1-x^2}$,
(10) $1-x^{\frac{2}{5}}$.

3.5 Maxima and Minima of Continuous Functions on Clsoed Interval

Global (absolute) maximun,minimun. *$f(c)$ is a global(absolute) maximum (minimum) value of f if $f(c)\geq(\leq)f(x)$ for all x in the domain of f.*

Example.

(1) $f(x)=2x$ on $0\leq x<1$,
(2) $g(x)=\frac{1}{x}$ for $0<x\leq 1$ and $g(0)=0$.

Local (relative) maximun,minimun. *$f(c)$ is a local(relative) maximum (minimum) value of f if there is an open interval I contains c such that $f(c)\geq(\leq)f(x)$ for all $x\in I$.*

Theorem. *If f is differentiable at c and $f(c)$ is a local extremum value of f , then $f'(c) = 0$.*

Critical point. *A point c is a critical point of f if either $f'(c) = 0$ or f is not differentiable at c.*

Theorem. *Suppose that $f(c)$ is a global extremum value of f on $[a, b]$, then either c is a critical point of f or c is a or b.*

Example.

(1) $\frac{3}{5}x(30 - x)$ on $[0, 30]$,
(2) $2x^3 - 3x^2 - 12x + 15$ on $[0, 3]$,
(3) $3 - |x - 2|$ on $[1, 4]$,
(4) $5x^{\frac{2}{3}} - x^{\frac{5}{3}}$ on $[-1, 4]$,
(5) $4x^4 - 11x^2 - 5x - 3$ on $[-3, 3]$.

3.6 Applied Optimization Problems

Example.

(1) Example 1 p.156 $2x + y = 200, A = xy$,
(2) Example 2 p.157 $V = x(8 - 2x)(5 - 2x)$,
(3) $C(x) = a + bx, x = m - np(x), P(x) = xp(x) - C(x)$
(4) Example 3 p.158 $a = 10,000, b = 8, p(7000) = 13, p(5000) = 15$,
(5) Example 4 p.159 $C = 300\pi, V = \pi r^2 h, 4\pi r^2 + 2\pi rh = C$,
(6) Example 5 p.160 $A = 4xy, x^2 + y^2 = r^2$,
(7) Principle of refraction Exercise 48 p.167.

3.7 Derivatives of Trigonometric Functions

Theorem.

(1) $D_x(\sin x) = \cos x$,
(2) $D_x(\cos x) = -\sin x$.

Corollary.

(1) $D_x \tan x = \sec^2 x$,
(2) $D_x \cot x = -\csc^2 x$,
(3) $D_x \sec x = \tan x \sec x$,
(4) $D_x \csc x = -\cot x \csc x$.

Example.

(1) $x^2 \sin x$,
(2) $\frac{\cos x}{1-\sin x}$,
(3) $\cos^3 t$,
(4) $(2 - \cos t)^{\frac{2}{3}}$,

(5) Tangent line to $y = \cos^2 x$ at $x = 0.5$,$y(0.5) \approx 0.7702, y'(0.5) \approx -0.8415$.
(6) $x \tan x, \cot^3 x, \frac{\sec z}{\sqrt{z}}$,
(7) $2 \sin 10t + 3 \cos \pi t$,
(8) $\sin^3 x \cos^4 5x$,
(9) $\sqrt{x} \cos \sqrt{x}$,
(10) $\sin^2(2x - 1)^{\frac{3}{2}}$,
(11) $\tan 2x^3, \cot^3 2t, \sec \sqrt{y}, \sqrt{\csc x}$,
(12) Example 12 p.174 $\tan \theta = \frac{y}{5}, \frac{d\theta}{dt} = 3°, \theta = 60°, \frac{dy}{dt} =?$,
(13) Example 13 p.175 $A = 2r \cos \theta \sin \theta$.

3.8 Exponential and Logarithimic Functions

Example.(1) $P(0) = 1, P(1) = 2$ then $P(\frac{q}{p}) = 2^{\frac{q}{p}}$, so $P(t) = 2^t$.

Laws of Exponents. $a^{r+s} = a^r \cdot a^s, a^{-r} = \frac{1}{a^r}, (a^r)^s = a^{rs}, a^r \cdot b^r = (a \cdot b)^r$

Derivate of a^x. $\lim_{h \to 0} \frac{a^{x+h} - a^x}{h} = a^x \lim_{h \to 0} \frac{a^h - 1}{h} = m(a)a^x$.
Let e be the real number such that $m(e) = 1$, then e^x also be written as $\exp(x)$

Example.
(1) a. $D_x(x^2 e^{-x})$, b. $D_x \frac{e^{2x}}{2x+1}$,
(2) Find the maxima of (1) a.

Inverse function. *f has an inverse function g if $f(x) = y$ iff $g(y) = x$. A function f has an inverse function iff f is an one to one function.*

Example.
(1) $f = x + 1, 2x, \frac{1}{x}$ then $g = y - 1, \frac{y}{2}, \frac{1}{y}$,
(2) x^2,
(3) x^3.

Increasing functions,Decreasing functions. *A function f defined on an interval$I$ is increasing (decreasing) function if for all $a < b, a, b \in I$ $f(a) < (>) f(b)$.*

Theorem. *Suppose that f is an one to one continuous function on an interval I, then f is either increasing or decreasing on I.*

Lemma. *Suppose that f is an one to one continuous function on an interval I. And suppose that $a < b, a, b \in I$ and $f(a) < f(b)$, then for $x, y, z \in I$ such that $x < a < y < b < z$ we have $f(x) < f(a) < f(y) < f(b) < f(z)$.*

Theorem. *Suppose that f is an one to one continuous function on an interval I, then f^{-1} is also continuous.*

Proof. Assume f is increasing and let $f(a) = b$, then $f^{-1}(b) = a$. For given small $\epsilon > 0$, let $b_1 = f(a - \epsilon), b_2 = f(a + \epsilon)$. Let $\delta = \min(|b - b_1|, |b - b_2|$, then for $|y - b| < \delta$, we have $b_1 < y < b_2$. Which implies $a - \epsilon < f^{-1}(y) < a + \epsilon$.

Theorem. *Suppose that f is an one to one continuous function on an interval I and f is differentiable at $a \in I$ and $f'(a) \neq 0$. Let $f(a) = b$, then f^{-1} is differentiable at b and $(f^{-1})'(b) = \frac{1}{f'(a)}$.*

Proof. Let $f(a) = b$, then $f^{-1}(b) = a$. Now let $h = f^{-1}(b+k) - f^{-1}(b)$, since b is fixed, h is a continuous function of k and $f(a+h) = b + k$.

$$\lim_{k \to 0} \frac{f^{-1}(b+k) - f^{-1}(b)}{k} = \lim_{k \to 0} \frac{h}{f(a+h - f(a)},$$

for $k \to 0$ implies $h \to 0$ and the last limit is just $\frac{1}{f'(a)}$.

Natural Logarithm. $\ln x = y$ *iff* $e^y = x$, $\ln ab = \ln a + \ln b, \ln \frac{1}{a} = -\ln a, \ln a^r = r \ln a$, $(\ln x)' = \frac{1}{x}$.

Example.

(1) $D_x(\frac{\ln x}{x})$, $D_x(\ln |x|) = \frac{1}{x}$,
(2) $D_x(\ln(1 + x^2))$,
(3) $D_x(\sqrt{1 + \ln x})$,
(4) $D_x \ln \sqrt{\frac{2x+3}{4x+5}}$.

Logerithmic Differentiation.

Example.

(1) $D_x \frac{(x^2+1)^{\frac{3}{2}}}{(x^3+1)^{\frac{4}{3}}}$,
(2) $D_x(x^{x+1})$.

3.9 Implicit Differentiation and Related Rates

Implicit Differentiation. *Suppose that $f(x, y) = c$ defines y as a function of x, apply differentiation laws to the equation , you can solve $\frac{dy}{dx}$.*

Example.

(1) $x - y^2 = 0$,
(2) $x^2 + y^2 = 100$,
(3) $x^3 + y^3 = 3xy$,
(4) $\sin(x + 2y) = 2x \cos y$.

Related Rates. *Suppose that $f(x(t), y(t)) = c$ defines a relation of $y(t)$ and $x(t)$, apply differentiation laws to the equation , you can get a relation of $\frac{dt}{dt}$ and $\frac{dy}{dt}$.*

Example.

(1) Example 5 p.197 $x^2 + y^2 = 25, x = 3, y = 4, x' = 12, y' = ?$,

(2) Example 6 p.197 $z^2 = 9 + y^2, z' = 500, z = 5, y' = ?$,
(3) Example 7 p.198 $\frac{18}{z} = \frac{6}{z-x}, x' = 8, z' = ?$,
(4) Example 8 p.199 $u^2 = x^2 + y^2, v^2 = (6-x)^2 + y^2, u(1) = 5 = v(1), u'(1) = 28, v'(1) = 4, x(1) = ?, y(1) = ?, x'(1) = ?, y'(1) = ?$.

3.10 Successive Approximation, Newton's Method

Convergence of Approximation. *We asy a sequence of approximation* $\{x_1, x_2, x_3, \cdots\}$ *converges to the number* r *provided for any* $\epsilon > 0$ *there is* $N(\epsilon)$ *such that* $n \geq N(\epsilon)$ *implies* $\|x_n - r\| < \epsilon$.

Newton's iteration formula. $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$.

Example. $f(x) = x^{\frac{1}{3}}$.

Theorem. *Suppose that* $|f"| < M, |f'| > \frac{1}{K}$, *then* $|x_{n+1} - x_0| \leq KM|x_n - x_0|^2$.