CHAPTER 14, VECTOR CALCULUS

14.1 Vector Field

Definition.

A vector field on a domain is a vector valued function $\mathbf{F}(x, y, z)$.

$$\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, 0\mathbf{j} \ \text{ in } \mathbf{R}^2,$$

$$\mathbf{F}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k} \ \text{ in } \mathbf{R}^3.$$

Definition. *(Gradient Field)*

$$\nabla f = (f_x, f_y, f_z).$$

Properties.

$$\nabla(af + bg) = a\nabla f + b\nabla g,$$

$$\nabla(fg) = g\nabla f + f\nabla g.$$

Example.

(1) $f(x, y) = x^2 - y^2$, $\nabla f = 2x\mathbf{i} - 2y\mathbf{j}$,
(2) $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$,
(3) $\mathbf{v} = \omega(-y\mathbf{i} + x\mathbf{j})$
(4) $\mathbf{F} = -k\frac{\mathbf{r}}{r^3}$,

Definition. $\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$.

Definition. *(Divergence of a vector field* $\mathbf{F} = (P, Q, R)$*)*

$$div(\mathbf{F}) = \nabla \cdot \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$

Propreties.

$$\nabla \cdot (a\mathbf{F} + b\mathbf{G}) = a\nabla \cdot \mathbf{F} + b\nabla \cdot \mathbf{G},$$

$$\nabla \cdot (f\mathbf{F}) = \nabla f \cdot \mathbf{F} + f\nabla \cdot \mathbf{F},$$

$$\nabla \cdot \nabla f = \triangle f.$$

Example. $\mathbf{F} = xe^y\mathbf{i} + z\sin y\mathbf{j} + xy\ln z\mathbf{k}$.

Definition. *(* $Curl(\mathbf{F}) = \nabla \times \mathbf{F} = (\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z})\mathbf{i} + (\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x})\mathbf{j} + (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})\mathbf{k}$*).*

Propreties.

$$\nabla \times (a\mathbf{F} + b\mathbf{G}) = a\nabla \times \mathbf{F} + b\nabla \times \mathbf{G},$$

$$\nabla \times (f\mathbf{F}) = \nabla f \times \mathbf{F} + \nabla f \times \nabla \cdot \mathbf{F},$$

$$\nabla \times (\nabla f) = \mathbf{0}.$$

Example. $\mathbf{F} = xe^y\mathbf{i} + z\sin y\mathbf{j} + xy\ln z\mathbf{k}$.

14.2 The Line Integrals

Definition. *The line integral of a function f along a curve $C : \{\mathbf{r}(t) : a \le t \le b\}$ with respect to arc length*

$$\int_C f ds = \int_a^b f(\mathbf{r}(t))|\mathbf{r}'(t)|dt.$$

(Application to mass, centroid, moment of initia.)

Example.

(1) $\int_C xyds, (\cos t, \sin t), 0 \le t \le \pi/2,$
(2) Centroid of $(3\cos t, 3\sin t, 4t), 0 \le t \le \pi, \delta = kz.$

Definition. *The line integral of a function f along a curve $C : \{\mathbf{r}(t) : a \le t \le b\}$ with respect to coordinate variables*

$$\int_C Pdx + Qdy + Rdz = \int_a^b P(\mathbf{r}(t))x'(t) + Q(\mathbf{r}(t))y'(t) + R(\mathbf{r}(t))z'(t)dt.$$

(Application to work.)

Example.

(1) $\int_{\mathbf{C}} ydx + zdy + xdz, (t, t^2, t^3), 0 \le t \le 1,$
(2) $\int_{\mathbf{C}} xyds, \int_{\mathbf{C}} ydx + xdy, (1 + 8t, 2 + 6t), 0 \le t \le 1, (9 - 4t, 8 - 3t), 0 \le t \le 2,$
(3) $\int_{\mathbf{C}_i} ydx + 2xdy, \mathbf{C}_1 : [(1, 1), (2, 4)], \mathbf{C}_2 : y = x^2, 1 \le x \le 2, \mathbf{C}_3 : [(1, 1), (2, 1), (2, 4)]$ (Path dependent).

Remark. $\int_{\mathbf{C}} fds = \int_{-\mathbf{C}} fds, \int_{\mathbf{C}} Pdx + Qdy = -\int_{-\mathbf{C}} Pdx + Qdy.$

Line Integral and Vector Field $\mathbf{r}(t) = (x(t), y(t), z(t)), \mathbf{F} = (P, Q, R)$**.**

$$\int_{\mathbf{C}} \mathbf{F} \cdot d\mathbf{r} = \int_{\mathbf{C}} Pdx + Qdy + Rdz.$$

Example.

(1) $\mathbf{F} = (y, z, x), (t, t^2, t^3), 0 \le t \le 1,$
(2) $\mathbf{F} = k\frac{\mathbf{r}}{r^3}, [(0, 4, 0), (0, 4, 3)].$

14.3 Fundamantal Theorem and Independent of Pathjs

Theorem. *(Fundamental Theorem)*

$$\int_{\mathbf{C}} \nabla(f) \cdot d\mathbf{r} = f(B) - f(A).$$

Example.
$f = k\frac{1}{|\mathbf{r}|}, \nabla(f) = -k\frac{\mathbf{r}}{|\mathbf{r}|^3}.$

Independent of Pathe.

$$\int_{\mathbf{C}} \mathbf{F} \cdot d\mathbf{r} = \int_A^B \mathbf{F} \cdot d\mathbf{r}.$$

Theorem. $\mathbf{F}$ *is independent of path if and only* $\oint_C \mathbf{F} \cdot d\mathbf{r} = o$ *for all closed path.*

Example.$\mathbf{F} = k\mathbf{w}$

(1) $\mathbf{w} = (10, 10), k = 0.5, \mathbf{C}_1 : (10t, 10t), \mathbf{C}_2 : y = \frac{x^2}{10}.,$
(2) $\mathbf{w} = (-2y, 2x), k = 0.5, \mathbf{C}_1 : -[-10, 10], \mathbf{C}_2 : (10\cos t, 10\sin t).$

Theorem. $\int_{\mathbf{C}} \mathbf{F} \cdot d\mathbf{r}$ *is independent of paths if and only if* $\mathbf{F} = \nabla(f)$.

Example.
$\mathbf{F} = (6xy - y^3, 4y + 3x^2 - 3xy^2).$

Definition. *A vector field* $\mathbf{F}$ *is a conservative vector field if* $\mathbf{F} = \nabla f$, *for some continuous differentiable function* f. f *is called a potential function of* $\mathbf{F}$.

Conservation of Total Energy.

$$f(\mathbf{b}) - f(\mathbf{a}) = \int_{\mathbf{C}} \mathbf{F} \cdot d\mathbf{r} = \frac{1}{2}mv(\mathbf{b})^2 - \frac{1}{2}mv(\mathbf{a})^2.$$

Example.$f = \frac{GMm}{r}, E = \frac{1}{2}mv^2 - \frac{GMm}{r}.$

Theorem. $\mathbf{F} = (P, Q)$ *on a convex domain in* $\mathbb{R}^2$ *is conservative if and only if* $P_y = Q_x$.

14.4 Green's Theorem

Theorem.

$$\oint_{\mathbf{C}} Pdx + Qdy = \iint_R Q_x - P_y dA.$$

Example.

(1) $\oint_{\mathbf{C}} (2y + \sqrt{9 + x^3})dx + (5x + e^{\tan y})dy, \mathbf{C} = (2\cos t, 2\sin t), 0 \le t \le 2\pi,$
(2) $\oint_{\mathbf{C}} 3xydx + 2x^2dy, \mathbf{C} = \{x = y, y = x^2 - 2x\}.$

Corollary.

$$A = \frac{1}{2}\oint -ydx + xdy = \oint xdy = \oint -ydx.$$

Example. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$

Theorem. *If* $\partial R = C_o - C_i$,

$$\oint_{\mathbf{C}_0} Pdx + Qdy - \oint_{\mathbf{C}_i} Pdx + Qdy = \iint_R Q_x - P_y dA.$$

Example. $\oint \frac{-ydx+xdy}{x^2+y^2} = 2\pi.$

$$\Phi = \oint \mathbf{F} \cdot \mathbf{n} ds.$$

Divergence. $\mathbf{n} = (y'(t), -x'(t))/s(t)$

Theorem. *(Divergence Theorem)*

$$\oint_{\partial D} \mathbf{F} \cdot \mathbf{n} ds = \iint_D div\mathbf{F} dA.$$

Example. $\mathbf{F} = (-y, x), \mathbf{C} = (a\cos t, a\sin t), 0 \le t \le 2\pi.$

14.5 Surface integrals

Parametric surface $\mathbf{S} = \{\mathbf{r}(u,v) : (u,v) \in R.\}$

Surface integral with respect to area element.

$$\iint_{\mathbf{S}} f dS = \iint_R f(\mathbf{r}(u,v))|\mathbf{r}_u \times \mathbf{r}_v| dudv.$$

$$|\mathbf{r}_u \times \mathbf{r}_v| = [(\frac{\partial(x,y)}{\partial(u,v)})^2 + (\frac{\partial(y,z)}{\partial(u,v)})^2 + (\frac{\partial(z,x)}{\partial(u,v)})^2]^{1/2},$$

in case $\mathbf{S}$ is the graph of $z = h(x,y)$,

$$dS = \sqrt{1 + h_x^2 + h_y^2} dxdy.$$

(Application to mass, centroid, moment of initia.)

Example.

(1) Centroid of $\delta = 1, z = \sqrt{a^2 - x^2 - y^2}, x^2 + y^2 \le a^2,$

(2) I_z of $x^2 + y^2 + z^2 = a^2$, $\delta = 1$,

$$\mathbf{r} = (a\sin\phi\cos\theta, a\sin\phi\sin\theta, a\cos\phi)$$

,

$$\mathbf{r}_\phi = (a\cos\phi\cos\theta, a\cos\phi\sin\theta, -a\sin\phi),$$

$$\mathbf{r}_\theta = (-a\sin\phi\sin\theta, a\sin\phi\cos\theta, 0),$$

$$\mathbf{r}_\phi \times \mathbf{r}_\theta = (a^2\sin^2\phi\cos\theta, a^2\sin^2\phi\sin\theta, a^2\sin\phi\cos\phi).$$

$dS = a^2 \sin\phi d\phi d\theta, I_z = \frac{8}{3}\pi a^4 = 2a^2 m.$

Surface Integrals with respect to Coordinate Elements.

$$\iint_{\mathbf{S}} Pdydz + Qdzdx + Rdxdy = \iint_{\mathbf{S}} [P\frac{\partial(y,z)}{\partial(u,v)} + Q\frac{\partial(z,x)}{\partial(u,v)} + R\frac{\partial(x,y)}{\partial(u,v)}]dudv$$
$$= \iint_{\mathbf{S}} \mathbf{F}\cdot\mathbf{n}dS.$$

Example. $\mathbf{S}$ is the graph of $z = h(x,y)$, $\iint_{\mathbf{S}} \mathbf{F}\cdot\mathbf{n}dS = \iint_D -Ph_x - Qh_y + Rdxdy$.

Flux of a Vector Field throught a Surface with continuous unit normal vector..

$$\Phi = \iint_{\mathbf{S}} \mathbf{F}\cdot\mathbf{n}dS.$$

Example.

(1) $\mathbf{F} = v_0\mathbf{k}, \mathbf{S} = \{z = \sqrt{a^2 - x^2 - y^2}, \mathbf{n}\}$ up, $\iint_{\mathbf{S}} \mathbf{F}\cdot\mathbf{n}dS = \int_0^{2\pi}\int_0^{\pi/2} v_0 z/adS = \pi a^2 v_0$,

(2) $\mathbf{F} = (x, y, 3), z = x^2 + y^2, z = 4$, $\iint_{\mathbf{s}_u} 3dS = 12\pi$, $\iint_{\mathbf{S}_l} \mathbf{F}\cdot\mathbf{n}dS = \iint_{r\le 2} 2x^2 + 2y^2 - 3dxdy = 8 - 6$,

(3) Heat flow $\mathbf{q} = -k\nabla\mathbf{u}$. $\mathbf{u}(x,y,z) = c(R^2 - x^2 - y^2 - z^2)$, $\mathbf{q} = -kc(-2x, -2y, -2z) = 2kc\mathbf{r}$, $\iint_{\mathbf{s}_a} \mathbf{q}\cdot\mathbf{n}dS = 2cka(4\pi a^2)$,

(4) $\mathbf{F} = k\frac{\mathbf{r}}{r^3}$, $\iint_{\mathbf{S}_a} \mathbf{F}\cdot\mathbf{n}dS = 4\pi$.

14.6 Divergence Theorem

Theorem. *(Divergence Theorem)*

$$\iint_{\mathbf{S}} \mathbf{F}\cdot\mathbf{n}dS = \iiint_{\mathbf{T}} \nabla\cdot\mathbf{F}dV.$$

Example.

(1) $\mathbf{T} = \{(0 \le z \le 1 - x^2) \cap (0 \le y \le 2)\}, \mathbf{F} = (x + \cos y, y + \sin z, z + e^x)$, $\nabla\cdot\mathbf{F} = 3, \iint \mathbf{F}\cdot\mathbf{n}dS = 8$,

(2) $\mathbf{T} = \{(0 \le z \le 3)\cap(x^2+y^2 \le 4)\}, \mathbf{F} = (x^2+y^2+z^2)(x,y,z), \nabla\cdot\mathbf{F} = 5(x^2+y^2+z^2)$, $\iint \mathbf{F}\cdot\mathbf{n}dS = 300\pi$,

(3) $\mathbf{V} = \frac{1}{3}\iint xdydz + ydzdx + zdxdy = \iint xdydz = \iint ydzdx = \iint zdxdy$,

(4) $\nabla\cdot\mathbf{F} = \lim_{r\to 0}\frac{1}{|B_r|}\iint_{\mathbf{S}_r} \mathbf{F}\cdot\mathbf{n}dS$,

(5) Ex 14.6 20 $\iint f\mathbf{n}dS = \iiint \nabla f dV$.

(6) Ex 14.6 21 $\mathbf{B} = -\iint \delta gzdS$.

Theorem. *(Divergence Theorem over general Domain)*

$$\iint_{\mathbf{S}_o} \mathbf{F}\cdot\mathbf{n}dS - \iint_{\mathbf{S}_i} \mathbf{F}\cdot\mathbf{n}dS = \iiint_{\mathbf{T}} \nabla\cdot\mathbf{F}dV.$$

Theorem. *(Gauss Theorem) Let* $\mathbf{F} = k\frac{\mathbf{r}}{r^3}$, *then*

$$\iint_{\mathbf{S}} \mathbf{F}\cdot\mathbf{n}dS = 4\pi k$$

for any closed surface $\mathbf{S}$ *containing the origin.*

14.7 Stokes Theorem

Theorem. *(Stokes Theorem)*

$$\oint_{\mathbf{C}} \mathbf{F} \cdot d\mathbf{r} = \iint_{\mathbf{S}} (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS.$$

Example.

(1) $\oint \mathbf{F} \cdot d\mathbf{r}, \mathbf{S} = \{(x, y, z) : (z = y + 3) \cap (x^2 + y^2 \leq 1)\}$, $\mathbf{F} = (3z, 5x, -2y)$, $\nabla \times \mathbf{F} = (-2, 3, 5), \mathbf{n} = \frac{1}{\sqrt{2}}(0, -1, 1)$, $|S| = \sqrt{2}\pi$,
(2) $\iint (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS$, $\mathbf{F} = (3z.5x. - 2y)$, $\mathbf{S} = \{(x, y, z) : (z = x^2 + y^2) \cap (z \leq 4)\}$, $\mathbf{r}(t) = (2\cos t, 2\sin t, 4)$, $\oint \mathbf{F} \cdot d\mathbf{r} = 20\pi$,
(3) Ex 14.7 18 $\oint f\mathbf{T} ds = \iint \mathbf{n} \times \nabla f dS.$,
(4) EX 14.7 20 $\iint \mathbf{n} \times \mathbf{F} dS = \iiint \nabla \times \mathbf{F} dV$
(5) $\lim_{r\to 0} \frac{1}{\pi r^2} \oint_{C_r} \mathbf{F} \cdot \mathbf{T} ds = \nabla \times F \cdot \mathbf{n}.$.

Definition. $\mathbf{F}$ *is irraotational if* $\nabla \times \mathbf{F} = \mathbf{0}$.

Theorem. *If* $\mathbf{F}$ *is irrotational on a simply connected domain* $\mathbf{D}$, *then* $\mathbf{F} = \nabla\phi$ *on* $\mathbf{D}$.

Example.$\mathbf{F} = (3x^2, 5z^2, 10yz)$.