CHAPTER 13, MULTIPLE INTEGRALS

13.1 Double Integral over Rectangle

Definition.

A function $f(x,y)$ is integrable on a rectangle $[a,b]\times[c,d]$ if there is a number I such that for any given $\epsilon>0$ there is a $\delta>0$ such that , fir any partition $P=\{a=x_0,x_1,\cdots,x_n=b\}$ on $[a,b]$ $Q=\{c=y_0,y_1,\cdots,y_m=d\}$ on $[c,d]$ with $|P|,|Q|<\delta$ for any set of representatives $(x_{ij},y_{ij})\in[x_{i-1},x_i]\times[y_{j-1},y_j]$

$$|\sum_{i,j} f(x_{ij},y_{ij})\triangle_i\triangle_j - I| < \epsilon.$$

I is written as

$$\iint_R f(x,y)dA.$$

Theorem. *Every continuous function is integrable on rectangle.*

Theorem. *If f is continuous on R , then*

$$\iint_R f(x,y)dA = \int_c^d [\int_a^b f(x,y)dx]dy = \int_a^b [\int_c^d f(x,y)dy]dx.$$

Proof. Let $F(y)=\int_a^b f(x,y)dx$, then F is a continuous function on $[c,d]$. Now for any partition $\{y_0,\cdots,y_m\}$ of $[c,d]$ and any partition $\{x_0,\cdots,x_n\}$ of $[a,b]$

$$\int_c^d\int_a^b f(x,y)dxdy = \int_c^d F(y)dy = \sum_1^m \int_{y_{j-1}}^{y_j} F(y)dy = \sum_1^m F(y_j^*)\triangle y_j,$$

the last equality comes from mean value theorem of integral. Again

$$F(y_j^*) = \int_a^b f(x,y_j^*)dx = \sum_1^n f(x_{ij}^*,y_j^*)\triangle x_i.$$

For given $\epsilon>0$ there is a $\delta>0$ such that for all $\triangle y_j<\delta, \triangle x_i<\delta$ we have

$$|\iint_R f(x,y)dA - \sum_{i,j} f(x_{ij}^*,y_j^*)\triangle x_i\triangle y_j| < \epsilon|,$$

hence

$$|\iint_R f(x,y)dA - \int_c^d \int_a^b f(x,y)dxdy| < \epsilon.$$

Example.

(1) $R = [1,3] \times [-2,1], f(x,y) = 4x^3 + 6xy^2$,
(2) $\int_0^\pi \int_0^{\frac{\pi}{2}} \cos x \cos y dydx$,
(3) $\int_0^1 \int_0^{\frac{\pi}{2}} [e^y + \sin x]dxdy$.

13.2 Double Integral over More General Region

Definition. *$f(x,y)$ is integrable over a region R with integral I if for any given $\epsilon > 0$ the is a $\delta > 0$ such that for any partition with $\triangle y_j < \delta, \triangle x_i < \delta$ any Riemann sum over the inner partition S satisfies $|S - I| < \epsilon$. Here the inner partition means those boxes which is contained in R.*

Theorem. *Let R be the region defined by $\{(x,y) : 0 \le y \le \phi(x), a \le x \le b\}$ where ϕ is a continuous function on $[a,b]$ and $f(x,y)$ is a continuous function R. Then f is integrable over R.*

Theorem. *R and f as above, then*

$$\iint_R f(x,y)dA = \int_a^b \int_0^{\phi(x)} f(x,y)dydx.$$

Properties of Double Integrals.

(1) $\iint_R cf(x,y)dA = c\iint_R f(x,y)dA$,
(2) $\iint_R [f(x,y) + g(x,y)]dA = \iint_R f(x,y)dA + \iint_R g(x,y)dA$,
(3) If $m \le f(x,y) \le M$ then $mA(R) \le \iint_R f(x,y)dA \le MA(R)$,
(4) If $R = R_1 \cup R_2$ then $\iint_R f(x,y)dA = \iint_{R_1} f(x,y)dA + \iint_{R_2} f(x,y)dA$.

Example.

(1) $\iint_R xy^2 dA, x^3 \le y \le \sqrt{x}$,
(2) $\iint_R 6x + 2y^2 dA, y^2 \le x \le 2 - y$,
(3) $\int_0^2 \int_{y/2}^1 ye^{x^3} dxdy$,
(4) Area of the region $y = 2 + \frac{1}{4}x^2, x = \pm(1+y^2), (-2,3), (-2,1), (-2,-1), (2,3), (2,1), (2,-1)$.

13.3 Area, Volume by Double Inetral

Example.

(1) $f(x,y) = 1 + xy, [0,2] \times [0,1]$,
(2) $y = x^2 - 2x, y = x$,
(3) $0 \le z \le x, x^2 + y^2 \le 4$,
(4) $2y \le z \le 6, y = x^2, y = 2 - x^2$.

13.4 Double Integrasl in Polar Coordinates

$$\int_{\alpha}^{\beta}\int_{0}^{\phi(\theta)} f(r,\theta) r dr d\theta.$$

Example.

(1) $0 \le z \le 25 - x^2 - y^2$,
(2) $r = 1, r = 2 + \cos\theta$,
(3) $x^2 + y^2 + z^2 \le 4, (x-1)^2 + y^2 \le 1$,
(4) $r^2 \le z \le 8 - r^2$,
(5) $\int_0^{\infty} e^{-x^2} dx$.

13.5 Application of Double Integrals

Center of Mass.

$$\text{Mass: } M = \iint_R \delta(x,y) dA,$$

$$\text{Centroid: } \overline{x} = \frac{1}{M}\iint_R x\delta(x,y) dA, \overline{y} = \frac{1}{M}\iint_R y\delta(x,y) dA.$$

Example.

(1) $\delta = 1, 0 \le r \le a, 0 \le \theta \le \pi$,
(2) $\delta = kx^2, x^2 \le y \le x + 2$,
(3) $\delta = kr, 0 \le r \le a, 0 \le \theta \le \frac{\pi}{2}$.

First Pappus theorem. $V = 2\pi\overline{r}A$.

Example.$x = b + r\cos\theta, y = r\sin\theta, a \le r \le c, 0 \le \theta \le 2\pi, a < c < b$.

Second Pappus theorem. $A = 2\pi\overline{r}s$.

Example.

(1) $(a\cos t, a\sin t), 0 \le t \le \pi$,
(2) $b + a\cos t, a\sin t), 0 \le t \le 2\pi$.

Moment of Initia.

$$I_o = \iint r^2\delta(x,y) dA,$$

$$I_x = \iint y^2\delta(x,y) dA,$$

$$I_y \iint x^2\delta(x,y) dA.$$

Example.

(1) $I_x = \int_{-1}^{1}\int_{-y^4}^{y^4} y^2 dxdy = \frac{4}{7}, I_x = \int_{-1}^{1}\int_{-1/5}^{1/5} y^2 dxdy = \frac{4}{15}$,
(2) $r \le a, \delta \equiv 1, I_o = \frac{\pi a^4}{2}$.

13.6 Triple Integral in Rectangular Coordinates

Example.

(1) $R = [-1,1]\times[2,3]\times[0,1], f(x,y,z) = xy + yz$,
(2) $\delta = z, R =< (0,0,0),(2,0,0),(0,1,0),(0,0,4) >$,
(3) $x^2 + y^2 \le z \le y + 2, \iint_{(x,y)} \int dzdxdy$,
(4) Centroid of $0 \le z \le 1 - x, x = y^2, \iiint_{zxy}, \iiint_{yzx}, \iiint_{xzy}$,
(5) $x^2 + y^2 \le z \le y + 2, \int\int\int dxdzdy$.

13.7 Triple Integrals in Cylindrical and Spherical Coordinates

Cylindrical Coordinates.

$$\iiint_R f(r,\theta,z) r dr d\theta dz.$$

Example.

(1) Centroid of $x^2 + y^2 + z^2 \le 1, 0 \le x, y, z$,
(2) $b(x^2 + y^2) \le z \le h$.

Spherical Coordinates.

$$\iiint_R f(\rho,\phi,\theta)\rho^2 \sin\phi d\rho d\phi d\theta.$$

Example.

(1) Volume and I_z of $x^2 + y^2 + z^2 \le 1$,
(2) $\overline{z}$ of $\phi \le \frac{\pi}{6}, \rho \le 2a\cos\phi$,
(3) Exercise 47,48.

$$\iiint \frac{(L-\rho\cos\phi)\rho^2\sin\phi}{(L^2+\rho^2-2L\rho\cos\phi)^{3/2}} d\theta d\phi d\rho = 2\pi \iint \frac{(L-\rho\cos\phi)\rho^2\sin\phi}{(L^2+\rho^2-2L\rho\cos\phi)^{3/2}} d\phi d\rho. \qquad (1)$$

Using $u = \rho\cos\phi$ and integrate the inner integral right hand side of (1) will get

$$2\pi\frac{(-\rho)}{L}[\frac{L-\rho\cos\phi}{(L^2+\rho^2-2L\rho\cos\phi)^{1/2}} - \frac{1}{L}(L^2+\rho^2-2L\rho\cos\phi)^{1/2}]|_0^{\pi},$$

$$= 2\pi\frac{(-\rho)}{L}[(\frac{L+\rho}{|L+\rho|} - \frac{|L+\rho|}{L}) - (\frac{L-\rho}{|L-\rho|} - \frac{|L-\rho|}{L})].$$

In case $L > \rho$, it is $4\pi\frac{\rho^2}{L^2}$ and in case $L < \rho$ it is 0. So when $L > R$ (1) is $\frac{4\pi R^3}{3L^2}$ and when $L < R$, (1) is $\frac{4}{3}\pi L$.

13.8 Surface Area of Parametric Surface

Parametric Surfaces.

$$\mathbf{r}(u,v) = (x(u,v),(y(u,v),z(u,v)).$$

Example.

(1) $(x,y,f(x,y))$,
(2) $(r\cos\theta, r\sin\theta, g(r,\theta))$,
(3) $(h(\phi,\theta)\sin\phi\cos\theta, h(\phi,\theta)\sin\phi\sin\theta, h(\phi,\theta)\cos\phi)$.

Surface Area.

$$A = \iint_R |\mathbf{N}|dudv = \iint_R |\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}|dudv.$$

Surface Area of Graph.

$$A = \iint_R \sqrt{1 + f_x^2 + f_y^2}dxdy.$$

Example. $z = 2x + 2y + 1$ inside $x^2 + y^2 \leq 1$.

Surface Area in Polar Coordinates.

$$A = \iint_R \sqrt{r^2 + (rz_r)^2 + z_\theta^2}drd\theta.$$

Example.

(1) $z = r^2, r \leq 1$,
(2) $z = \theta, 0 \leq \theta \leq \pi, 0 \leq r \leq 1$.

Example. $(x-b)^2 + z^2 = a^2$ revolute about z-axis.

13.9 Change Variables in Multiple Integrals

Jacobian in 2-dimension.

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix}.$$

Theorem.

$$\iint_R f(x,y)dxdy = \iint_S f(x(u,v),y(u,v))\frac{\partial(x,y)}{\partial(u,v)}dudv.$$

Example.

(1) $\iint f(x,y)dxdy = \iint f(r\cos\theta, r\sin\theta)rdrd\theta$,
(2) I_o of $1 \leq xy \leq 3, 1 \leq x^2 - y^2 \leq 4$,
(3) Area of $1 \leq xy \leq 3, 1 \leq xy^{1.4} \leq 2$.

Jacobian in 3-dimension.

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \\ \frac{\partial x}{\partial w} & \frac{\partial y}{\partial w} & \frac{\partial z}{\partial w} \end{vmatrix}.$$

Theorem.

$$\iiint_R f(x,y,z)dxdydz = \iiint_S f(x(u,v,w,y(u,v,w),z(u,v,w))\frac{\partial(x,y,z)}{\partial(u,v,w)}dudvdw.$$

Example.

(1) Sperical coordinates,
(2) Revolution of $(x-b)^2 + z^2 = a^2$ about z-axis.