# CHAPTER 12, PARTIAL DIFFERENTIATION

## 12.2 Function of Several Variables

**Example.**

(1) $\sqrt{25 - x^2 - y^2}$,
(2) $\frac{y}{\sqrt{x-y^2}}$,
(3) $\frac{x+yz}{\sqrt{x^2+y^2}}$,
(4) $\frac{\exp[-\frac{x^2+y^2+z^2}{4kt}]}{4\pi kt}$.

**Graph.** $\{(x, y, f(x, y)) : (x, y) \in D\}$

**Example.**

(1) $z = 2 - \frac{1}{2}x - \frac{1}{2}y$,
(2) $z = y^2 + x^2$,
(3) $z = \frac{1}{2}\sqrt{4 - 4x^2 - y^2}$,

**Level Curves, Surfaces.** $C_d = \{((x, y) : f(x, y) = d\}$, $S_d = \{(x, y, z) : f(x, y, z) = d\}$.

**Example.**

(1) $25 - x^2 - y^2$,
(2) $y^2 - x^2$,
(3) $x^2 + y^2 - z^2$.

## 12.3 Limit and Continuouity

**Limits.**

**Definition.**

$$\lim_{(x,y)\to(a,b)} f(x, y) = L$$

*if and only if for any $\epsilon > 0$ there is a $\delta > 0$ such that,*

$$0 < |(x, y) - (a, b)| < \delta, \quad \textit{implies} \quad |f(x, y) - L| < \epsilon.$$

**Example.**

(1) $\lim_{(x,y)\to(2,3)} xy = 6$,
(2) $\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2} = 1$.

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**Continuity.**

**Definition.** *$f(x,y)$ is continuous at $(a,b)$ if*

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b).$$

**Laws of limits and Continuity.**

**Theorem.** *Suppose that $\lim_{(x,y)\to(a,b)} f(x,y) = L, \lim_{(x,y)\to(a,b)} g(x,y) = M$, then*

(1) $\lim_{(x,y)\to(a,b)}(f(x,y)+g(x,y)) = L+M$,
(2) $\lim_{(x,y)\to(a,b)} f(x,y)\cdot g(x,y) = LM$,
(3) $\lim_{(x,y)\to(a,b)} f(x,y)/g(x,y) = L/M$ *if* $M \neq 0$.

**Remark.** *The squeeze law and the substituition law is also true for function of several variables.*

**Example.**

(1) Polynomials $2x^4y^2 - 7xy + 4x^2y^3 + 35$,
(2) $\frac{\sin(x^2+y^2)}{x^2+y^2}$ if $(x,y)\neq(0,0)$ and 1 if $(x,y)=(0,0)$,
(3) $e^{xy} + \sin\frac{y}{4} + xy\ln\sqrt{y-x}$,
(4) $\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = 0$,
(5) $\frac{xy}{x^2+y^2}$,
(6) $\frac{xy^2}{x^2+y^4}$.

## 12.4 Partial Derivatives

**Definition.** $f_x(x,y) = \frac{\partial f}{\partial x}(x,y) = \lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h}$, *provide the limit exists, and is called the partial derivative of $f$ with respect to $x$.*

**Example.**

(1) $x^2 + 2xy^2 - y^3$,
(2) $(x^2+y^2)e^{-xy}$.

**Instaneous Rates of Change.**

**Example.**

(1) $V = \frac{(82.06)T}{p}$.

**Geometric Interpretation.**

**Example.**

(1) $5xye^{-x^2-2y^2}$.

**Theorem.** *Suppose that $f$ has continuous partial derivatives near $(a,b)$, then the plane contains $(1,0,f_x(a,b))$ and $(0,1,f_y(a,b))$ contains all vectors that is tangent to a curve on $z=f(x,y)$. This plane is called the tangent plane of the graph $z=f(x,y)$ at $(a,b)$.*

Since $\mathbf{n}=(1,0,f_x)\times(0,1,f_y)=(-f_x,-f_y,1)$ the equation of the tangent is

$$z=f(a,b)+f_x(a,b)(x-a)+f_y(a,b)(y-b).$$

If $r(t)$ is a curve on on the graph of $f(x,y)$ such that $r(0)=(x_0,y_0,f(x_0,y_0))$, then $r'(0)=(x'(0),y'(0),f_x(x_0,y_0)x'(0)+f_y(x_0,y_0)y'(0))$.

**Example.**

(1) $z=5-2x^2-y^2$ at $(1,1,2)$,
(2) $z=\sin xye^{uv}$.

**Higher Derivatives.**

**Example.**

(1) $x^2+2xy^2-y^3$,
(2) $\frac{x^3y-xy^3}{x^2+y^2}$.

**Remark.** *If $f_x,f_y,f_{xy}.f_{yx}$ are continuous, then $f_{xy}=f_{yx}$.*

## 12.5 Multivariables Optimization Problems

**Theorem.** *Suppose that $f$ is a continuous function on a bounded closed region $D$ , then $f$ takes both maximum and minimum values on $D$.*

**Example.**(Closed domain)

(1) $D=\{(x,y):x^2+y^2\le 1\}$,
(2) $D=\{(x,y):x^2\le 1,y^2\le 1\}$.

**Definition.** *Local extremum.*

**Theorem.** *Suppose that $f$ has local extremum at $(a,b)$ and the partial derivatives exist at $(a,b)$, then $f_x(a,b)=0=f_y(a,b)$.*

**Example.**

(1) $f(x,y)=x^2+y^2, g(x,y)=1-x^2-y^2, h(x,y)=x^2-y^2$,
(2) $z=\frac{3}{4}y^2+\frac{1}{24}y^3-\frac{1}{32}y^4-x^2$,
(3) Global extrema of $\sqrt{x^2+y^2}$ over $\{(x,y):x^2+y^2\le 1\}$,
(4) Global extrema of $xy-x-y+3$ over $\{(x,y):0\le x\le 2),0\le y\le 4+2x\}$,
(5) Highest point of $z=\frac{8}{3}x^3+4y^3-x^4-y^4$,
(6) When $V=48$, FB=1\$, TB=2\$, LR=3\$,
(7) $f(x,y,z)=xy+yz-xz$.

## 12.6 Increment, Linear Approximation and Differentiablity

**Increment.**

$$\triangle f = f(x+h, y+k) - f(x,y).$$

**Differential.**

$$df = f_x(x,y)dx + f_y(x,y)dy.$$

**Example.**

(1) $x^2 + 3xy - 2y^2, (3,5) \to (3.2, 4.9), \triangle f = 5.26, df = 5.3,$
(2) $\sqrt{2(2.02)^3 + (2.97)^2},$
(3) $dV = yzdx + xzdy + xydz,$

**Definotion.** *$\nabla f = (f_x, f_y)$ is called the gradient of $f$.*

**Theorem.** *Suppose that $f$ has continuous partial derivatives near $\mathbf{a}$, then*

$$f(\mathbf{a}+\mathbf{h}) = f(\mathbf{a}) + \nabla f(\mathbf{a}) \cdot \mathbf{h} + \epsilon(\mathbf{h}) \cdot \mathbf{h},$$

*where $\epsilon(\mathbf{h})$ is a vector valued function which goes to zero as $\mathbf{h} \to \mathbf{0}$.*

**Definition.** *$f$ is differentiable at $\mathbf{a}$ if*

$$f(\mathbf{a}+\mathbf{h}) = f(\mathbf{a}) + \nabla f(\mathbf{a}) \cdot \mathbf{h} + \epsilon(\mathbf{h}),$$

*where $\frac{\epsilon(\mathbf{h})}{|\mathbf{h}|} \to 0 \quad as \quad \mathbf{h} \to 0.$*

**Remark.** *If $f$ has continuous partial derivatives near $\mathbf{a}$ then $f$ is differentiable at $\mathbf{a}$.*

**Example.** $f(x,y) = \frac{x^3+y^3}{x^2+y^2}$ for $(x,y) \neq (0,0)$ and $f(0,0) = 0$

## 12.7 Multivariables Cahin Rule

**Theorem.** *Suppose that $f$ is differentiable at $(a,b)$ and $\phi(t), \psi(t)$ are differentiable at $t_0$ with $\phi(t_0) = a, \psi(t_0) = b$, then $g(t) = f(\phi(t), \psi(t))$ is differentiable at $t_0$ and*

$$\frac{dg}{dt}(t_0) = f_x(\phi(t_0), \psi(t_0))\phi'(t_0) + f_y(\phi(t_0), \psi(t_0))\psi'(t_0).$$

**Example.**

(1) $w = e^{xy}, x = t^2, y = t^3,$
(2) $V = \pi r^2 h, \frac{dh}{dt} = -3, \frac{dr}{dt} = -1, h = 40, r = 15, \frac{dV}{dt} = ?,$
(3) $w = x^2 + ze^y + \sin xz, x = t, y = t^2, z = t^3.$

**Theorem.** *(Chain Rule for Several Variables) Suppose that $f(u_1, u_2, \cdots, u_m)$ is differentiable and $u_j(x_1, x_2, \cdots, x_n)$ are differentiable , then $g(x_1, x_2, \cdots, x_n) = f(\cdots, u_j(x_1, x_2, \cdots, x_n), \cdots)$ is differentiable and*

$$\frac{\partial g}{\partial x_i}(x_1, x_2, \cdots, x_n) = \sum_{j=1}^{m} f_{u_j}(\cdots, u_k(x_1, x_2, \cdots, x_n)\cdots)(u_j)_i(x_1, x_2, \cdots, x_n).$$

**Example.**

(1) $z = f(u, v), u = 2x + y, v = 3x - 2y, (u, v) = (3, 1), (x, y) = (1, 1), \frac{\partial z}{\partial u} = 3, \frac{\partial z}{\partial v} = -2,$
(2) $w = f(x, y), \frac{\partial w}{\partial r}, \frac{\partial w}{\partial \theta}, \frac{\partial^2 w}{\partial r^2},$
(3) $w = f(u, v, x, y), u(x, y), v(x, y),$
(4) $z = f(x, y), x(t), y(t), z(t), T = (x', y', z'), n = (z_x, z_y, -1).$

**Theorem.** *(Mean Value Theorem) Suppose that $f(x, y)$ is differentiable on a covex domain $D$. For $P, Q \in D$*

$$F(P) - f(Q) = \nabla f(R) \cdot (P - Q),$$

*for some $R$ on the line from $Q$ to $P$.*

**Remark.** *The key part of the Mean value theorem is that $f$ is differentiable on the line from $P$ to $Q$. Consider the example $f(x, y) = \frac{1}{x^2+y^2}$, for $(x, y) \neq (0, 0)$ $P = (-1, 0), Q = (1, 0)$.*

**Theorem.** *(Implicit Function Theorem) Suppose that $F(x_1, x_2, \cdots, x_n, y)$ is continuously differentiable near $(\mathbf{a}, b)$ and $F(\mathbf{a}, b) = 0$. If $F_y(\mathbf{a}, b) \neq 0$ , then there exists a continuously differentiable function $g(x_1, x_2, \cdots, x_n)$ near $\mathbf{a}$ such that $g(\mathbf{a}) = b$ and $F(\mathbf{x}, g(\mathbf{x})) \equiv 0$. Further more*

$$\frac{\partial g}{\partial x_i}(\mathbf{x}) = -\frac{F_{x_i}(\mathbf{x}, g(\mathbf{x}))}{F_y(\mathbf{x}, g(\mathbf{x}))}.$$

**Corollary.** *Suppose that $\nabla f$ does not vanish on the level surface $f = 0$, then the level surface is a smooth surface.*

**Example.**

(1) $x^3 + y^3 - 3xy = 0,$
(2) $x^4 + y^4 + z^4 + 4x^2y^2z^2 - 34 = 0.$

**Chain Rule in Matrix Form.** Suppose that $f(u_1, u_2, \cdots, u_m)$ is differentiable and $u_j(x_1, x_2, \cdots, x_n)$ are differentiable , then $g(x_1, x_2, \cdots, x_n) = f(\cdots, u_j(x_1, x_2, \cdots, x_n), \cdots)$ is differentiable and

$$\nabla_x g = \nabla_u f \cdot \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \cdots & \frac{\partial u_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial u_m}{\partial x_1} & \cdots & \frac{\partial u_m}{\partial x_n} \end{bmatrix}.$$

**Example.** $\left[\frac{\partial w}{\partial r} \quad \frac{\partial w}{\partial \theta}\right] = \begin{bmatrix} \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \end{bmatrix} \cdot \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}.$

## 12.8 Directional Derivatives and Gradient

**Definition.**

$$D_{\mathbf{u}} f(\mathbf{x}) = \lim_{t \to 0} \frac{f(\mathbf{x} + t\mathbf{u}) - f(\mathbf{x})}{t},$$

is the directional derivative of $f$ along the direction $\mathbf{u}$. It is the instantaneous rate of change of $f$ at $\mathbf{x}$ along the direction $\mathbf{u}$.

**Theorem.** *If $f$ is differentiable at $\mathbf{x}$, then $D_{\mathbf{u}} f(\mathbf{x}) = \nabla f(\mathbf{x}) \cdot \mathbf{u}$.*

**Example.**

(1) $T = \frac{1}{180}[7400 - 4x - 9y - 0.03xy]$, P(200,200), $\mathbf{v} = (3, 4)$,
(2) $\nabla f, f = yz + \sin xz + e^{xy}, (0, 7, 3)$,
(3) As in (1) $s = 5$,
(4) $T = \frac{1}{180}[7400 - 4x - 9y - 0.03xy] - 2z$, P(200,200,5), $\mathbf{v} = (3, 4, -12), s = 3$,
(5) As in (4) The direction with the most ripid increasing.

**Normal Direction of $F = c$.**

**Example.**

(1) $2x^2 + 4y^2 + z^2 = 45$ at P(2,-3,-1),
(2) Tangent at P(1,-1,2) to the intersection of $x^2 + y^2 - z = 0$ and $2x^2 + 3y^2 + z^2 - 9 = 0$,
(3) Tangent at (1,2) to $2x^3 + 2y^3 - 9xy = 0$.

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## 12.9 Lagrange Multiplier and Cinstrained Optimizatiom

**Theorem.** *(Lagrange Multiplier) Suppose that $f(x, y), g(x, y)$ are continuously differentiable functions. If the maximum (minimum) value of $f$ subject to tne constraint $g(x, y) = 0$ cccurs at a point $P$ where $\nabla g(P) \neq \mathbf{0}$, then*

$$\nabla f(P) = \lambda \nabla g(P)$$

*for some $\lambda$.*

**Example.**

(1) $f = \sqrt{x^2 + y^2}, xy - 1 = 0$,
(2) $f = 4xy, \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$,
(3) $f = 8xyz, \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

**Theorem.** *(Lagrange Multiplier with Two Constraints) Suppose that $f(x, y, z), g(x, y, z), h(x, y, z)$ are continuously differentiable functions. If the maximum (minimum) value of $f$ subject to tne constraint $g(x, y, z) = 0, h(x, y, z) = 0$ cccurs at a point $P$ where $\nabla g(P)$ and $\nabla h(P)$ are independent, then*

$$\nabla f(P) = \lambda \nabla g(P) + \mu \nabla h(P)$$

*for some* $\lambda, \mu$.

**Example.**

(1) $x + y + z = 12, z = x^2 + y^2 \ f = z$,
(2) $x + y + z = 1, x^2 + y^2 + z^2 = 1$ and $f = x^3 + y^3 + z^3$,
(3) $\frac{x_1 + \cdots + x_n}{n} \geq (x_1 \cdots x_n)^{1/n}$ for $x_i > 0, i = 1, \cdots n$,
(4) $f(x, y, z) = 2x + 2y + z, \sqrt{x^2 + y^2} \leq z \leq \sqrt{4 - x^2 - y^2}$.

## 12.10 Critical Point of Function of Two Variables

**Taylor Polynomial of Function of Two variables.**

$$P_n(x, y) = f(0, 0) + \sum_{n=1}^{m} \frac{1}{n!}[\sum_{0}^{n} \binom{n}{k} \frac{\partial^n f}{\partial^k x \partial^{n-k} y}(0, 0) x^k y^{n-k}.$$

**Theorem.** *Suppose that* $f_{xy}$ *and* $f_{yx}$ *are continuous, then* $f_{xy}(a, b) = f_{yx}(a, b)$.

**Critical Point of Function of Two Variables.** Suppose that (0,0) is a critical point of $f$, let $A = f_{xx}(0, 0), B = f_{xy}(0, 0), C = f_{yy}(0, 0)$ and $\triangle = AC - B^2$. Then the Taylor polynomial of $f$ at (0,0) of order two is

$$f(0, 0) + \frac{1}{2} A[(x + \frac{B}{A} y)^2 + \frac{\triangle}{A^2}],$$

so we have the following conclusions

(1) If $\triangle > 0, A > 0$ then $f(0, 0)$ is local minimum.
(2) If $\triangle > 0, A < 0$ then $f(0, 0)$ is local maximum.
(3) If $\triangle < 0$ then $f(0, 0)$ is neither local maximum nor local minimum.

**Example.**

(1) $3x - x^3 - 3xy^2$,
(2) $6xy^2 - 2x^3 - 3y^4$.