CHAPTER 11, VECTORS, CURVES AND SURFACES IN SPACE $\mathbb{R}^3$

11.2 Three Dimensional Rectangular Coordinates and Vectors

Rectangular Coordinates. $P(x, y, z)$

Distance in $\mathbb{R}^3$.

$$|P_1P_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$

Vectors in $\mathbb{R}^3$.

$$\mathbf{a} = (a_1, a_2, a_3), \text{ length } |\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}.$$

$$\mathbf{i} = (1, 0, 0), \mathbf{j} = (0, 1, 0), \mathbf{k} = (0, 0, 1).$$

Dot Product of two Vectors.

$$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3.$$

Example. $(3, 4, 12) \cdot (-4, 3, 0)$

Properties of Dot Product.

(1) $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$,
(2) $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$,
(3) $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$,
(4) $(\lambda\mathbf{a}) \cdot \mathbf{b} = \lambda(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot (\lambda\mathbf{b})$.

Theorem. $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta$, *or* $\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$. *Where* θ *is the angle between* $\mathbf{a}, \mathbf{b}$.

Example.

(1) $(8, 5) \cdot (-11, 17)$,
(2) $(8, 5, -1) \cdot (-11, 17, 3)$.

Directional Angles, directional Cosine of a.

(1) $\alpha = \angle\mathbf{ai}, \quad \cos\alpha$
(2) $\beta = \angle\mathbf{aj}, \quad \cos\beta$
(3) $\gamma = \angle\mathbf{ak}, \quad \cos\gamma$.

Hence $\cos\alpha = \frac{a_1}{|\mathbf{a}|}, \cos\beta = \frac{a_2}{|\mathbf{a}|}, \cos\gamma = \frac{a_3}{|\mathbf{a}|}$.

Example. $(2, 3, -1)$.

DEFINITION.Let a, b be two nontrivial vectors.

(1) $\mathbf{a}_{\parallel} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|^2}\mathbf{b}$, **which is the component of a in the direction of b.**
(2) $\mathbf{a}_{\perp} = \mathbf{a} - \mathbf{a}_{\parallel}$ **which is the component of a perpendicular to b.**

Example. $(4,5,3),(2,1,-2)$.

Remark. *Let* $\mathbf{a},\mathbf{b}$ *be any pair of othogonal vectors in* $\mathbb{R}^2$*, then any vector* $\mathbf{x}$ *in* $\mathbb{R}^2$ *can be written as*

$$\mathbf{x}=\frac{\mathbf{x}\cdot\mathbf{a}}{|\mathbf{a}|^2}\mathbf{a}+\frac{\mathbf{x}\cdot\mathbf{b}}{|\mathbf{b}|^2}\mathbf{b}. \tag{*}$$

11.3 Cross Product of Vectors

DEFINITION.

$$\mathbf{a}\times\mathbf{b}=(a_2b_3-a_3b_2,a_3b_1-a_1b_3,a_1b_2-a_2b_1)$$

also

$$\mathbf{a}\times\mathbf{b}=\begin{vmatrix}a_2 & a_3\\ b_2 & b_3\end{vmatrix}\mathbf{i}+\begin{vmatrix}a_3 & a_1\\ b_3 & b_1\end{vmatrix}\mathbf{j}+\begin{vmatrix}a_1 & a_2\\ b_1 & b_2\end{vmatrix}\mathbf{k}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}$$

Example.
(1) $\mathbf{i}\times\mathbf{j}=\mathbf{k},\mathbf{j}\times\mathbf{k}=\mathbf{i},\mathbf{k}\times\mathbf{i}=\mathbf{j}$,
(2) $(3,-1,2)\times(2,2,-1)$.

Theorem. $|\mathbf{a}\times\mathbf{b}|=|\mathbf{a}||\mathbf{b}|\sin\theta$.

Example. $\triangle ABC, A(3,0,-1), B(4,2,5), C(7,-2,4)$

Properties of cross product.
(1) $\mathbf{a}\times\mathbf{b}=-\mathbf{b}\times\mathbf{a}$.
(2) $(\lambda\mathbf{a})\times\mathbf{b}=\lambda(\mathbf{a}\times\mathbf{b})=\mathbf{a}\times(\lambda\mathbf{b})$.
(3) $\mathbf{a}\times(\mathbf{b}+\mathbf{c})=\mathbf{a}\times\mathbf{b}+\mathbf{a}\times\mathbf{c}$.
(4) $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})=(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}$.
(5) $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=(\mathbf{a}\cdot\mathbf{c})\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\mathbf{c}$.

Proof of(5). (I) Assume $\mathbf{b}\perp\mathbf{c}$. Since $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})\perp(\mathbf{b}\times\mathbf{c})$, it is in the plane generated by $\mathbf{b}$ and $\mathbf{c}$. In order to apply (*) , let's compute

$$(\mathbf{a}\times(\mathbf{b}\times\mathbf{c}))\cdot\mathbf{b}=\mathbf{a}\cdot((\mathbf{b}\times\mathbf{c})\times\mathbf{b})=(\mathbf{a}\cdot\mathbf{c})|\mathbf{b}|^2.$$

And similarly

$$(\mathbf{a}\times(\mathbf{b}\times\mathbf{c}))\cdot\mathbf{c}=\mathbf{a}\cdot((\mathbf{b}\times\mathbf{c})\times\mathbf{c})=-(\mathbf{a}\cdot\mathbf{b})|\mathbf{c}|^2.$$

Now by (*) we prove (5) in this case.

(II) In general

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=\mathbf{a}\times(\mathbf{b}\times(\mathbf{c}_\perp+\mathbf{c}_\parallel)=\mathbf{a}\times(\mathbf{b}\times\mathbf{c}_\perp).$$

Apply (I) we get

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c}_\perp)\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}_\perp.$$

But the right hand side is equal to $(\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$.

DEFINITION. Scalar triple product $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

Example.

(1) $(\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}) \times (3\mathbf{i} + 2\mathbf{j} - 4\mathbf{k})$,
(2) $\mathbf{a} = 2\mathbf{i} - 3\mathbf{k}, \mathbf{b} = \mathbf{i} + \mathbf{j} + \mathbf{k}, \mathbf{c} = 4\mathbf{j} - \mathbf{k}$.

Theorem. $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ *is the volume of the parallelpiped generated by* $\mathbf{a}, \mathbf{b}, \mathbf{c}$.

Example.

(1) $O(0,0,0), P(3,2,-1), Q(-2,5,1), R(2,1,5)$,
(2) $A(1,-1,2), B(2,0,1), C(3,2,0), D(5,4,-2)$ coplanar.

11.4 Lines and Planes in Space $\mathbb{R}^3$

Line in Parametric Equation. Given a point $P(x_0, y_0, z_0)$ and a direction $\mathbf{u} = (a, b, c)$, the line pass P along the direction $\mathbf{u}$ is represented by $x = x_0 + at, y = y_0 + bt, z = z_0 + ct$

Example.

(1) $P_1(1,2,2), P_2(3,-1,3)$,
(2) $x = 1 + 2t, y = 2 - 3t, z = 2 + t$.

Line in Symmetric Equation.

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}, \quad \text{or if } c = 0, z = z_0.$$

Example. $P_1(3,1,-2), P_2(4,-1,1)$.

Plan. Given a point $\mathbf{r}_0 = (x_0, y_0, z_0)$ and a direction $\mathbf{n} = (a, b, c)$, the plane pass $\mathbf{r}_0$ and normal to $\mathbf{n}$ is represented by

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0. \text{ or } ax + by + cz = ax_0 + by_0 + cz_0$$

Example.

(1) $\mathbf{r}_0 = (-1,5,2), \mathbf{n} = (1,-3,2)$,
(2) $P(2,4,-3), Q(3,7,-1), R(4,3,0)$.

Parallel Planes, Angle between two nonparallel planes. Two planes are parallel if $\mathbf{n}_1 = \lambda \mathbf{n}_2$. The angle between two non parallel planes is $\cos\theta = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1`||\mathbf{n}_2|}$.

Example. $2x + 3y - z = 3, 4x + 5y + z = 1$.

The intersection line of two nonparallel planes.

$$\mathbf{r}_0 + t\mathbf{n}_1 \times \mathbf{n}_2.$$

Example. $\frac{x-3}{1} = \frac{y-1}{-2} = \frac{z+2}{3}$ or $2x + y = 7, 3x - z = 11, 3y + 2z = -1$.

Distance from a point to a line. For given point P and line $Q + t\mathbf{u}$, the distance is given by

$$|\overrightarrow{PQ} - [\overrightarrow{PQ} \cdot \frac{\mathbf{u}}{|\mathbf{u}|}]\frac{\mathbf{u}}{|\mathbf{u}|}|.$$

Distance from a point to a Plane. For given point P and plane $\mathbf{u} \cdot (\mathbf{r} - \mathbf{r}_0) = 0$, the distance is given by

$$d = |\frac{\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0)}{|\mathbf{n}|}|$$

or

$$d = \frac{ax + by + cz - ax_0 - by_0 - cz_0|}{\sqrt{a^2 + b^2 + c^2}}.$$

Distance between two nonparallel nonintersecting lines. Let $P + t\mathbf{u}$ and $Q + s\mathbf{v}$ be two nonparallel lines, then the distance between them can be represented by

$$|\overrightarrow{PQ} \cdot \frac{\mathbf{u} \times \mathbf{v}}{|\mathbf{u} \times \mathbf{v}|}|.$$

11.5 Curves and Motions in Space

$$x = f(t), y = g(t), z = h(t).$$

Example.

(1) $x(t) = (2 + \cos\frac{3}{2}t)\cos t, y(t) = (2 + \cos\frac{3}{2}t)\sin t, z(t) = \sin\frac{3}{2}t$,
(2) $x(t) = 4\cos t, y(t) = 4\sin t, z(t) = 0$,
(3) $x(t) = 5\cos t, y(t) = 0, z(t) = 3\sin t$,
(4) $x(t) = 0, y(t) = 3\cos t, z(t) = 5\sin t$.

Vector Valued Function.

$$\mathbf{r}(t) = (x(t), y(t), z(t)).$$

Example. $\mathbf{r}(t) = (\cos t, \sin t, t)$

Limits, Continuity , Differentiation and Integration of Vector Valued Function. The Limits, Continuity , Differentiation and Integration of Vector Valued Functions are consider componentwisely.

Example. The tangent of $(\cos t, \sin t, t)$.

Theorem.

(1) $[\mathbf{u}+\mathbf{v}]'(t) = \mathbf{u}'(t) + \mathbf{v}'(t)$,
(2) $[c\mathbf{u}]'(t) = c\mathbf{u}'(t)$,
(3) $[f\mathbf{u}]'(t) = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$,
(4) $[\mathbf{u}\cdot\mathbf{v}]'(t) = \mathbf{u}'(t)\cdot\mathbf{v}(t) + \mathbf{u}(t)\cdot\mathbf{v}'(t)$,
(5) $[\mathbf{u}\times\mathbf{v}]'(t) = \mathbf{u}'(t)\times\mathbf{v}(t) + \mathbf{u}(t)\times\mathbf{v}'(t)$.

Example. If $|\mathbf{r}(t)| = c$ then $\mathbf{r}(t)\cdot\mathbf{r}'(t) = 0$ and if $\mathbf{u}(t)\cdot\mathbf{v}(t) = 0$, then $\mathbf{u}'(t)\cdot\mathbf{v}(t) = -\mathbf{u}(t)\cdot\mathbf{v}'(t)$.

Velocity vector and Acceleration vector.

(1) $\mathbf{v}(t) = \mathbf{r}'(t)$ is the velocity vector,
(2) $\mathbf{a}(t) = \mathbf{v}'(t)$ is the accelration vector,
(3) $|\mathbf{v}(t)|$ is the speed function and $|\mathbf{a}(t)|$ is the scalar acceleration function.

Example.

(1) $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j}$,
(2) $\mathbf{r}(t) = a\cos\omega t\mathbf{i} + a\sin\omega t + bt\mathbf{k}$ is the solution to $\mathbf{F} = m\mathbf{a}, \mathbf{F} = (q\mathbf{v})\times\mathbf{B}$, $\mathbf{B} = b\mathbf{k}$.
(3) $\mathbf{a}(t) = 2\mathbf{i} + 6t\mathbf{j}, \mathbf{r}(0) = 2\mathbf{i}, \mathbf{v}(0) = \mathbf{i} - \mathbf{j}$,
(4) $\mathbf{a}(t) = -32\mathbf{j}, \mathbf{r}(0) = 1600, \mathbf{v}(0) = 220, \mathbf{r}(?) = 0$,
(5) $\mathbf{a}(t) = 2\mathbf{i} - 32\mathbf{k}, \mathbf{r}(0) = (0,0,0), \mathbf{v}(0) = 80\mathbf{j} + 80\mathbf{k}, z(?) = 0$.

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11.6 Curvature and Acceleration

Arc Length. $s = \int_a^b |\mathbf{v}(t)| dt$.

Example. $\mathbf{r}(t) = (a\cos\omega t, a\sin\omega t, bt)$

Paramatrized by arclength.

Example. As above.

Curvature of Plane Curve. *Let* $\mathbf{T} = (\cos\phi, \sin\phi)$ *be the unit tangent vector. Then* $\frac{d\mathbf{T}}{ds} = (-\sin\phi\mathbf{i} + \cos\phi\mathbf{j})\frac{d\phi}{ds}$, *and let* $\kappa = |\frac{d\mathbf{T}}{ds}| = |\frac{d\phi}{ds}|$, *is the curvature of the curve, and it can be represented by*

$$\kappa = \frac{|y''x' - x''y'|}{(\sqrt{(x')^2 + (y')^2})^3}.$$

In the case of graph $(x, f(x))$,

$$\kappa = \frac{|y''|}{(1 + (y')^2)^{32}}.$$

Example. $(a\cos t, a\sin t), \kappa = \frac{1}{a}$ Write

$$\frac{d\mathbf{T}}{ds} = \kappa\mathbf{N}$$

$\mathbf{N}$ is called the principal normal. $\frac{d\mathbf{N}}{ds} = -\kappa\mathbf{T}$.

Theorem. *Suppose that $\kappa \equiv 0$, then $\mathbf{r}(s) = \mathbf{r}(0) + sT$.*

Proof. Since $\kappa \equiv 0$, $\mathbf{T}(s)$ is a constant vector $\mathbf{T}$. Hence $\mathbf{r}(s) = \mathbf{r}(0) + \int_0^s \mathbf{T}ds = \mathbf{r}(0) + s\mathbf{T}$.

Theorem. *Let $\mathbf{r}$ and $\mathbf{r}_1$be two curves paramatrized by arclength, suppose that $\kappa \equiv \kappa_1$ then $\mathbf{r}$ and $\mathbf{r}_1$ are congruent.*

Proof. Since $\frac{d}{ds}[|\mathbf{T} - \mathbf{T}_1|^2 + |\mathbf{N} - \mathbf{N}_1|^2] = 0$, $|\mathbf{T} - \mathbf{T}_1|^2 + |\mathbf{N} - \mathbf{N}_1|^2$ is a constant. After translation, rotation and reflection(if necessary), we may assume that $\mathbf{r}(0) = (0,0) = \mathbf{r}_1(0), \mathbf{T}(0) = \mathbf{i} = \mathbf{T}_1(0), \mathbf{N}(0) = \mathbf{j} = \mathbf{N}_1(0)$. Hence $|\mathbf{T} - \mathbf{T}_1| \equiv 0 \equiv |\mathbf{N} - \mathbf{N}_1|$. So $\mathbf{r}(s) = \int_0^s \mathbf{T}ds = \int_0^s \mathbf{T}_1 ds = \mathbf{r}_1(s)$.

Osculation circle (Circle of Curvature), Radius of Curvature, Center of Curvature. $\rho = \frac{1}{\kappa}$ is the radius of curvature and $\gamma = \mathbf{r} + \rho\mathbf{N}$ is the center of curvature.

Example. $y = x^2$

Curvature of Space Curves. *Let $\mathbf{T}$ be the unit tangent of a space curve , then $\frac{d\mathbf{T}}{ds} = \kappa\mathbf{N}$ define the curvature κ and the pricipal normal $\mathbf{N}$ and then $\mathbf{B} = \mathbf{T} \times \mathbf{N}$ is the binormal. And also $\frac{d}{ds}\mathbf{B} = -\tau\mathbf{N}$, τ is the torsion .*

Example. $(a\cos\omega t, a\sin\omega t, bt)$.

Theorem. *If $\tau \equiv 0$, then $\mathbf{r}(s)$ is a plane curve.*

Proof. Since $\tau \equiv 0$ implies that $\frac{d}{ds}\mathbf{B} \equiv 0$, that means $\mathbf{B}$ is a constant vector. After rotation we may assume $\mathbf{B} = \mathbf{k}$, then $\mathbf{T}(s) = \cos\theta(s)\mathbf{i} + \sin\theta(s)\mathbf{j}$. Then $\mathbf{r}(s) = \mathbf{r}(0) + \int_0^s \mathbf{T}(s)ds$ is a plane curve.

Theorem.

$$\frac{d\mathbf{N}}{ds} = -\kappa\mathbf{T} + \tau\mathbf{B}.$$

Theorem. *Let $\mathbf{r}(s)$ and $\mathbf{r}_1(s)$ be two curve paramatrized by arclength ,if $\kappa \equiv \kappa_1, \tau \equiv \tau_1$, then $\mathbf{r}(s)$ and $\mathbf{r}_1(s)$ are congruent.*

Tangent and Normal Component of Acceleration.

$$\mathbf{a} = \frac{d}{dt}(v\mathbf{T}) = \frac{dv}{dt}\mathbf{T} + \kappa v^2\mathbf{N} = a_T\mathbf{T} + a_N\mathbf{N}$$

where $a_T = \frac{dv}{dt}, a_N = \kappa v^2$ are the tangent and normal components respectively of the acceleration.

Example. $\mathbf{r}(t) = (\frac{3}{2}t^2, \frac{4}{3}t^3)$

Computation in terms of $\mathbf{v}, \mathbf{a}$.

(1) $\mathbf{v} \cdot \mathbf{a} = (v\mathbf{T}) \cdot (a_T\mathbf{T} + a_N\mathbf{N}) = a_T, a_T = \frac{\mathbf{v}\cdot\mathbf{a}}{v}$,
(2) $\mathbf{v} \times \mathbf{a} = va_N\mathbf{B} = \kappa v^2\mathbf{B}, \kappa = \frac{|\mathbf{v}\times\mathbf{a}|}{v^3}, a_N = \frac{|\mathbf{v}\times\mathbf{a}|}{v}$,
(3) $\mathbf{N} = \frac{\mathbf{a}-a_T\mathbf{T}}{a_N}$.

Example. $\mathbf{r}(t) = (t, \frac{1}{2}t^2, \frac{1}{3}t^3)$.

Kepler vs Newton.

Kepler's Law.

(1) *The orbit of each planlet is an ellpise with the sun at one focus.*
(2) *The radius vector from the sun to a planlet sweeps out area at constant rate.*
(3) *The square of the period of revolution of planlet about the sun is proportional to cube of ther major semiaxis of the elliptic orbit.*

In short

Kepler's Law.

(1) *The orbit can be represented by* $r = \frac{pe}{1+r\cos\theta}$.
(2) $r^2(t)\theta'(t)$ *is a constant* $h = r(0)v(0)$
(3) $\frac{T^2}{a^3}$ *is constant*

Definition. *A force field is called center force field if* $\mathbf{F} \parallel \mathbf{r}$.

Theorem. *The motion under a center force field is a planar motion.*

Proof. Sinc $\mathbf{F} \parallel \mathbf{r}$, then $\mathbf{a} \parallel \mathbf{r}$.

$$\frac{d(\mathbf{r}\times\mathbf{v})}{dt} = \mathbf{v}\times\mathbf{v} + \mathbf{r}\times\mathbf{a} = \mathbf{0},$$

so $\mathbf{r}\times\mathbf{v}$ is a constant vector, after a rotation we may assume that $\mathbf{r}\times\mathbf{v} = c\mathbf{k}$. So $\mathbf{r}(t) \perp \mathbf{k}$, hence we have $\mathbf{r}(t) = \phi(t)\mathbf{i} + \psi(t)\mathbf{j}$.

Definition. $\mathbf{u}_r = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}$ *is the radial unit vector and* $\mathbf{u}_\theta = -\sin\theta\mathbf{i} + \cos\theta\mathbf{j}$ *is the transverse unit vector. It is clear*

$$\frac{d\mathbf{u}_r}{dt} = \theta'\mathbf{u}_\theta, \frac{d\mathbf{u}_\theta}{dt} = -\theta'\mathbf{u}_r. \tag{0}$$

Now for planar motion

$$\mathbf{r} = r\mathbf{u}_r,$$

$$\mathbf{v} = r'\mathbf{u}_r + r\theta'\mathbf{u}_\theta, \tag{1}$$

$$\mathbf{a} = (r'' - r(\theta')^2)\mathbf{u}_r + (2r'\theta' + r\theta'')\mathbf{u}_\theta. \tag{2}$$

The force is center force if and only if

$$(2r'\theta' + r\theta'')=0. \tag{3}$$

But that means

$$\frac{1}{r}\frac{dr^2\theta'}{dt} = 0$$

and hence

$$r^2\theta' = h \tag{4}$$

is a constant, that is the Kepler's 2nd law,

Kepler implies Newton.

Proof. From Kepler's 2nd law $r^2\theta' = h$ is a constant, we get that

$$\mathbf{a} = (r'' - r(\theta')^2)\mathbf{u}_r. \tag{4}$$

And from Kepler's 1st law

$$r = \frac{pe}{1 + e\cos\theta}. \tag{6}$$

Then

$$r' = \frac{pee\sin\theta\theta'}{(1 + e\cos\theta)^2}, \tag{7}$$

$$r'' = \frac{pe2(e\sin\theta\theta')^3}{(1 + e\cos\theta)^3} + \frac{pee\cos\theta(\theta')^2}{(1 + e\cos\theta)^2} + \frac{pee\sin\theta\theta''}{(1 + e\cos\theta)^2}. \tag{8}$$

By (3) the first and third terms on the right hand side of (8) cancel out each other, that gives

$$r'' = \frac{pee\cos\theta(\theta')^2}{(1 + e\cos\theta)^2}. \tag{9}$$

By (3),(4),(5),(6),and (9) we get

$$\mathbf{a} = -\frac{pe(\theta')^2}{(1 + e\cos\theta)^2}\mathbf{u}_r = -\frac{h^2/(pe)}{r^2}\mathbf{u}_r$$

with $\mu = \frac{h^2}{pe}$ we complete the proof of Newton's gravitation law

$$\mathbf{a} = -\frac{\mu}{r^2}\mathbf{u}_r. \tag{10}$$

Newton implies Kepler.

Proof. From Newton's law (10) we have from (2),(3) and (4) $r^2\theta' = h$ and

$$\frac{d\mathbf{v}}{dt} = \mathbf{a} = -\frac{\mu}{r^2}\mathbf{u}_r = \frac{\mu}{r^2\theta'}\frac{d\mathbf{u}_\theta}{dt} = \frac{\mu}{h}\frac{d\mathbf{u}_\theta}{dt}.$$

Hence

$$\mathbf{v} = \frac{\mu}{h}\mathbf{u}_\theta + \mathbf{C}.$$

Now assume that $r(0)$ is the minimum of $r(t)$,hence $r'(0) = 0$ and $\mathbf{r}(0) = r(0)\mathbf{i}$ and then by (1) $\mathbf{v}(0) = v(0)\mathbf{j} = r(0)\theta'(0)\mathbf{j}$, hence $h = (r(0))^2\theta(0) = r(0)v(0)$. Let $t = 0$ we have

$$\mathbf{C} = (v(0) - \frac{\mu}{h})\mathbf{j}.$$

So

$$\mathbf{v} = \frac{\mu}{h}\mathbf{u}_\theta + (v(0) - \frac{\mu}{h})\mathbf{j}.$$

Take the dot product with $\mathbf{u}_\theta$ we get

$$r\theta' = \frac{\mu}{h} + (v(0) - \frac{\mu}{h})\cos\theta,$$

hence

$$\frac{h}{r} = \frac{\mu}{h} + (v(0) - \frac{\mu}{h})\cos\theta.$$

That gives

$$r = \frac{h^2/\mu}{1 + (v(0)h/\mu - 1)\cos\theta} = \frac{pe}{1 + e\cos\theta}$$

with $pe = \frac{h^2}{\mu}$ and $e = \frac{v(0)h}{\mu} - 1 = \frac{r(0)(v(0))^2}{\mu} - 1$.

To prove (3) of Kepler, since $Th = \pi ab$ and $a = \frac{pe}{1-e^2}, b = \frac{pe}{\sqrt{1-e^2}}$.

$$T^2h^2 = \pi^2\frac{(pe)^4}{(1-e^2)^3} = \pi^2a^3pe$$

so $\frac{T^2}{a^3} = \pi^2\frac{pe}{h^2}$.

11.7 Cylinders and Quadratic Surfaces

Cylinder.

Example.

(1) $x^2 + y^2 = a^2$,
(2) $x = \sin t, y = \sin 2t$,
(3) $4y^2 + 9z^2 = 36$,
(4) $z = 4 - x^2$.

Surface of revolution.

Example. $4y^2 + z^2 = 4$, revolution

(1) about y-axis is $4x^2 + 4y^2 + z^2 = 4$,
(2) about z-axis is $x^2 + 4y^2 + z^2 = 4$.

For a curve given by $f(x, y) = 0$ the surface of revolution about x-axis is $f(\sqrt{x^2 + z^2}, y) = 0$.

Quadratic Surface.

Example.

(1) $z^2 = x^2 + y^2$ is a quadratic cone.
(2) $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ is an ellpisoid.
(3) $\frac{x^2}{a^2} + \frac{x^2}{a^2} = \frac{z}{c}$ is elliptic parabaloid.
(4) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$ is an ellptic cone.
(5) $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ is a hyperboloid of one sheet.
(6) $\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$is a hyperboloid of two sheets.
(7) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{z}{c}$ is a hyperbolic paraboloid.

11.8 Cylindrical and Spherical Coordinates

Cylindrical Coordinates.

$$x = r\cos\theta, y = r\sin\theta, z = z$$

$$r^2 = x^2 + y^2, \tan\theta = \frac{y}{x}, z = z$$

Example.

(1) $(4, \frac{5\pi}{3}, 7)$,
(2) $(-2, 2, 5)$,
(3) $x^2 + y^2 + z^2 = a^2$,
(4) $z^2 = x^2 + y^2$,
(5) $z = x^2 + y^2$,
(6) $(\frac{x}{3})^2 + (\frac{y}{3})^2 + (\frac{z}{2})^2 = 1$.

Spherical Coordinates.

$$x = \rho\sin\phi\cos\theta, y = \rho\sin\phi\sin\theta, z = \rho\cos\phi$$

Example.

(1) $(8, \frac{5\pi}{6}, \frac{\pi}{3})$,
(2) $(-3, -4, -12)$,
(3) $z = x^2 + y^2$,
(4) $\rho = 2\cos\phi$,
(5) $\rho = \sin\phi\sin\theta$.