CHAPTER 10, INFINITE SERIES

10.2 Infinite Sequence

Definition.

$$a_1, a_2, a_3, \cdots, \{a_n\}_{n=1}^{\infty}, \{a_n\}_1^{\infty}, \{a_n\}.$$

$a_n = f(n).$

Example.

(1) $\{\frac{1}{n}\}_1^{\infty}, \{10^n\}_1^{\infty}, \{\sqrt{3n-7}\}_3^{\infty}, \{\sin\frac{n\pi}{2}\}, \{3+(-1)^n\}.$
(2) $\{2, 3, 5, 7, 11, 13, 17, 23, \cdots\},$
(3) $\{3, 1, 4, 1, 5, 9, 2, 6, 5, 3, 5, \cdots\},$
(4) Fibonacii sequemce $F_1 = 1, F_2 = 1, F_{n+1} = F_n + F_{n-1}, n > 2$(recurrently definded) ,
(5) $A_0 = 100, A_{n+1} = A_n * (1.1)$(recurrently definded).

Limit of Sequence.

Definition. *A sequence $\{a_n\}$ converges to a limit L if for any $\epsilon > 0$ there is $N(\epsilon)$ such that $n \geq N(\epsilon)$ implies that $|a_n - L| < \epsilon$, in this case we write $\lim_{n\to\infty} a_n = L$. A sequence diverges if it does not converge.*

Example.($\lim_{n\to\infty} \frac{1}{n} = 0$. $\{(-1)^n\}$ diverges.

Limit Laws.

Theorem. *Suppose that $\lim_{n\to\infty} a_n = A, \lim_{n\to\infty} b_n = B$, then*

(1) $\lim_{n\to\infty} ca_n = cA$
(2) $\lim_{n\to\infty}(a_n + b_n) = A + B$
(3) $\lim_{n\to\infty} a_n b_n = AB$
(4) $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{A}{B}$ *if* $B \neq 0$.

Theorem. *(Substitution law) If* $\lim_{n\to\infty} a_n = a$ *and* $\lim_{n\to\infty} f(a_n) = L$ *, then* $\lim_{n\to\infty} f(a_n) = L$.

Theorem. *(Squeeze law) If $a_n \leq b_n \leq c_n$ and $\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} c_n$, then $\lim_{n\to\infty} b_n = L$.*

Example.

(1) $\lim_{n\to\infty} \frac{7n^2}{5n^2-3} = \frac{7}{5},$
(2) $\lim_{n\to\infty} \frac{\cos n}{n} = 0,$
(3) If $a > 0$, $\lim_{n\to\infty} a^{\frac{1}{n}} = 1,$

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(4) $\lim_{n\to\infty} \sqrt{\frac{4n-1}{n+1}} = 2$,
(5) If $|r| < 1$ then $\lim_{n\to\infty} r^n = 0$.
(6) $\lim_{n\to\infty} \frac{\ln n}{n} = 0$,
(7) $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$,
(8) $\lim_{n\to\infty} \frac{3n^3}{e^{2n}} = 0$.

Definition. *A sequence $\{a_n\}$ is bounded sequence if there exists M such that $|a_n| \leq M$ for all n.*

Theorem. *Convergent sequences are bounded sequence.*

Definition. *A sequence $\{a_n\}$ is monotonic sequence if $a_n \geq (\leq) a_{n+1}$ for all n.*

Bounded monotonic sequences.

Theorem. *Bounded monotonic sequences are convergent sequences.*

Example. $a_1 = \sqrt{6}, a_{n+1} = \sqrt{6 + a_n}, n \geq 1$ In general

Example.(Exploration 1 page 732) $a_1 = \sqrt{q}, a_{n+1} = \sqrt{q + pa_n}, n \geq 1$

Example. Exer. 56 page 731.

Example.(Exploration 2 page 732) $a_1 = p, a_{n+1} = p + \frac{q}{a_n}$.

Example.$a_1 = a \geq 0, a_{n+1} = \frac{1+a_n^2}{2}$, converges if and only if $0 \leq a \leq 1$.

Definition. *A sequence $\{b_n\}$ is a subsequence of $\{a_n\}$ if $b_n = a_{\lambda(n)}$ for $n = 1, 2, 3 \cdots$ and λ is an increasing function from $\mathbb{N}$ to $\mathbb{N}$.*

Example. $a_n = \frac{1}{n}, b_n = \frac{1}{2n}, c_n = \frac{1}{2^n}$.

Theorem. *If $\{a_n\}$ converges to L, then every subsequence also converges to L*

Definition. *A sequence $\{a_n\}$ is a Cauchy sequence if for any $\epsilon > 0$ there is $N(\epsilon)$ such that $m \geq n \geq N(\epsilon)$ implies that $|a_n - a_m| < \epsilon$.*

Theorem. *A sequence is a convergent sequence iff it is a Cauchy sequence.*

Proof.

(1) Choose c, d such that $c \leq a_n \leq d$ for all n, and let $L = d - c$.
(2) Let $c_1 = c, d_1 = d$ and choose N_1 such that for all $m > n \geq N_1$ we have $|a_m - a_n| < \frac{L}{4}$.
(3) Let $c_2 = \max(c, a_{N_1} - \frac{L}{4}), d_2 = \min(d, a_{N_1} + \frac{L}{4})$, then $c_2 \leq a_n \leq d_2$ for all $n \geq N_1$ and $d_2 - c_2 \leq \frac{L}{2}$. Choose N_2 such that for all $m > n \geq N_2$ we have $|a_m - a_n| < \frac{L}{8}$.
(4) Assume we have done k step. Let $c_{k+1} = \max(c, a_{N_k}) - \frac{L}{2^k}, d_2 = \min(d, a_{N_k}) + \frac{L}{2^k}$, then $c_{k+1} lea_n \leq d_{k+1}$ for all $n \geq N_{k+1}$ and $d_{k+1} - c_{k+1} \leq \frac{L}{2^{k+1}}$. Choose N_{k+2} such that for all $m > n \geq N_{k+2}$ we have $|a_m - a_n| < \frac{L}{2^{k+2}}$.

(5) Now we have $c_1 \leq c_2 \leq cdotsc_n \leq \cdots \leq d_n \leq \cdots \leq d_2 \leq d_1$. So $\{c_n\}$ converges to the *l.u.b.* C and $\{d_n\}$ converges to the *g.l.b.*D and $D \geq C$. But since $d_n \geq D \geq C \geq c_n$ for all n, so $D = C$.
(6) Let $\mathcal{L} = D = C$, then for $n \geq N_k$ we have $|a_n - \mathcal{L}| \leq \frac{L}{2^k}$.

□

Remark. *In case of a recursiveely definded sequence* $a_{n+1} = f(a_n)$*, you can use the solution* $x = f(x)$ *to determine the limit only after you show that the sequence converges.*

10.3 Infinite series and convergence

Infinite series and their partial sum.

$$\text{Infinite series} \quad \sum_1^\infty a_n$$

$$\text{partial sum up to m-th terms} \quad S_m = \sum_1^m a_n$$

Remark. *Every sequence can be written as the squence of partial sums of some series.*

Definition. *A series* $\sum_1^\infty a_n$ *converges (or is convergent) to a sum* S*, provided that the limit* $S = \lim_{m\to\infty} S_m$ *exists. Otherwise the series* $\sum_1^\infty a_n$ *diverges.*

Example.
(1) $\sum_1^\infty (\frac{1}{2})^n = 1$,
(2) $\sum_1^\infty (-1)^n$ diverges,
(3) $\sum_1^\infty \frac{1}{n(n+1)} = 1$.

Definition. *Geometry series* $a_0 = a, a_{n+1} = ra_n$.

Example. $\sum_1^\infty \frac{2}{3^n}$

Theorem. *A geometric series converges to* $\frac{a}{1-r}$ *if* $|r| < 1$*. Diverges if* $|r| \geq 1$.

Example.
(1) $a = 1, r = -\frac{2}{3}$,
(2) $\sum_1^\infty \frac{2^{2n-1}}{3^n}$.

Theorem. *If* $\sum_1^\infty a_n = A, \sum_1^\infty b_n = B$*, then*
(1) $\sum_1^\infty (a_n + b_n) = A + B$,
(2) $\sum_1^\infty ca_n = cA$.

Example.
(1) $0.555\cdots$,
(2) $0.72828\cdots$,
(3) Paul and Mary, $r = (\frac{5}{6})^2, a_1 = \frac{1}{6}, b_1 = \frac{1}{6}\frac{5}{6}$.

Remark. *A real number is a rational number if and only if its decimal form is periodic.*

Theorem. *(n-th term test) If* $\sum_1^\infty a_n$ *converges then* $\lim_{n\to\infty} a_n = 0$.

Example. $\sum_1^\infty (-1)^{n-1}n^2$, $\sum_1^\infty \frac{n}{3n+1}$ both diverge

Theorem. $\sum_1^\infty \frac{1}{n}$ *diverges*

Theorem. *If* $a_n = b_n$ *for all* $n \geq N$, *then either both* $\sum_1^\infty a_n$, $\sum_1^\infty b_n$ *converge or both diverge.*

Theorem. $\sum_1^\infty a_n$ *converges if and only if for any* $\epsilon > 0$ *there is* $N(\epsilon)$ *such that for all* $N(\epsilon) \leq n < m$, *implies* $|\sum_n^m a_i| < \epsilon$.

10.4 Taylor polynomials and Taylor series

Taylor polynomials.

$$\sum_0^m \frac{f^{(k)}(a)}{k!}(x-a)^k$$

is the Taylor polynomial of f *center at* a *up to* m*-th terms.*

Example.

(1) $\frac{1}{1-x}$, $a = 0$, $\sum_0^m x^k$,
(2) $\ln x$, $a = 1$, $\sum_1^m \frac{(-1)^{k-1}(x-1)^k}{k}$,
(3) e^x, $a = 0$, $\sum_0^m \frac{x^k}{k!}$.

Theorem. *(Taylor's formulae) Suppose that* f *is* $(n+1)$*-times differentiable on an open interval* I *contains* a, *then for each* $x \in I$

$$f(x) = \sum_0^n \frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{f^{(n+1)}(z)}{(n+1)!}(x-a)^{(n+1)}$$

where z *is between* a *and* x.

Proof. Let $F(t) = \frac{(x-t)^{n+1}}{(n+1)!}$, then $F(x) - F(a) = -\frac{(x-a)^{n+1}}{(n+1)!}$ and $F'(z) = -\frac{(x-z)^n}{n!}$.

Let $G(t) = f(x) - \sum_0^n \frac{f^{(k)}(t)}{k!}(x-t)^k$, then $G(x) - G(a) = f(x) - \sum_0^n \frac{f^{(k)}(a)}{k!}(x-a)^k$, and $G'(z) = -\frac{f^{n+1}(z)}{n!}(x-z)^n$.

Apply the generalized M.V.T. to F, G we get $[F(x) - F(a)]G'(z) = [G(x) - G(a)]F'(z)$ to complete the proof.

Example. $\ln 1.1 = (0.1) - \frac{(0.1)^2}{2} + \frac{(0.1)^3}{3} - \frac{(0.1)^4}{4z}$. $\ln 1.1 \approx 0.095333$.

Definition. *Taylor series of f center at a is*

$$\sum_0^\infty \frac{f^{(k)}(a)}{k!}(x-a)^k$$

.

Example.

(1) $e^x = \sum_0^\infty \frac{x^k}{k!}$,
(2) $\cos x = \sum_0^\infty \frac{(-1)^n x^{2n}}{2n!}$, $\sin x = \sum_0^\infty \frac{(-1)^n x^{2n+1}}{(2n+1)!}$,
(3) $e^{-t^2} = \sum_0^\infty \frac{(-1)^n t^{2n}}{n!}$,
(4) $\sin 2t = \sum_0^\infty \frac{(-1)^n 2^{2n+1} t^{2n+1}}{(2n+1)!}$.

From (1),(2) we have the Euler formulae $e^{i\theta} = \cos\theta + i\sin\theta$. The following example is to find an approximation π.

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots + (-1)^n x^n + \frac{(-x)^{n+1}}{1+x}$$

$$\frac{1}{1+t^2} = 1 - t^2 + t^4 - t^6 + \cdots + (-1)^n t^{2n} + \frac{(-t^2)^{n+1}}{1+t^2}$$

$$\tan^{-1} t = 1 - \frac{t^3}{3} + \frac{t^5}{5} \cdots \frac{(-1)^n t^{2n+1}}{2n+1} + \int_0^t \frac{(-t^2)^{n+1}}{1+t^2} dt$$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} \cdots + \frac{(-1)^n}{2n+1} + R_n$$

with $|R_n| \leq \frac{1}{2n+3}$.

10.5 Integral Test

Theorem. *Let $\sum_1^\infty a_n$ be a series with $a_n \geq 0$ for all n. Then $\{S_m\}$ is an increasing sequence, so it converges if and only if it is bounded.*

Theorem. *(Integral test) Suppose that f is a positive decreasing continuous function on $[1,\infty)$ and $a_n = f(n)$ for all n. Then the series $\sum_1^\infty a_n$ and the improper integral $\int_1^\infty f(x)dx$ are either both convergent or both divergent.*

Proof. Since

$$a_1 + \int_1^m f(x)dx \geq \sum_1^m a_n \geq \int_1^{m+1} f(x)dx,$$

so they are either both bounded or both unbounded.

Example.

(1) $\sum_1^\infty \frac{1}{n}$,
(2) $\sum_1^\infty \frac{1}{n^p}$.

Theorem. *(Remainder estimate)*

$$\int_{n+1}^{\infty} f(x)dx \leq R_n \leq \int_{n}^{\infty} f(x)dx.$$

Example. $\frac{\pi}{6} \approx 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots \frac{1}{n^2}$ with $\frac{1}{n+1} \leq Rn \leq \frac{1}{n}$.

10.6 Comparison Test

Theorem. *If $\sum_1^\infty a_n$ and $\sum_1^\infty b_n$ are two series with positive terms such that $a_n \leq b_n$ for all n, then*

(1) *If $\sum_1^\infty b_n$ convegres then $\sum_1^\infty a_n$ also converges.*
(2) *If $\sum_1^\infty a_n$ divegres then $\sum_1^\infty b_n$ also diverges.*

Theorem. *If $\sum_1^\infty a_n$ and $\sum_1^\infty b_n$ are two series with positive terms such that $a_n \leq b_n$ for all $n \geq N$ for some N and $\sum_1^\infty b_n$ convegres, then $\sum_1^\infty a_n$ also converges.*

Example.

(1) $\sum_1^\infty \frac{1}{n(n+1(n+2)}$,
(2) $\sum_1^\infty \frac{1}{\sqrt{2n-1}}$,
(3) $\sum_1^\infty \frac{1}{n!}$.

Limit comparison test.

Theorem. *If $\sum_1^\infty a_n$ and $\sum_1^\infty b_n$ are two series with positive terms such that $\lim_{n\to\infty} \frac{a_n}{b_n} = L$ for some $0 < L < \infty$. Then either both $\sum_1^\infty a_n$ and $\sum_1^\infty b_n$ are both convergent or $\sum_1^\infty a_n$ and $\sum_1^\infty b_n$ are both divergent.*

Proof. There exists N such that ,for all $n \geq N$, $\frac{L}{2} \leq \frac{a_n}{b_n} \leq \frac{3L}{2}$. That means $a_n \leq \frac{3L}{2} b_n$ and $b_n \leq \frac{2}{L} a_n$ for $n \geq N$.

Remark. *Suppose that $\sum_1^\infty a_n$ converges and $b_n \leq C a_n$ for all $n \geq N$, then $\sum_1^\infty b_n$ converges.*

Example.

(1) $\sum_1^\infty \frac{3n^2+n}{n^4+\sqrt{n}}$,
(2) $\sum_1^\infty \frac{1}{2n+\ln n}$,
(3) $\sum_1^\infty \frac{1}{n^2+\sqrt{n}}$ $|R_n| \leq \frac{1}{n}$.

Rearrangement and Grouping.

Definition. *Aseries $\sum_1^\infty b_n$ is a rearrangement of a series $\sum_1^\infty a_n$, if there is a one to one onto function $\sigma : \mathbb{N} \to \mathbb{N}$ such that $b_n = a_{\sigma(n)}$ for all n.*

Theorem. *If $\sum_1^\infty a_n$ is a convergent series with positive terms, then any rearrangement of it will converge to the same limit.*

Proof. Suppose that $b_n = a_{\sigma(n)}$ be a rearrangement of $\sum_1^\infty a_n$. Let $\lambda(n) = \max_1^n\{\sigma(j)$, it is clear that $B_n \leq A_{\lambda(n)}$ for all n. Hence $B_n \leq \sum_1^\infty a_n$ for all n, which implies

$$\sum_1^\infty b_n \leq \sum_1^\infty a_n.$$

Conversely $\sum_1^\infty a_n$ is also a rearrangement of $\sum_1^\infty b_n$. We also have

$$\sum_1^\infty a_n \leq \sum_1^\infty b_n.$$

Theorem. *If $\sum_1^\infty a_n$ os a convergent series, then any grouping of terms will also converge to the same limit.*

Example. $\sum_0^\infty (-1)^n$.

10.7 Alternting series and Absolute convergence

Definition. *Suppose that $a_n \geq 0$,then $\sum_1^\infty (-1)^{n-1} a_n$ is an alternting series .*

Theorem. *(Alternating series theorem) Suppose that (1) $a_n \geq a_{n+1}$ for all n and (2) $\lim_{n\to\infty} a_n = 0$, then $\sum_1^\infty (-1)^{n-1} a_n$ converges.*

Proof.

(1) Since $S_{2k+1} = S_{2k-1} - (a_{2k} - a_{2k+1})$,then $\{S_{2k+1}\}$ is decreasing.
(2) Since $S_{2k+2} = S_{2k} + (a_{2k+1} - a_{2k+2})$, then $\{S_{2k}\}$ is increasing.
(3) Since $S_{2k+1} = S_{2k} + a_{2k+1}$, then $S_{2k+2} \geq S_{2k}$.
(4) For $l < k$, $S_{2l+1} \geq S_{2k+1} \geq S_{2k}$ and for $l > k$, $S_{2l+1} \geq S_{2l} \geq S_{2k}$.
(5) Every S_{2k} is a lower bound of $\{S_{2n+12}\}$, so$\{S_{2n+1}\}$ converges to S_o which is the g.l.b. of $\{S_{2n+1}\}$.
(6) Every S_{2k+1} is an upper bound of $\{S_{2n}\}$, so $\{S_{2n}\}$ converges to S_e, mwhich is the l.u.b of $\{S_{2n}\}$.
(7) $S_o \geq S_{2n}$ for all n and $S_e \leq S_{2n+1}$ for all n and $S_o \geq S_e$.
(8) From $0 \leq S_o - S_e \leq S_{2k+2} - S_{2k} = a_{2k+1}$, let $k \to \infty$ we get $S_o = S_e$.

□

Example.

(1) $\sum_1^\infty \frac{(-1)^{n+1}}{2n-1}$,
(2) $\sum_1^\infty \frac{(-1)^{n+1} n}{2n-1}$.

Theorem. *(Remainder estimate)*

$$|R_n| \leq a_{n+1}.$$

Example. $e^{-1} \approx 1 - \frac{1}{2!} + \frac{1}{3!} \cdots \frac{(-1)^{n-1}}{n!}$ $|R_n| < \frac{1}{(n+1)!}$.

Definition. *If $\sum_1^\infty |a_n|$ converges, we say $\sum_1^\infty a_n$ converges absolutely. If $\sum_1^\infty a_n$ converges but $\sum_1^\infty |a_n|$ diverges, we say $\sum_1^\infty a_n$ converges conditionally.*

Theorem. *If $\sum_1^\infty |a_n|$ converges, then $\sum_1^\infty a_n$ converges .*

Proof. Let $a_n^+ = \frac{|a_n|+a_n}{2}, a_n^- = \frac{|a_n|-a_n}{2}$, $(a_n^+ = \max(a_n, 0), (a_n^- = \min(-a_n, 0))$.

Then $|a_n| = a_n^+ + a_n^-, a_n = a_n^+ - a_n^-$ and $0 \leq a_n^+, a_n^- \leq |a_n|$.

If $\sum_1^\infty |a_n|$ converges, then both $\sum_1^\infty a_n^+$ and $\sum_1^\infty a_n^-$ converge. Which implies that $\sum_1^\infty a_n$ converges.

Remark. *If $\sum_1^\infty a_n$ converges conditionally, then both $\sum_1^\infty a_n^+$ and $\sum_1^\infty a_n^-$ diverge.*

Example.

(1) $\sum_0^\infty (-\frac{1}{3})^n = \frac{3}{4}$,
(2) $\sum_0^\infty (\frac{1}{3})^n = \frac{3}{2}$,
(3) $\sum_1^\infty \frac{\cos n}{n^2}$.

Theorem. *If $\sum_1^\infty a_n$ converges conditionally, then for any given real number L, there is a rearrangement $\sum_1^\infty b_n$ of $\sum_1^\infty a_n$, such that $\sum_1^\infty b_n = L$.*

Ratio Test.

Theorem. *If $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \rho$, then*

(1) *$\rho < 1$ implies $\sum_1^\infty |a_n|$ converges,*
(2) *$\rho > 1$ implies $\sum_1^\infty |a_n|$ diverges,*
(3) *$\rho = 1$ no conclusion.*

Theorem. *If $|\frac{a_{n+1}}{a_n}| \leq \rho < 1$,for all $n \geq N$ for some N, then $\sum_1^\infty |a_n|$ converges.*

Example.

(1) $\sum_1^\infty \frac{(-2)^n}{n!}$,
(2) $\sum_1^\infty \frac{n}{2^n}$,
(3) $\sum_1^\infty \frac{3^n}{n^2}$.

Root Test.

Theorem. *If $\lim_{n\to\infty} |a_n|^{\frac{1}{n}} = \rho$, then*

(1) *$\rho < 1$ implies $\sum_1^\infty |a_n|$ converges,*
(2) *$\rho > 1$ implies $\sum_1^\infty |a_n|$ diverges,*
(3) *$\rho = 1$ no conclusion.*

Theorem. *If $|a_n|^{\frac{1}{n}} \leq \rho < 1$,for all $n \geq N$ for some N, then $\sum_1^\infty |a_n|$ converges.*

Example. $\sum_1^\infty \frac{1}{2^{n+(-1)^n}}$.

10.8 Power series

$$\sum_{0}^{\infty} a_n x^n.$$

Domain of convergence. *The set* $\{x : \sum_0^\infty a_n x^n$ *converges*$\}$.

Theorem. *Suppose that* $\sum_0^\infty a_n x_0^n$ *converges for some* $x_0 \neq 0$, *then* $\sum_0^\infty a_n x^n$ *converges absolutely for all* $\{x : |x| < |x_0|\}$.

Proof. Since $\sum_0^\infty a_n x_0^n$ converges, $a_n x_0^n \to 0$ as $n \to \infty$. In particalar there exist N such that for $n \geq N$, $|a_n x_0^n| < 1$. Now for fix x with $|x| < |x_0|$, $|a_n x^n| = |a_n x_0^n| (\frac{|x|}{|x_0|})^n$. So $\sum_N^\infty |a_n x^n| \leq \sum_N^\infty (\frac{|x|}{|x_0|})^n$, which converges due to $|x| < |x_0|$. The proof is complete.

Remark. The domain of convergence must be one of the following

(1) $\{0\}, \sum_0^\infty n! x^n$,
(2) $\mathbb{R}, \sum_0^\infty \frac{x^n}{n!}$,
(3) $[-R, R], \sum_0^\infty \frac{x^n}{R^n n^2}$,
(4) $[-R, R), \sum_0^\infty \frac{x^n}{R^n n}$
(5) $(-R, R], \sum_0^\infty \frac{(-x)^n}{R^n n}$
(6) $(-R, R), \sum_0^\infty x^n$,

R is called the radius of convergence.

Theorem. *Either* $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \rho$ *or* $\lim_{n\to\infty} |a_n|^{\frac{1}{n}} = \rho$, *then* $R = \frac{1}{\rho}$.

Example.

(1) $\sum_1^\infty \frac{x^n}{n 3^n}$,
(2) $\sum_1^\infty \frac{(-2x)^n}{(2n)!}$,
(3) $\sum_1^\infty n^n x^n$,
(4) $\sum_0^\infty \frac{(-1)^n x^{2n}}{2n!}$.

Power series center at c.

$$\sum_{0}^{\infty} a_n (x-c)^n.$$

Example. $\sum_0^\infty \frac{(-1)^n (x-3)^n}{n 4^n}$.

Power series representation of function center at a.

$$f(x) = \sum_{0}^{\infty} \frac{f^{(k)}(a)}{k!} (x-a)^k.$$

Taylor series if $a = 0$ is called Maclaurin series.

Example.

(1) $e^{-x}, \cosh x, \sinh x, e^{-x^2}$,
(2) Bessel function $J_0(x) = \sum_0^\infty \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$,
(3) Binomial series $(1+x)^\alpha = 1 + \sum_0^\infty \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)}{n!} x^n$.

Continuity, integration and differentiation of power series.

Theorem. *Power series define a continuous function on the interior of the interval of conveggence.*

Since the partial sums of a power series are polynomials, hence are continuous functions. We will use the following theorem to get the continuity of the power series from its partial sums. We define the distance between two functions f, g by

$$|f - g| = l.u.b._{x\in(-R,R)}|f(x) - g(x)|.$$

Definition. *A sequence of functions $\{f_k\}$ converges uniformly to a limit function f if for any given $\epsilon > 0$ there is a $N(\epsilon)$ such that $n \geq N(\epsilon)$ implies that $|f_k - f| < \epsilon$. In this case we will write*

$$\lim_{k\to\infty} f_k = f.$$

Theorem. *Suppose that sequence of continuous functions $\{f_k\}$ on $(-R, R)$ converges uniformly to a limit function f. Then f is a continuous function on $(-R, R)$.*

Proof. For any $a \in (-R, R)$, since

$$|f(x) - f(a)| \leq |f(x) - f_k(x)| + |f_k(x) - f_k(a)| + |f_k(a) - f(a)|.$$

For given $\epsilon > 0$, we can fix some large k such that $|f_k - f| < \frac{\epsilon}{3}$. And since f_k is continuous function , there is a $\delta > 0$ such that , $|x - a| < \delta$ implies $|f_k(x) - f_k(a)| < \frac{\epsilon}{3}$. Put them together we have

$$|f(x) - f(a)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

Proof of the continuity of the power series. For any $x \in (-R, R)$, fix $r < R$ such that $x \in [-r, r]$. Then

$$|\sum_0^\infty a_n x^n - \sum_0^m a_n x^n| \leq |\sum_{m+1}^\infty |a_n x^n| \leq \sum_{m+1}^\infty |a_n||r^n|.$$

As we know from $r < R$,$\sum_0^\infty |a_n||r^n|$ converges, so the $\sum_{m+1}^\infty |a_n||r^n|$ goes to zero as m goes to ∞. So $\sum_0^\infty a_n x^n$ is continuous on $[-r, r]$ and this is true for all $0 < r, R$. So $\sum_0^\infty a_n x^n$ is continuous on $(-R, R)$.

Theorem. *The power series* $\sum_0^\infty a_n x^n$*, its term by trem differentiation power series* $\sum_1^\infty n a_n x^{n-1}$*, and its term by term integration power series* $\sum_0^\infty \frac{a_n}{n+1} x^{n+1}$ *have same radius of convergence.*

Proof. Letr their radius convergence be R, R_d, R_i respectively. Since the series $\sum_0^\infty a_n x^n$ and $\sum_1^\infty a_n x^n$ have same radius R. The series $\sum_0^\infty \frac{a_n}{n+1} x^{n+1}$, $\sum_1^\infty \frac{a_n}{n+1} x^{n+1}$, and $x \sum_1^\infty \frac{a_n}{n+1} x^n$ have the same radius of convergence R_i. Hence $R_i \geq R$ and also $R \geq R_d$ since $\sum_0^\infty a_n x^n$ is the term by term integration series of $\sum_1^\infty n a_n x^{n-1}$. In order to show that $R = R_d$, we will show that for any $x \in (-R, R)$ $\sum_1^\infty n a_n x^{n-1}$ converges absolutely. Fix some r such that $|x| < r < R$ and apply the limit comarison theorem sine

$$\frac{n|a_n x^{n-1}|}{|a_n r^n} = \frac{n}{r} (\frac{|x|}{r})^{n-1} \to 0 \quad \text{as} \quad n \to \infty,$$

which prove that $R = R_d$ and similarly $R = R_i$.

Next we will show that integration of power series can be done by term by term integration.

Theorem. $\int_0^x \sum_0^\infty a_n t^n dt = \sum_0^\infty \frac{a_n}{n+1} x^{n+1}$.

Proof. For given $x \in (-R, R)$,

$$|\int_0^x \sum_0^\infty a_n t^n dt - \sum_0^m \frac{a_n}{n+1} x^{n+1}| = |\int_0^x \sum_{m+1}^\infty a_n t^n dt| \leq \int_0^x \sum_{m+1}^\infty |a_n||x|^n dt,$$

and last integral is $|x| \sum_{m+1}^\infty |a_n||x|^n$ which will goes to zero as $m \to \infty$ due to the absolute convergence of $\sum_0^\infty a_n x^n$

In order to show that differentiation of power series can be done by term by term differentiation, we need the following theorem (we will skip the proof).

Theorem. *Suppose that a sequence of differentiable function* $\{f_k\}$ *converges uniformly to a limit function* f*, and* $\{f_k'\}$ *also converges to a limit function* g*. Then* f *is differentiable and* $f' = g$*.*

Theorem. $\frac{d}{dx}(\sum_0^\infty a_n x^n) = \sum_1^\infty n a_n x^{n-1}$.

Proof. Inorder to apply the theorem above , we have to make sure that the derivetives of the partial sums converges uniformly on $[-r, r]$ for all $r < R$. But since $R - d_R$, this is true, so the proof is complete.

Remark. *A function defined by a power series is call a real analytic function. Suppose that* $f(x) = \sum_0^\infty a_n x^n$*, then* $f^{(k)}(0) = k! a_k$*. So real analytic function can only be deinned by unique powerseries.*

Example.

(1) $\frac{1}{(1-x)^2} = (\frac{1}{(1-x)})' = \sum_1^\infty n x^{n-1}$

(2) $\ln(1+x) = \int_0^x \frac{dt}{1+t} = \sum_1^\infty (-1)^{n-1} \frac{x^n}{n}$,

(3) $\tan^{-1} x = \int_0^x \frac{dt}{1+t^2} = \sum_0^\infty (-1)^n \frac{x^{2n+1}}{2n+1}$,

(4) $\sin^{-1} x = \int_0^x \frac{dt}{\sqrt{1-t^2}}$.

10.9 Power series computation

Example.

(1) $\sqrt{105} = 10\sqrt{1.05}$,
(2) $A = \int_{-\pi}^{\pi} \frac{\sin x}{x} dx$.

Algebra of power series.

Theorem. *Suppose that* $\sum_0^\infty a_n x^n, \sum_0^\infty b_n x^n$ *have radius of convergence* $R > 0$, *then*

(1) $\sum_0^\infty (a_n x^n + b_n x^n) = (\sum_0^\infty a_n x^n) + (\sum_0^\infty b_n x^n)$,
(2) $\sum_0^\infty c_n x^n = (\sum_0^\infty a_n x^n)(\sum_0^\infty b_n x^n)$ *where* $c_n = \sum_0^n a_j b_{n-j}$,

have radius of convergence R.

Proof. (1) is clear ,we only have to prove (2). Let

(1) $A = \sum_0^\infty a_n x^n, B = \sum_0^\infty b_n x^n, d_n = \sum_0^n |a_j b_{n-j}|$,
(2) $\tilde{A} = \sum_0^\infty |a_n x^n|, \tilde{A}^m = \sum_0^m |a_n x^n|, \tilde{B} = \sum_0^\infty |b_n x^n|, \tilde{B}^m = \sum_0^m |b_n x^n|$,
(3) $\tilde{C} = \sum_0^\infty |c_n x^n|, \tilde{C}^m = \sum_0^m |c_n x^n|, \tilde{D} = \sum_0^\infty |d_n x^n|, \tilde{D}^m = \sum_0^m |d_n x^n|$.

Since

$$\tilde{C}^m \le \tilde{D}^m \le \tilde{A}^m \tilde{B}^m \le \tilde{A}\tilde{B},$$

that proves $\sum_0^\infty c_n x^n$ converges absolutely. Also

$$|AB - \sum_0^m c_n x^n| \le \tilde{A}\tilde{B} - \tilde{D}^{[\frac{m}{2}]},$$

that proves (2).

Example. $\tan x \cos x = \sin x$

Indeterminate forms.

Example.

(1) $\lim_{x\to 0} \frac{\sin x - \tan^{-1} x}{x^2 \ln(1+x)}$,
(2) $\lim_{x\to 1} \frac{\ln x}{1-x}$.

Series solution of difrferential equation

Example.

(1) $y' + 2y = 0$,
(2) $(x-3)y' + 2y = 0$,
(3) $x^2 y' = y - x - 1$,
(4) $y'' + y = 0$,
(5) $y'' - xy = 0$,
(6) $xy'' + y' + xy = 0$ Bessel equation.