CHAPTER 0

0.1 Two Examples

Find root by bisection method. *Let $P(x)$ be a polynomial such that $P(a) < 0$ and $P(b) > 0$. Find a $c \in (a,b)$ such that $P(c) = 0$.*

Bisection method.

(1) If $P(\frac{a+b}{2}) < 0$, let $a_1 = \frac{a+b}{2}, b_1 = b$. If $P(\frac{a+b}{2}) > 0$, let $a_1 = a, b_1 = \frac{a+b}{2}$.
(2) If $P(\frac{a_1+b_1}{2}) < 0$, let $a_2 = \frac{a_1+b_1}{2}, b_2 = b_1$. If $P(\frac{a_1+b_1}{2}) > 0$, let $a_2 = a_1, b_2 = \frac{a_1+b_1}{2}$.
(3) If $P(\frac{a_n+b_n}{2}) < 0$, let $a_{n+1} = \frac{a_n+b_n}{2}, b_{n+1} = b_n$. If $P(\frac{a_n+b_n}{2}) > 0$, let $a_{n+1} = a_n, b_{n+1} = \frac{a_n+b_n}{2}$.

Let $I_n = [a_n, b_n]$, by induction we have $|I_n| = b_n - a_n = (\frac{1}{2})^n(b-a)$. We also have $a_n \le a_{n+m} \le b_{n+m} \le b_n$, hence $|a_n - a_{n+m}| \le (\frac{1}{2})^n(b-a)$ for all n, m. Since there is a root c_n in I_n,so we have an effective way to find a good approximation of a root.($|a_n - c_n|, |b_n = c_n| < (\frac{1}{2})^n(b-a)$.)

Find the area of the unit dick (π).

Approximations. Let A_n be the area of the regular n-gon inscribed in the unit disc, and let B_n be the area of the regular n-gon out-tangent to the unit disc, It is easy to see that $A_n = n\sin\frac{\pi}{n}\cos\frac{\pi}{n}, B_n = n\tan\frac{\pi}{n}$.

$$B_n - A_n = n\sin\frac{\pi}{n}(\frac{1}{\cos\frac{\pi}{n}} - \cos\frac{\pi}{n}) = n\frac{(\sin\frac{\pi}{n})^3}{\cos\frac{\pi}{n}}.$$

Since $\cos\frac{\pi}{n} > \cos\frac{\pi}{3} = \frac{1}{2}$ and $\sin\frac{\pi}{n} \le \frac{\pi}{n}$, we hvae $B_n - A_n \le \frac{2\pi^3}{n^2}$ and $A_n \le \pi \le B_n$. So we have an effective way to find a good approximation of π.

In the two examples above, we believe that there exist solutions (the limit points exist). In fact we can prove it over real number system.

0.2 Completeness of real number system $\mathbf{R}$

Theorem. *Let $\{I_n = [a_n, b_n]\}$ be a sequence of bounded closed intervals such that $I_{n+1} \subset I_n$ for all n, and $|I_n| = b_n - a_n$ goes to 0 as n goes to ∞. Then $\cap_{n=1}^{\infty} I_n$ is a single point.*

Proof. First we show that $\cap_{n=1}^{\infty} I_n$ contains at most one point. We prove it by contradiction, assume that $\cap_{n=1}^{\infty} I_n$ contains c, c' such that $c \ne c'$. Then $|c-c'| > 0$, and for n large enough, $|I_n| < \frac{1}{2}|c-c'|$. By our assumption that $c \in I_n$, but then c' can not be in I_n, which is a contradiction.

For the existence of $c \in \cap_{n=1}^{\infty} I_n$, we need the following property of $\mathbf{R}$.

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX

Bounded above, Upper bound, Least upper bound. For a nonempty $S \subset \mathbf{R}$ to be bounded above if there is M such that $x \leq M$ for all $x \in S$. In this case we say that M is an upper bound of S.

Least upper bound. *For a nonempty set S which is bounded above, an upper bound N is called the least upper bound if $N \leq M$ for all upper bound M of S denoted by* $\mathbf{l.u.b.}S$.

Remark. *($\forall\epsilon$ is for every ϵ, and $\exists\delta$ is there exists δ.)*

(1) *There are no $x \in S$ such that $x > \mathbf{l.u.b.}S$.*
(2) *For $\forall\epsilon > 0, \exists x \in S$ such that $\mathbf{l.u.b.}S - \epsilon < x \leq \mathbf{l.u.b.}S$.*

$\mathbf{l.u.b.}(0,1) = 1, \mathbf{l.u.b.}[0,1] = 1$, the first case $\mathbf{l.u.b.}S \notin S$ and second case $\mathbf{l.u.b.}S \in S$. It is clear that the least upper bound is unique.(Exercise: prove it.)

AXIOM. *(Least upper property of $\mathbf{R}$) Every nonempty subset S of $\mathbf{R}$, if S is bounded above, then S has the least upper bound.*

Remark. *The set of all rational numbers $\mathbf{Q}$ does not have this property. The set of all integers $\mathbf{Z}$ has this priperty.*

Now we can complete the proof of the existence of $c \in \cap_{n=1}^{\infty} I_n$. Let $S = \{a_n\}$, then b_n is upper bound of S for all n. If we let c be the least upper bound of S , then we have $a_n \leq c \leq b_n$ for all n, which implies that $c \in \cap_{n=1}^{\infty} I_n$. Q.E.D

0.3 Some inequalities

Lemma. *Let $a > 0$, then $|b| \leq a$ if and only if $-a \leq b \leq a$.*

Lemma.

(1) $|a + b| \leq |a| + |b|$,
(2) $||a| - |b|| \leq |a - b|$

proof.

(1) Since $-|a| \leq a \leq |a|, -|b| \leq b \leq |b|$, we have $-(|a| + |b|) \leq a + b \leq (|a| + |b|)$, then by the Lemma above $|a + b| \leq |a| + |b|$.
(2) From (1) let $b \to (b - a)$, we get $|b| \leq |a| + |a - b|$. That means $-|a - b| \leq |a| - |b|$. By $a \to b, b \to a$ we get $-|b-a| \leq |b|-|a|$ multiply byb -1 we get $|a-b| \geq |a|-|b|$. Again by the Lemma above, we hvae $||a| - |b|| \leq |a - b|$.

0.4 Some logic statements

The following propositions are equirvalent

(1) A implies B,
(2) $A \implies B$,
(3) $-B$ implies $-A$,
(4) A is a sufficient cvondition of B,
(5) B is a necessary condition of A,
(6) B if A(If A is true then B is true.),
(7) A only if B.(A will be true only if B is true.)

The following propositions are equirvalent

(1) A is equivalent to B,
(2) $A \Leftrightarrow B$,
(3) $A \equiv B$,
(4) A is a necessary and sufficient condition of B,
(5) A if only if B,
(6) A iff B.

Mathematical induction. The method of mathematical induction in proving some statment involving n say $P(n)$ is true for all n from n_0(most cases $n_0 = 1$) on, consisting of two steps.

(1) Initial step: show that $P(n_0)$ is true.
(2) Induction step: assume that $P(n_0), \cdots, P(k)$ are ture, show that $P(k+1)$ is true.