Recall what is $\vec{X_u}$ & $\vec{X_v}$?

$\vec{X}: (u,v)^{\in \mathbb{R}^2} \longrightarrow (x(u,v), y(u,v), z(u,v))_{\in S \subset \mathbb{R}^3}$ a parametrization around $\vec{X}(u_0,v_0) = p \in S$

Then $\vec{X_u} := \dfrac{\partial \vec{X}}{\partial u}$, $\vec{X_v} := \dfrac{\partial \vec{X}}{\partial v}$

$\Rightarrow$ we see that the domain for $\vec{X}$ is u-v plane, so one should write $\vec{X_u}(u,v)$ to represent the vector in $\mathbb{R}^3$, because $\vec{X_u}(u,v) := \left(\frac{\partial x}{\partial u}(u,v), \frac{\partial y}{\partial u}(u,v), \frac{\partial z}{\partial u}(u,v)\right)$

However, what we usually do, is to write $\vec{X_u}(p)$, where $p \in S$ is a point on S! So what this notation $\vec{X_u}(p)$ means, is really $\vec{X_u}(u_0,v_0)$ if $p = \vec{X}(u_0,v_0)$!

so for a curve on the u-v plane:

$\alpha: I \longrightarrow \underbrace{\mathbb{R}^2}_{\text{u-v plane}}$, $t \mapsto \alpha(t) = (u(t), v(t))$

with $\alpha(0) = (u(0), v(0)) = (u_0, v_0)$, $\vec{X}(\alpha(0)) = \vec{X}(u_0,v_0) = p \in S$

The tangent vector to the curve on S at p

$\vec{X}(\alpha(t)): I \longrightarrow S \subset \mathbb{R}^3$, $t \mapsto \vec{X}(u(t), v(t))$

[i.e. $\vec{X}(u(t),v(t)) = (x(u(t),v(t)), y(u(t),v(t)), z(u(t),v(t)))$]

$$\left.\frac{d}{dt}\right|_{t=0} \vec{X}(\alpha(t)) = \left.\frac{d}{dt}\right|_{t=0} \vec{X}(u(t), v(t)) = \frac{\partial \vec{X}}{\partial u}(u_0,v_0) \cdot u'(0) + \frac{\partial \vec{X}}{\partial v}(u_0,v_0) \cdot v'(0)$$

(Recall $u(0) = u_0$, $v(0) = v_0$)

This is why we wrote : for $v \in T_pS$, by definition $\exists\ \alpha : I \to \mathbb{R}^2$ s.t. $\vec{X}(\alpha(0)) = p$, $\left.\frac{d}{dt}\right|_{t=0} \vec{X}(\alpha(t)) = v$

Then in parametrization v can be rewritten as

$$\underset{T_pS}{v} = \left.\frac{d}{dt}\right|_{t=0} \vec{X}(\alpha(t)) = \vec{X}_u(u_0,v_0)\cdot u'(0) + \vec{X}_v(u_0,v_0)\cdot v'(0)$$

and for convenince, write as $\vec{X}_u(p)\cdot u'(0) + \vec{X}_v(p)\cdot v'(0)$

Now let's look at the proof we did in the class:

claim: $\langle d\vec{N}_p(\vec{X}_u), \vec{X}_v \rangle = \langle \vec{X}_u, d\vec{N}_p(\vec{X}_v) \rangle$

The left hand side $\langle d\vec{N}_p(\vec{X}_u), \vec{X}_v \rangle$ means

$\langle d\vec{N}_p(\vec{X}_u(p)), \vec{X}_v(p) \rangle$ where $\vec{X}_u(p)$ & $\vec{X}_v(p)$ are explained above.

which is really

$\langle \underbrace{d\vec{N}_p(\vec{X}_u(u_0,v_0))}, \vec{X}_v(u_0,v_0) \rangle$, where $p = \vec{X}(u_0,v_0)$ (so $\vec{X}_u(u_0,v_0) \in T_pS$ and $\vec{X}_v(u_0,v_0) \in T_pS$)

Now recall how do we calculate differential:

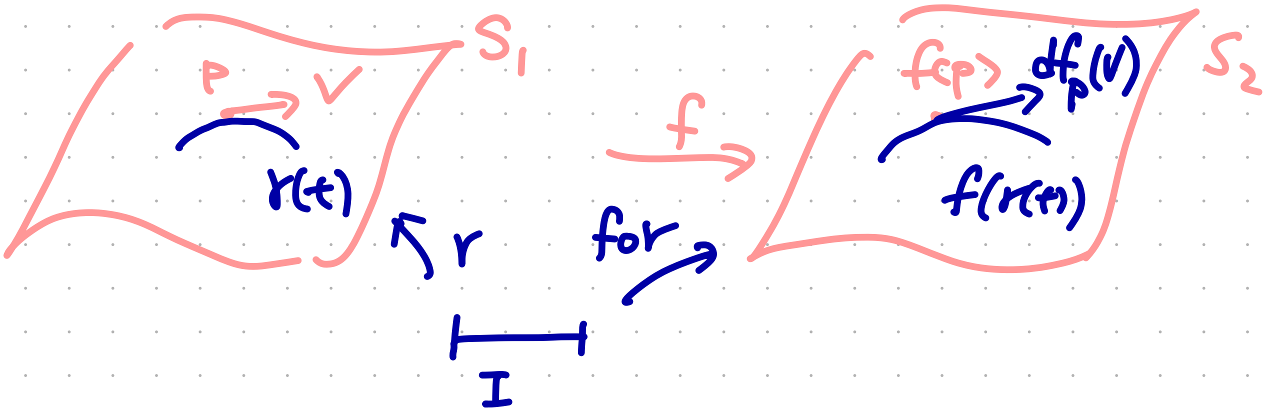
If we have a map between two smooth surfaces

and $v \in T_p S_1$, then $df_p(v) \in T_{f(p)} S_2$ is defined as

1° pick a curve $\gamma(t) : I \longrightarrow S$ such that $\gamma(0) = p$, $\gamma'(0) = v$

2° $df_p(v) := \left.\frac{d}{dt}\right|_{t=0} f(\gamma(t))$

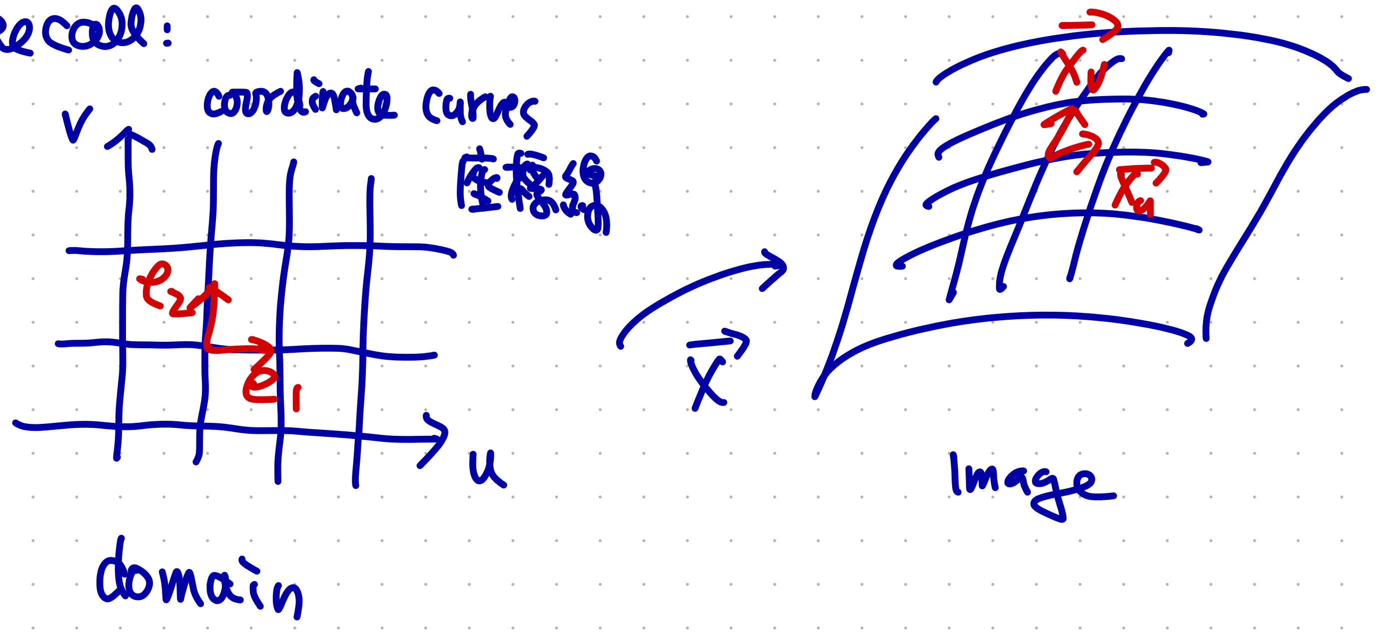
i.e. sending the curve from the domain to the image, then take the derivative.



So to calculate $dN_p(\vec{X_u}(u_0, v_0))$, we need to find a curve on S such that the tangent vector to this curve at the pt is $\vec{X_u}(u_0, v_0)$.

So what is this curve ?

Recall:



and $\vec{X}_u(u_0,v_0) := \dfrac{\partial \vec{X}}{\partial u}(u_0,v_0) = \dfrac{d}{dt}\Big|_{t=0} \vec{X}(\alpha(t))$

where $\alpha(t) = (u(t), v(t)) = (u_0+t, v_0)$

is u-curve passing (u_0,v_0)

So $\vec{X}_u(u_0,v_0) = \dfrac{d}{dt}\Big|_{t=0} \vec{X}(u_0+t, v_0)$

$$= \frac{\partial \vec{X}}{\partial u}(u_0,v_0)\cdot\frac{d(u_0+t)}{dt} + \frac{\partial \vec{X}}{\partial v}(u_0,v_0)\cdot\frac{d(v_0)}{dt}$$

$$= \frac{\partial \vec{X}}{\partial u}(u_0,v_0)\cdot 1 + \frac{\partial \vec{X}}{\partial v}(u_0,v_0)\cdot 0$$

So $d\vec{N}_p(\vec{X}_u(u_0,v_0)) = \dfrac{d}{dt}\Big|_{t=0} \vec{N}(\underbrace{\vec{X}(\alpha(t))}_{\text{curve on } S})$

$$= \frac{d}{dt}\Big|_{t=0} \vec{N}\circ\vec{X}(u_0+t, v_0)$$

For simplicity, denote the composition $\vec{N} \circ \vec{X}$ by $\vec{N}$ for taking derivative

$$= \frac{d}{dt}\Big|_{t=0} \vec{N}(u_0+t, v_0)$$

$$= \frac{\partial \vec{N}}{\partial u}(u_0,v_0) \cdot \frac{d(u_0+t)}{dt} + \frac{\partial \vec{N}}{\partial v}(u_0,v_0) \cdot 0$$

$$= \vec{N}_u(p)$$

i.e. again, we abuse the notation $\vec{N}_u(p)$ to mean $(\vec{N} \circ \vec{X})_u(u_0,v_0)$ if $\vec{X}(u_0,v_0) = p$.

So to compute $< dN_p(\vec{X}_u), \vec{X}_v >$, it's really

$$< d\vec{N}_p(\vec{X}_u(u_0,v_0)), \vec{X}_v(u_0,v_0) >$$

$$= < \frac{d}{dt}\Big|_{t=0} \vec{N}(\vec{X}(\alpha(t))), \vec{X}_v(u_0,v_0) > \qquad \alpha(t) = (u_0+t, v_0)$$

$$= \frac{d}{dt}\Big|_{t=0} < \vec{N}(\vec{X}(\alpha(t))), \vec{X}_v(\alpha(t)) >$$

$$- < \vec{N}(\vec{X}(\alpha(0))), \frac{d}{dt}\Big|_{t=0} \vec{X}_v(\alpha(t)) >$$

$\vec{N} \perp \vec{X}_v$

$$= 0 - < \vec{N}(p), \frac{\partial \vec{X}_v}{\partial u}(\alpha(0)) \cdot \underbrace{u'(0)}_{=1} + \frac{\partial \vec{X}_v}{\partial v}(\alpha(0)) \cdot \underbrace{v'(0)}_{=0} >$$

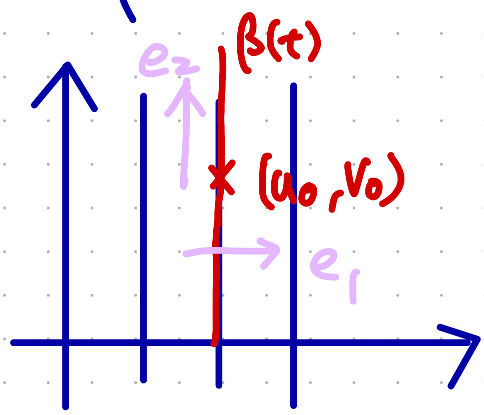
$$= - < \vec{N}(p), \frac{\partial \vec{X}_v}{\partial u}(p) > = - < \vec{N}(p), \vec{X}_{vu}(p) >$$

Similarly, to compute $\langle \vec{X}_u , d\vec{N}_p(X_v)\rangle$, we start

with $\vec{X}_v(p)$, which is really $\vec{X}_v(u_0,v_0)$

and define $\beta(t): I \to \mathbb{R}^2$, $t \mapsto (u(t), v(t))$ for $\vec{X}_v(u_0,v_0)$

Then $\beta(0) = (u_0,v_0)$, $\beta(t) = (u_0, v_0 + t)$, $\beta'(0) = e_2$



$$\text{and } \vec{X}_v(p) := \vec{X}_v(u_0,v_0) = \frac{\partial \vec{X}}{\partial v}(u_0,v_0)$$
$$= \frac{d}{dt}\Big|_{t=0} \vec{X}(\beta(t))$$

If also denote $\beta(t) = (u(t), v(t))$, then $u(t) = u_0$ and $v(t) = v_0 + t$

so

$$\frac{d}{dt}\Big|_{t=0} \vec{X}(\beta(t)) = \frac{\partial \vec{X}}{\partial u}(u_0,v_0)\cdot u'(0) + \frac{\partial \vec{X}}{\partial v}(u_0,v_0)\cdot v'(0)$$
$$= \frac{\partial \vec{X}}{\partial v}(u_0,v_0)\cdot 1$$

So $\langle \vec{X}_u(p), d\vec{N}_p(\vec{X}_v)(p)\rangle$

$$:= \Big\langle \vec{X}_u(u_0,v_0), \frac{d}{dt}\Big|_{t=0} \vec{N}\big(\overbrace{\vec{X}(\beta(t))}^{\text{curve on } S}\big)\Big\rangle$$
$$= \frac{d}{dt}\Big|_{t=0}\Big\langle \vec{X}_u(\beta(t)), \vec{N}(\vec{X}(\beta(t)))\Big\rangle$$
$$- \Big\langle \frac{d}{dt}\Big|_{t=0} \vec{X}_u(\beta(t)), \vec{N}\big(\overbrace{\vec{X}(\beta(0))}^{p \,=\, \vec{X}(u_0,v_0)}\big)\Big\rangle$$

$$= 0 - \left\langle \frac{\partial \vec{X_u}}{\partial u}(u(0),v(0)) \cdot u'(0) + \frac{\partial \vec{X_u}}{\partial v}(u(0),v(0)) \cdot v'(0), \vec{N}(p) \right\rangle$$

$\vec{N} \perp \vec{X_u}$

$$= -\left\langle \frac{\partial \vec{X_u}}{\partial v}(p), \vec{N}(p) \right\rangle \quad \text{because for } \beta(t), \begin{cases} u'(0)=0 \\ v'(0)=1 \end{cases}$$

$$= -\langle \vec{X_{uv}}(p), \vec{N}(p) \rangle$$

Now $\vec{X}$ is C^∞, so $\vec{X_{uv}} = \vec{X_{vu}}$, thus

$$\langle \vec{X_{uv}}(p), \vec{N}(p) \rangle = \langle \vec{N}(p), \vec{X_{vu}}(p) \rangle$$

$$\Rightarrow \langle \vec{X_u}(p), d\vec{N_p}(\vec{X_v})(p) \rangle = \langle d\vec{N_p}(\vec{X_u})(p), \vec{X_v}(p) \rangle$$

and since $\{\vec{X_u}(p), \vec{X_v}(p)\}$ form a basis for T_pS,

we conclude that

$$\langle d\vec{N_p}(v), w \rangle = \langle v, dN_p(w) \rangle, \ \forall v, w \in T_pS$$

i.e. $d\vec{N_p}$ is a self-adjoint linear map. #

✓(Also as in the textbook)

Notice that, by abusing the notation (as we did in class)

the proof can be simplified as follows:

$$\langle d\vec{N_p}(\vec{X_u})(p), \vec{X_v}(p) \rangle := \left\langle \frac{\partial}{\partial u}(\vec{N} \circ \vec{X})(u_0,v_0), \vec{X_v}(u_0,v_0) \right\rangle$$

denote $\vec{N}\circ\vec{X}$ by $\vec{N}$ $\searrow$

$$:= \langle \vec{N}_u(u_0,v_0), \vec{X}_v(u_0,v_0) \rangle$$

$$= \frac{\partial}{\partial u}\langle \vec{N}, \vec{X}_v \rangle (u_0,v_0)$$

$$- \langle \vec{N}(u_0,v_0), \frac{\partial \vec{X}_v}{\partial u}(u_0,v_0) \rangle$$

again, abuse notations $\searrow$

$$= 0 - \langle \vec{N}(p), \vec{X}_{vu}(p) \rangle$$

and $\langle \vec{X}_u(p), d\vec{N}_p(\vec{X}_v)(p) \rangle := \langle \vec{X}_u(u_0,v_0), \frac{\partial}{\partial v}(\vec{N}\circ\vec{X})(u_0,v_0) \rangle$

denote again $\vec{N}\circ\vec{X}$ by $\vec{N}$ $\searrow$

$$:= \langle \vec{X}_u(u_0,v_0), \vec{N}_v(u_0,v_0) \rangle$$

$$= \frac{\partial}{\partial v}\langle \vec{X}_u, \vec{N} \rangle (u_0,v_0) - \langle \frac{\partial \vec{X}_u}{\partial v}(u_0,v_0), \vec{N}(u_0,v_0) \rangle$$

$$= - \langle \vec{X}_{uv}(p), \vec{N}(p) \rangle$$

So by $\vec{X}$ is C^∞ $\Rightarrow$ $\vec{X}_{uv}(p) = \vec{X}_{vu}(p)$

\#

So you can see that they are really the same thing, except that one looks simpler by directly taking derivative for u and for v!