LECTURE NOTES ON MATHEMATICAL ANALYSIS

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Dedicated to the memory of Professor Chao-Liang Shen

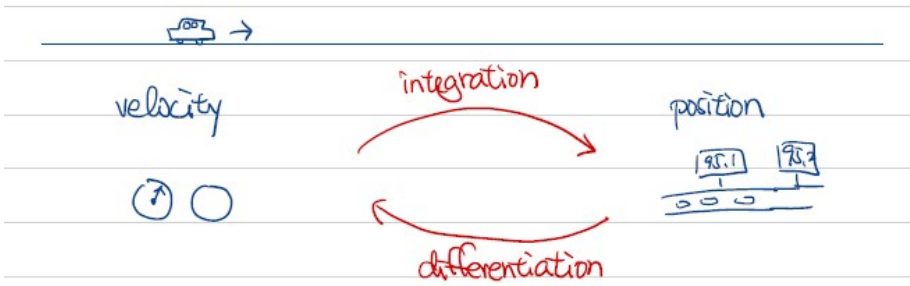
ABSTRACT. Lecture notes for courses for Calculus and Advanced Calculus.

CONTENTS

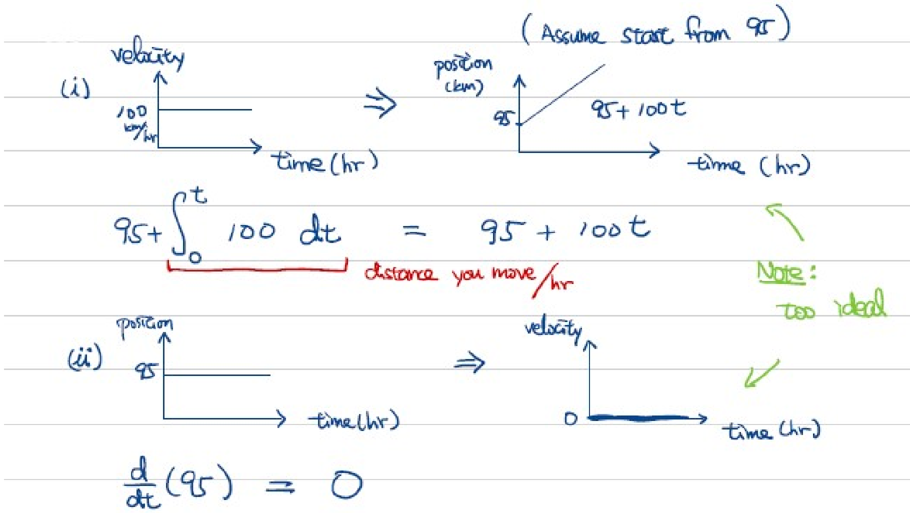
Part 1. Calculus

Introduction

The main topics in Calculus are "differentiation" and "integration." As an example of Calculus problem, we consider a car driving on a highway:



As more concrete examples, let us assume the car starts from 95 to the south at velocity 100 km/hr in case (i), and assume the car stops at 95 in case (ii). Then the relations between velocity-time graph, position-time graph, differentiation and integration are shown as follows:



In this course, we will discuss these two transformations — differentiation and integration — and the relevant techniques and applications.

The textbook in Spring, 2021 was [3]. The textbook in Fall, 2022 was [1]. The textbook in Fall, 2023 is [3].

Notations and conventions. The notation "∞" means *"infinity."* The notation "$\in$" means *"in."* The notation " $\forall$ " means *" for all."* The notation "$\exists$ " means *" exist."* The notation "$\because$" means *"because."* The notation "$\therefore$" means *"therefore."*

The notation " $\mathbb{Q}$ " means *"the set of all rational numbers."* The notation " $\mathbb{R}$ " means *"the set of all real numbers."* The notation "$\mathbb{C}$ " means *"the set of all complex numbers."*

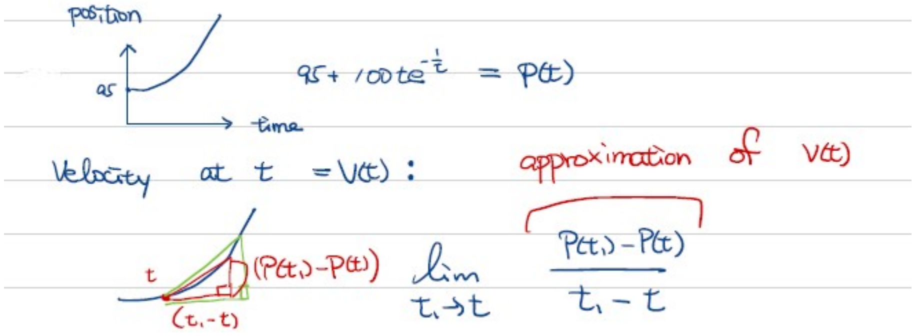
The notation "s.t." means *"such that."* The notation "i.e." means *"that is"* or *"in other words."* The notation "e.g." means *"for example."* The notation "cf." means *"compare."*

The notation "$\Leftrightarrow$" means *"if and only if"* (or one may write *"iff"*). The statement *"P if and only if Q"* means *"P is true implies Q is true"* and *"Q is true implies P is true."* For example, if $x \in \mathbb{R}$, then x is rational if and only if x is not irrational.

Acknowledgments. I would like to thank Lin-Hsiang Weng and Cheng-Yi Hung for the assistance of editing the latex file.

1. Limits of sequences

Back to the velocity problem, let us consider a more real situation:



Note that we need "*limit*" in the study of velocity. In fact, limit is one of the most important notions in Calculus.

1.1. **Intuitive examples.**
We will see the following 3 types of limits in the course of Calculus:

- function-type A : $\lim_{x\to c} f(x)$ (c is a fixed number)
- function-type B : $\lim_{x\to\infty} f(x)$ or $\lim_{x\to-\infty} f(x)$ (∞ is infinity)
- sequence type : $\lim_{n\to\infty} a_n$

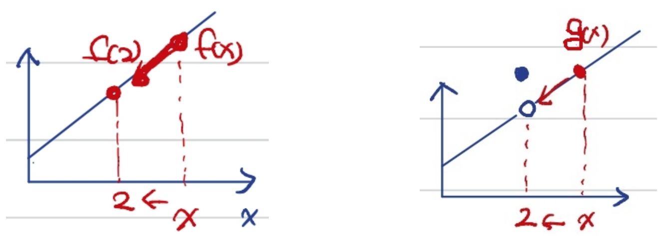
where $f(x)$ is a function , $(a_n)_{n=1}^{\infty} = a_1, a_2, a_3, \cdots$ is a sequence.

We will need "function-type A" in differentiation. Let us start with examples of this type. (One may find more examples in [3, Section 2.1] and [1, Section 2.2].)

Example 1.1. *Let*

$$f(x) = 4x + 5 \qquad \textit{and} \qquad g(x) = \begin{cases} 4x+5, & x \neq 2 \\ 15, & x = 2. \end{cases}$$

We have $\lim_{x\to 2} f(x) = f(2) = 13$*, and* $\lim_{x\to 2} g(x) = 13 \neq 15 = g(2)$.



(a) $\lim_{x\to 2} f(x) = f(2) = 13$ (b) $\lim_{x\to 2} g(x) \neq g(2)$

Figure 1. Example 1.1

Example 1.2. *Let*

$$f(x) = \sin(\frac{\pi}{x}).$$

All the (one-side) limits $\lim_{x\to 0^-} f(x)$, $\lim_{x\to 0^+} f(x)$ *and* $\lim_{x\to 0} f(x)$ *do NOT exist.*

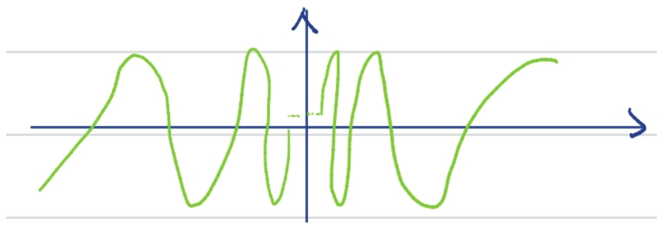


FIGURE 2. $\lim_{x\to 0} f(x)$ does NOT exist.

For "function-type B," let us consider following functions:

Example 1.3. *Let*

$$f(x) = 1 + \frac{1}{x} \qquad \textit{and} \qquad g(x) = 1 + \frac{\sin x}{x}.$$

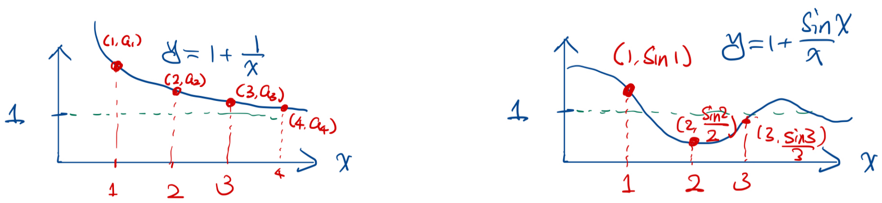
We have $\lim_{x\to\infty} f(x) = 1$*, and* $\lim_{x\to\infty} g(x) = 1$.

For the sequence type, we can consider the parallel sequences:

Example 1.4. *Let*

$$a_n = 1 + \frac{1}{n} \qquad \textit{and} \qquad b_n = 1 + \frac{\sin n}{n}$$

We have $\lim_{x\to\infty} f(x) = 1$*, and* $\lim_{x\to\infty} g(x) = 1$.



(a) $\lim_{x\to\infty} f(x) = 1$ (b) $\lim_{x\to\infty} g(x) = 1$

FIGURE 3. Example 1.3 and Example 1.4.

It may seem that we can guess the limits of functions or sequences from their graphs, but are those guesses really correct? To be sure, we need precise definitions of limits.

Since sequences can be counted one by one, it is more fundamental to first consider sequence-type limits. We will develop the theory of limits by introducing sequences first, and then investigate function-type limits (partly through sequences).

1.2. **Sequences and their limits.** Recall that a **function** $f : A \to B$ has 3 ingredients:

- Domain = A is the set of possible inputs.
- Codomain = B is the set of possible outputs.
- How it works.

It is common for people to omit specifying the domain and codomain of a function. Usually, these can be inferred from the context.

Example 1.5. *In previous examples, we considered*

$$f(x) = 4x + 5, \qquad g(x) = \begin{cases} 4x+5, & x \neq 2 \\ 15, & x = 2, \end{cases} \qquad \text{and} \qquad h(x) = \sin(\frac{\pi}{x}).$$

The domains and the codomains of f and g were understood to be the set of real numbers:

$$\mathbb{R} = \{\text{real numbers}\}.$$

The codomain of h was also understood to be $\mathbb{R}$, but the domain of h cannot be $\mathbb{R}$ since $\frac{\pi}{0}$ is not defined. In fact, the domain of h was understood to be

$$\mathbb{R} \setminus \{0\} = \{x \in \mathbb{R} \mid x \notin \{0\}\} = \{x \in \mathbb{R} \mid x \neq 0\}.$$

Definition 1.6 ([3, Definition 11.2.1])**.** *A **sequence** (of real numbers) is a function from $\mathbb{N}$ to $\mathbb{R}$, where $\mathbb{N}$ is the set of positive integers:*

$$\mathbb{N} = \{1, 2, 3, 4, \cdots, n, n+1, \cdots\}.$$

*If $a : \mathbb{N} \to \mathbb{R}$ is a sequence, we usually write $a(n)$ as a_n, called the **n-th term** of this sequence. It is common to denote the sequence $a : \mathbb{N} \to \mathbb{R}$ by*

$$(a_n)_{n=1}^{\infty} = a_1, a_2, \cdots, a_n, \cdots.$$

Example 1.7. *Following are a few examples of sequences:*

$$((-1)^n)_{n=1}^{\infty} = -1, 1, -1, 1, -1, \cdots;$$
$$(n)_{n=1}^{\infty} = 1, 2, 3, 4, 5, \cdots;$$
$$1, 1, 2, 3, 5, 8, 13, 21, \cdots.$$

Definition 1.8. *A sequence $(a_n)_{n=1}^{\infty}$ is said to be **convergent** if there exists a number L with the property: for each $\epsilon > 0$, there exists $N = N_\epsilon$ such that*

$$|a_n - L| < \epsilon \qquad \text{whenever} \qquad n \geq N.$$

*In this case, we say L is the **limit** of $(a_n)_{n=1}^{\infty}$, denoted by*

$$\lim_{n\to\infty} a_n = L, \qquad \text{or} \qquad a_n \to L \quad (\text{as } n \to \infty).$$

*A sequence $(a_n)_{n=1}^{\infty}$ is said to be **divergent** if it is not convergent.*

Example 1.9. *Prove that $\lim_{n\to\infty} \frac{1}{n} = 0$.*

Example 1.10. *Show that $\lim_{n\to\infty} r^n = 0$, for any $|r| < 1$.*

Example 1.11. *Let $r \in \mathbb{R}$ be a real number such that $|r| < 1$, and $(\xi_n)_{n=1}^{\infty}$ be a sequence such that $\xi_n \in \{\pm 1\}$ for all n. Define $(a_n)_{n=1}^{\infty}$ to be the sequence: $a_n = \xi_n r^n$. Prove that $(a_n)_{n=1}^{\infty}$ is convergent, and $\lim_{n\to\infty} a_n = 0$.*

Example 1.12. *Let $r \in \mathbb{R}$ be a real number such that $|r| < 1$. Define $(s_n)_{n=1}^{\infty}$ to be the sequence: $s_n = 1 + r + r^2 + \cdots + r^n$. Prove that $(s_n)_{n=1}^{\infty}$ is convergent, and $\lim_{n\to\infty} s_n = \frac{1}{1-r}$.*

We will consider two basic limit questions:

(1) Does the limit exist?
(2) If so, what is its value?

1.3. **Computation of limits.** We start with the second basic question: computation of limits. In general, it is difficult to compute limits directly from Definition 1.8. We need a few theorems to help with this.

Proposition 1.13. *If a limit exists, then it is unique. That is, if* $\lim_{n\to\infty} a_n = L$ *and* $\lim_{n\to\infty} a_n = M$*, then* $L = M$.

Theorem 1.14 ([3, Theorem 11.3.7])**.** *Let* $(a_n)_{n=1}^{\infty}$, $(b_n)_{n=1}^{\infty}$ *be convergent sequences, and* γ *a real number. Then*

(1) $\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$;
(2) $\lim_{n\to\infty}(\gamma \cdot a_n) = \gamma \cdot \lim_{n\to\infty} a_n$;
(3) $\lim_{n\to\infty}(a_n b_n) = (\lim_{n\to\infty} a_n)(\lim_{n\to\infty} b_n)$;
(4) *if* $\lim_{n\to\infty} b_n \neq 0$ *and* $b_n \neq 0$ *for all* n*, then* $\lim_{n\to\infty}(a_n/b_n) = (\lim_{n\to\infty} a_n)\big/(\lim_{n\to\infty} b_n)$.

Example 1.15. *Compute the limits.*

(1) $\lim_{n\to\infty} \dfrac{3n^4 - 2n^2 + 1}{n^5 - 3n^3} = 0$.
(2) $\lim_{n\to\infty} \dfrac{1 - 4n^7}{n^7 + 12n} = -4$.

Theorem 1.16 (Squeeze Theorem for sequences, [3, Theorem 11.3.9])**.** *Suppose that for all* n *sufficiently large*

$$a_n \le b_n \le c_n.$$

If $\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} c_n$*, then* $\lim_{n\to\infty} b_n = L$.

Example 1.17. *Prove the following:*

(1) $\lim_{n\to\infty} \dfrac{\cos n}{n} = 0$.
(2) $\lim_{n\to\infty} \sqrt{4 + (1/n)^2} = 2$*, since* $2 \le \sqrt{4 + (1/n)^2} \le 2 + 1/n$.
(3) $\lim_{n\to\infty} \dfrac{a^n}{n!} = 0$*, for any real number* a*. (For* $n > |a|$, $\dfrac{|a|^n}{n!} \le \Big(\dfrac{|a|^{[|a|]}}{[|a|]!}\Big)\dfrac{|a|}{n}$*.)*
(4) *Assume* $a > 0$*. Prove* $\lim_{n\to\infty} a^{\frac{1}{n}} = 1$*. (***Case 1:** $a > 1$*. Since* $x^n - 1 = (x-1)(x^{n-1} + \cdots + 1)$*, we have* $|a^{\frac{1}{n}} - 1| = \Big|\frac{a-1}{a^{\frac{n-1}{n}} + a^{\frac{n-2}{n}} + \cdots + 1}\Big| \le \frac{a-1}{n} \to 0$. **Case 2:** $0 < a < 1$. $\lim_{n\to\infty}\Big(1\big/(\dfrac{1}{a})^{\frac{1}{n}}\Big) = 1$*.)*
(5) *Assume* $a > 1$*. Prove that* $\lim_{n\to\infty} \dfrac{n}{a^n} = 0$*. (*$\dfrac{n}{a^n} = \dfrac{n}{(1 + (a-1))^n} = \dfrac{n}{1 + n\cdot(a-1) + \frac{n(n-1)}{2}(a-1)^2 + \cdots}$ *which is bounded by* $\dfrac{n}{\frac{n(n-1)}{2}(a-1)^2} \to 0$*.)*

1.4. **Convergence of sequences.** If a sequence is not a well-known one, it can be very difficult to determine the value of its limit. Sometimes, even if we cannot compute the exact value, it is still possible to decide whether the sequence converges. To do this, we need some definitions and a few theorems about sequences.

1.4.1. *Bounded monotone sequences.* We say that a sequence $(a_n)_{n=1}^\infty$ is

bounded above	if	$\exists M$ such that $a_n \le M,\ \forall n$;
bounded below	if	$\exists N$ such that $a_n \ge N,\ \forall n$;
bounded	if	it is both bounded above and bounded below;
strictly increasing (= **increasing** in [3])	if	$a_n < a_{n+1},\ \forall n$;
increasing (= **nondecreasing** in [3])	if	$a_n \le a_{n+1},\ \forall n$;
strictly decreasing (= **decreasing** in [3])	if	$a_n > a_{n+1},\ \forall n$;
decreasing (= **nonincreasing** in [3])	if	$a_n \ge a_{n+1},\ \forall n$;
constant	if	$a_n = a_{n+1},\ \forall n$.

Example 1.18. *(Explain and graph.) Following are a few examples of sequences:*

(1) The sequence $(n)_{n=1}^\infty$ is strictly increasing and bounded below.
(2) The sequence $(1)_{n=1}^\infty$ is constant and, in particular, bounded.
(3) The sequence $(\frac{n}{n+1})_{n=1}^\infty$ is strictly increasing, bounded below by $\frac{1}{2}$ and bounded above by 1.

Theorem 1.19 ([3, Section 11.3])**.** *The following properties hold:*

(1) Every convergent sequence is bounded. (Explain.)
(2) Every unbounded sequence is divergent.
(3) A bounded above increasing sequence is convergent.
(4) A bounded below decreasing sequence is convergent.

Example 1.20. *Determine the sequence is convergent or divergent. Prove your answers.*

(1) $a_n = n$.
(2) $a_n = 1/n$.
(3) $a_n = n/(n+1)$.
(4) $a_n = n/2^n$.
(5) $\displaystyle\lim_{n\to\infty} \frac{n^4 - 3n^2 + n + 2}{n^3 - 7n}$ *doesn't exist (divergent).*

Theorem 1.19 (3) and (4) are equivalent by considering $(-a_n)_{n=1}^\infty$ for a given sequence $(a_n)_{n=1}^\infty$. The proof Theorem 1.19 (3) is closely related to the definition of real numbers. Here a real number is understood to be an infinite decimal: $\alpha_0.\alpha_1\alpha_2\alpha_3\cdots$.

We need a few lemmas.

Lemma 1.21. *If $(a_n)_{n=1}^\infty$ is a bounded increasing sequence of integers, then there exists N such that $a_n = a_N$ for any $n \ge N$.*

Proof. Since $(a_n)_{n=1}^\infty$ is bounded, there exists M such that $a_n \le M$ for any $n = 1, 2, 3, \cdots$.

Assume the conclusion fails. Then, for any N, there exists $n(N) > N$ such that $a_{n(N)} > a_N$. Since a_n are integers, $a_{n(N)} - a_N \ge 1$. We write $n^0(N) = N$ and

$$n^j(N) = n(n^{j-1}(N)) = \cdots = n(n(\cdots n(N)\cdots)).$$

Then $a_{n^k(N)} - a_{n^{j-1}(N)} \geq 1$. Consider $N = 1$ and $K = \lfloor M - a_1 \rfloor + 1$. We have

$$a_{n^K(1)} = a_1 + \sum_{j=1}^{K} (a_{n^j(1)} - a_{n^{j-1}(1)}) \geq a_1 + K > a_1 + M - a_1 = M$$

which is a contradiction. □

Proof of Theorem 1.19 (3). Let $(a_n)_{n=1}^\infty$ be an increasing sequence, and let M be an upper bound of it. Since a_n and M are real numbers, they can be expressed in decimal expansions:

$$\begin{aligned} a_1 &= a_{10}.a_{11}a_{12}a_{13}\cdots, \\ a_2 &= a_{20}.a_{21}a_{22}a_{23}\cdots, \\ \vdots & \qquad \vdots \\ M &= \alpha_0.\alpha_1\alpha_2\alpha_3\cdots. \end{aligned}$$

Since $(a_{n0})_{n=1}^\infty$ is a bounded increasing sequence of integers (bounded above by α_0), Lemma 1.21 implies that there exists N_0 such that $a_{n0} = a_{N_0 0}$ for all $n \geq N_0$. Similarly, there exists a sequence of integers $N_0 \leq N_1 \leq N_2 \leq \cdots$ such that

$$a_{nk} = a_{N_k k}, \quad \forall n \geq N_k,$$

for each $k = 0, 1, 2, \ldots$

Define

$$L = a_{N_0 0}.a_{N_1 1}a_{N_2 2}a_{N_3 3}\cdots.$$

We claim that $L = \lim_{n\to\infty} a_n$. To prove this, let $\epsilon > 0$ be given. Set

$$K = 1 + \max\{1, \lfloor -\log_{10}\epsilon \rfloor\}, \qquad N = N_K.$$

For $n \geq N$, since $a_{nj} = a_{N_j j}$ for all $j \leq K$, we have

$$\begin{aligned} |a_n - L| &= \left| a_{n0}.a_{n1}a_{n2}a_{n3}\cdots - a_{N_0 0}.a_{N_1 1}a_{N_2 2}a_{N_3 3}\cdots \right| \\ &= \left| 0.0\cdots0\, a_{n(K+1)}a_{n(K+2)}\cdots - 0.0\cdots0\, a_{N_{K+1}(K+1)}a_{N_{K+2}(K+2)}\cdots \right| \\ &\leq 10^{-K} < 10^{\log_{10}\epsilon} = \epsilon. \end{aligned}$$

Thus $\lim_{n\to\infty} a_n = L$, as claimed. □

By the construction of L in the proof, we obtain the following proposition.

Proposition 1.22. *If $(a_n)_{n=1}^\infty$ is increasing and bounded above by M, then*

$$\lim_{n\to\infty} a_n \leq M.$$

In fact, Proposition 1.22 can be strengthened as follows.

Proposition 1.23. *Let $(a_n)_{n=1}^\infty$ and $(b_n)_{n=1}^\infty$ be sequences such that there exists N with $a_n \leq b_n$ for all $n \geq N$. If both sequences converge, then $\lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n$.*

Proof. Suppose, for contradiction, that $\lim_{n\to\infty} a_n > \lim_{n\to\infty} b_n$. For $\epsilon = \frac{1}{2}\Big(\lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n\Big) > 0$, there exists a positive integer $N_\epsilon \geq N$ such that

$$a_n > \lim_{n\to\infty} a_n - \frac{1}{2}\Big(\lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n\Big) = \lim_{n\to\infty} b_n + \frac{1}{2}\Big(\lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n\Big) > b_n$$

for all $n \geq N_\epsilon$. This contradicts to the assumption $a_n \leq b_n$ for all $n \geq N$. Hence $\lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n$. □

1.4.2. *Subsequences of bounded sequences.*

Definition 1.24. *A* ***subsequence*** *of a sequence $(a_n)_{n=1}^\infty$ is a sequence of the form*

$$(a_{n_k})_{k=1}^\infty \quad = \quad a_{n_1}, a_{n_2}, a_{n_3}, \cdots,$$

where $(n_k)_{k=1}^\infty$ is a strictly increasing sequence of positive integers.

Example 1.25. *The constant sequences*

$$1, 1, 1, \cdots \qquad \textit{and} \qquad -1, -1, -1, \cdots$$

are subsequences of the sequence

$$(a_n)_{n=1}^\infty \quad = \quad 1, -1, 1, -1, \cdots .$$

(Take $n_k = 2k - 1$ and $n_k = 2k$.)

Proposition 1.26. *A sequence converges to L iff (i.e. if and only if) every subsequence of it converges to L.*

This proposition provides a way to recognize divergent sequences.

Example 1.27. *The sequence $a_n = (-1)^{n+1}$ diverges, because it has two subsequences which converge to different numbers -1 and 1.*

Recall that every convergent sequence is bounded, but the converse does not hold. The correct statement is the following:

Theorem 1.28 (Bolzano–Weierstrass Theorem, [3, Theorem 11.5.1])**.** *Every bounded sequence has a convergent subsequence.*

Proof. Let $(a_n)_{n=1}^\infty$ be a bounded sequence, and let α, β be real numbers such that $\alpha \leq a_n \leq \beta$ for all $n \in \mathbb{N}$. We divide the interval $I_0 := [\alpha, \beta]$ into two subintervals:

$$I_{01} := \Big[\alpha, \frac{\alpha+\beta}{2}\Big] \qquad \text{and} \qquad I_{02} := \Big[\frac{\alpha+\beta}{2}, \beta\Big].$$

Note that at least one of I_{01} or I_{02} contains infinitely many terms of the sequence $(a_n)_{n=1}^\infty$. We denote this subinterval by $I_1 := [\alpha_1, \beta_1]$.

Again, we divide I_1 into two subintervals: $I_{11} := [\alpha_1, \frac{\alpha_1+\beta_1}{2}]$ and $I_{12} := [\frac{\alpha_1+\beta_1}{2}, \beta_1]$. One of them contains contains infinitely many terms of the sequence $(a_n)_{n=1}^\infty$. We denote this subinterval by $I_2 := [\alpha_2, \beta_2]$.

In this way, we obtain a sequence of intervals:

$$I_0 = [\alpha,\beta] \supset I_1 = [\alpha_1,\beta_1] \supset I_2 = [\alpha_2,\beta_2] \supset I_3 = [\alpha_3,\beta_3] \supset \cdots .$$

Let a_{n_1} be a term in I_1. Since each interval I_m contains infinitely many terms of the sequence $(a_n)_{n=1}^{\infty}$, we can choose $a_{n_m} \in I_m$, i.e.

$$\alpha_m \leq a_{n_m} \leq \beta_m,$$

such that $n_1 < n_2 < \cdots < n_{m-1} < n_m < \cdots$.

We claim that $(a_{n_m})_{m=1}^{\infty}$ is a convergent subsequence of $(a_n)_{n=1}^{\infty}$.

To prove the claim, notice that the endpoints α_m and β_m of I_m satisfy the inequality:

$$\alpha \leq \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_m \leq \cdots \leq \cdots \leq \beta_m \leq \cdots \leq \beta_2 \leq \beta_1 \leq \beta.$$

This shows that the sequence $(\alpha_m)_{m=1}^{\infty}$ is increasing and bounded above, and the sequence $(\beta_m)_{m=1}^{\infty}$ is decreasing and bounded below. Thus, these two sequences are convergent. Furthermore, since

$$\beta_m - \alpha_m = \frac{\beta - \alpha}{2^m},$$

we have $\lim_{m\to\infty} \alpha_m = \lim_{m\to\infty} \beta_m$. Therefore, by the pinching theorem, we conclude that $(a_{n_m})_{m=1}^{\infty}$ is convergent, and $\lim_{m\to\infty} a_{n_m} = \lim_{m\to\infty} \alpha_m = \lim_{m\to\infty} \beta_m$. This completes the proof. □

As we saw in Example 1.27, a (bounded) sequence can convergent subsequences with different limits. The largest one is called the "limit superior" and the smallest one is called the "limit inferior." It is convenient to consider these number when we study divergent sequences.

Definition 1.29. *Let S be a bounded subset of $\mathbb{R}$. The **least upper bound** of S, denoted by $\sup S$, is the number $\alpha \in \mathbb{R}$ with the properties:*

(1) For all $s \in S$, $s \leq \alpha$. (That is, α is an upper bound of S.)
(2) For each $\epsilon > 0$, there exists $s_\epsilon \in S$ such that $\alpha - \epsilon < s_\epsilon \leq \alpha$. (That is, any number that is smaller than α is not an upper bound of S.)

*Dually, the **greatest lower bound** of S, denoted by $\inf S$, is the number $\beta \in \mathbb{R}$ with the properties:*

(1) For all $s \in S$, $s \geq \beta$. (That is, β is a lower bound of S.)
(2) For each $\epsilon > 0$, there exists $t_\epsilon \in S$ such that $\beta \leq t_\epsilon < \beta + \epsilon$. (That is, any number that is bigger than β is not a lower bound of S.)

Example 1.30. *Let $S = (0, 1) \subset \mathbb{R}$ and $T = \{1 + (-\frac{1}{2})^n \mid n = 0, 1, 2, \cdots\}$. Then $\sup S = 1$, $\inf S = 0$, $\sup T = 2$, and $\inf T = \frac{1}{2}$.*

Remark 1.31. *The following facts follow from the completeness axiom of real numbers:*

(1) If S is nonempty and **bound above** *(i.e. there exists $M \in \mathbb{R}$ such that $x \leq M$ $\forall x \in S$), then $\sup S$ exists in $\mathbb{R}$. (This statement is called the completeness axiom of real numbers.)*
(2) If S is nonempty and **bound below** *(i.e. there exists $m \in \mathbb{R}$ such that $x \geq m$ $\forall x \in S$), then $\inf S$ exists in $\mathbb{R}$.*

Let $(a_n)_{n=1}^{\infty}$ be a bounded sequence of real numbers, and let $A_n = \{a_n, a_{n+1}, \cdots\}$. Since $A_{n+1} \subset A_n$, the sequence $(\sup A_n)_{n=1}^{\infty}$ is decreasing, and $(\inf A_n)_{n=1}^{\infty}$ is increasing. Since $(a_n)_{n=1}^{\infty}$ is assumed to be bounded, so are $(\sup A_n)_{n=1}^{\infty}$ and $(\inf A_n)_{n=1}^{\infty}$. By Theorem 1.19, they are convergent sequences.

Definition 1.32. *The* ***limit superior*** *of $(a_n)_{n=1}^{\infty}$ is the limit $\lim_{n\to\infty}(\sup A_n)$, denoted by*

$$\limsup_{n\to\infty} a_n = \overline{\lim_{n\to\infty}}\, a_n = \lim_{n\to\infty}(\sup A_n).$$

Dually, the ***limit inferior*** *of $(a_n)_{n=1}^{\infty}$ is the limit $\lim_{n\to\infty}(\inf A_n)$, denoted by*

$$\liminf_{n\to\infty} a_n = \underline{\lim_{n\to\infty}}\, a_n = \lim_{n\to\infty}(\inf A_n).$$

Proposition 1.33. *Let $(a_n)_{n=1}^{\infty}$ be a bounded sequence, and $\alpha = \overline{\lim_{n\to\infty}}\, a_n$, $\beta = \underline{\lim_{n\to\infty}}\, a_n$. There exist subsequences $(a_{n_j})_{j=1}^{\infty}$ and $(a_{\tilde{n}_k})_{k=1}^{\infty}$ of $(a_n)_{n=1}^{\infty}$ such that $\lim_{j\to\infty} a_{n_j} = \alpha$ and $\lim_{k\to\infty} a_{\tilde{n}_k} = \beta$.*

Furthermore, if $(a_{\check{n}_l})_{l=1}^{\infty}$ is an arbitrary convergent subsequence of $(a_n)_{n=1}^{\infty}$, then $\beta \le \lim_{l\to\infty} a_{\check{n}_l} \le \alpha$.

Proposition 1.34. *A bounded sequence $(a_n)_{n=1}^{\infty}$ converges iff $\overline{\lim_{n\to\infty}}\, a_n = \underline{\lim_{n\to\infty}}\, a_n$. In that case, $\lim_{n\to\infty} a_n = \overline{\lim_{n\to\infty}}\, a_n = \underline{\lim_{n\to\infty}}\, a_n$.*

Proposition 1.35. *Let $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ be bounded (not necessarily convergent) sequences. Suppose there exists $N \in \mathbb{N}$ such that*

$$a_n \le b_n, \qquad \forall n \ge N. \tag{1}$$

Then

$$\overline{\lim_{n\to\infty}}\, a_n \le \overline{\lim_{n\to\infty}}\, b_n,$$
$$\underline{\lim_{n\to\infty}}\, a_n \le \underline{\lim_{n\to\infty}}\, b_n.$$

Proof. To prove the proposition, let $A_n = \{a_n, a_{n+1}, \cdots\}$, $B_n = \{b_n, b_{n+1}, \cdots\}$ and $C_n = \{c_n, c_{n+1}, \cdots\}$. Since $\sup B_n$ is an upper bound of B_n, it follows from the assumption (1) that $\sup B_n$ is also an upper bound of A_n. Thus $\sup A_n \le \sup B_n$ for all $n \in \mathbb{N}$, and by Proposition 1.23, we have $\overline{\lim_{n\to\infty}}\, a_n \le \overline{\lim_{n\to\infty}}\, b_n$. The other inequality $\underline{\lim_{n\to\infty}}\, a_n \le \underline{\lim_{n\to\infty}}\, b_n$ can be proved similarly. □

1.4.3. *Completeness of the real line.*

Definition 1.36. *A sequence $(a_n)_{n=1}^{\infty}$ is called a* ***Cauchy sequence*** *if for each $\epsilon > 0$, there exists $N = N_\epsilon$ such that*

$$|a_n - a_m| < \epsilon \qquad \text{whenever} \qquad n \ge N.$$

Proposition 1.37. *Every convergent sequence is a Cauchy sequence.*

Theorem 1.38 (Completeness of the Real Line)**.** *Every Cauchy sequence (of real numbers) is convergent.*

Proof. Let $(a_n)_{n=1}^{\infty}$ be a Cauchy sequence. By taking $\epsilon = 1$, we can see that $(a_n)_{n=1}^{\infty}$ is bounded. By the Bolzano–Weierstrass theorem, there exists a convergent subsequence $(a_{n_j})_{j=1}^{\infty}$ of $(a_n)_{n=1}^{\infty}$. Let $L = \lim_{j\to\infty} a_{n_j}$. Furthermore, since $(a_n)_{n=1}^{\infty}$ is Cauchy, for each $\epsilon > 0$, there exists $N = N_\epsilon$ such that

(1) $|a_{n_j} - L| < \frac{\epsilon}{2}$, $\forall j \geq N$;
(2) $|a_n - a_m| < \frac{\epsilon}{2}$, $\forall n, m \geq N$.

Note that since $(n_j)_{j=1}^{\infty}$ is a strictly increasing sequence of positive integers, we have $n_j \geq j$ for all $j \in \mathbb{N}$. Therefore,

$$|a_j - L| \leq |a_n - a_{n_j}| + |a_{n_j} - L| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \qquad \forall j \geq N.$$

This proves that $\lim_{n\to\infty} a_n = L$. □

Example 1.39. *Let $(a_n)_{n=1}^{\infty}$ be a sequence with the property:*

$$|a_{n+1} - a_n| \leq \frac{1}{2^n}$$

for any $n \in \mathbb{N}$. Prove that $(a_n)_{n=1}^{\infty}$ is convergent.

1.5. **Relative rate of growth.** We state the following definitions in terms of sequences. Similar definitions also hold for functions (replacing n by x).

Definition 1.40 ([1, Page 470, Section 7.4])**.** *Let a_n and b_n be positive for sufficiently large n.*

*(1) We say that b_n **grows faster than** a_n (as $n \to \infty$) if*

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 0.$$

*Equivalently, we say that a_n **grows slower than** b_n (as $n \to \infty$), or that a_n is **of at most the order of** b_n (as $n \to \infty$). We denote this by $a_n = o(b_n)$ (read as "a_n is little-oh of b_n").*

*(2) We say that a_n and b_n **grow at the same rate** (as $n \to \infty$) if*

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0,$$

where L is finite and positive.

*(3) We say that a_n is **of at most the order of** b_n (as $n \to \infty$) if there exists a positive number M such that*

$$\frac{a_n}{b_n} \leq M$$

for all sufficiently large n. We write this as $a_n = O(b_n)$ (read as "a_n is big-oh of b_n").

Remark 1.41. *From the definitions, we have:*

(1) $a_n = o(b_n)$ implies $a_n = O(b_n)$.
(2) If a_n and b_n grow at the same rate, then $a_n = O(b_n)$ and $b_n = O(a_n)$. (Exercise.)

Example 1.42. *We compare the growth rates of the following sequences:*

(1) $n = o(n^2)$ as $n \to \infty$.
(2) If $k < l$, then $n^k = o(n^l)$ as $n \to \infty$.
(3) If $k \leq l$, then $n^k = O(n^l)$ as $n \to \infty$.
(4) $n^2 = o(n^3 + 1)$ as $n \to \infty$.
(5) $n + \sin n = O(n)$ as $n \to \infty$.
(6) $2^n = o(n!)$ as $n \to \infty$.
(7) If $0 < a < b$, then $a^n = o(b^n)$ as $n \to \infty$.

(8) $n + 2^n = O(2^n)$ *as* $n \to \infty$.

Example 1.43 (Searching algorithms)**.** *Question: Find a given word in a dictionary. (Design algorithms.) Assume there are n words in the dictionary.*

Algorithm 1: *Check each word one by one. In the worst case, the desired word is the last one we check. Thus, we need n steps to find the word. We say that the* time complexity *of Algorithm 1 is* $O(n)$.

Algorithm 2 (binary search): *Since the words in a dictionary are in alphabetical order, we open a page in the middle (dividing the words into two groups) and check whether the word comes before or after this page. We repeat this process until we find the word.*

In Algorithm 2, the worst case requires $\log_2 n$ *steps. Thus, the time complexity of Algorithm 2 is* $O(\log n)$.

Since it is a fact that $\dfrac{\log n}{n} \to 0$ *as* $n \to \infty$, *Algorithm 1 requires more steps in the worst case. Hence, Algorithm 1 is considered slower than Algorithm 2 for large n.*

2. Limit and continuity

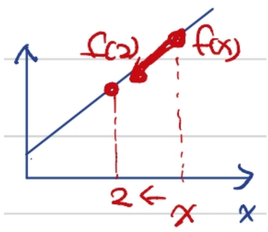
Remarks on organization. We put the precise definitions of limits and one-side limits ([1, Section 2.3 & 2.4]) in Section 2.2, put how limits interplay with basic operations (part of [1, Section 2.2 & 2.4] and more) in Section 2.3, and put the pinching theorem ([1, Theorem 4 in Section 2.2]) and limits of trigonometric functions (second half of [1, Section 2.4]) in Section 2.4.

2.1. Intuitive examples.

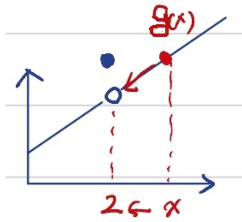
Example 2.1. *Let*

$$f(x) = 4x + 5 \qquad \text{and} \qquad g(x) = \begin{cases} 4x + 5, & x \neq 2 \\ 15, & x = 2. \end{cases}$$

We have $\lim_{x\to 2} f(x) = f(2) = 13$, *and* $\lim_{x\to 2} g(x) = 13 \neq 15 = g(2)$.



(a) $\lim_{x\to 2} f(x) = f(2) = 13$



(b) $\lim_{x\to 2} g(x) \neq g(2)$

Figure 4. Example 2.1

Example 2.2. *Let*

$$f(x) = \frac{x^2 - 9}{x - 3}.$$

Note that $f(x)$ *is NOT defined at* $x = 3$, *because denominator cannot be* 0. *However,*

$$f(x) = \frac{(x + 3)(x - 3)}{x - 3} = x + 3$$

if $x \neq 3$. *Thus,* $\lim_{x\to 3} f(x) = 3 + 3 = 6$.

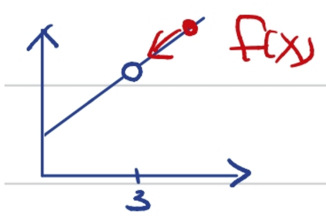


Figure 5. $\lim_{x\to 3} f(x) = 6$

Example 2.3. *Let*

$$f(x) = \begin{cases} 1, & x > 0, \\ 0, & x = 0, \\ -1, & x < 0. \end{cases}$$

The left-hand limit *of f is* $\lim_{x\to 0^-} f(x) = -1$, *and the* right-hand limit *of f is* $\lim_{x\to 0^+} f(x) = 1$. *Since* $\lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x)$, *the limit* $\lim_{x\to 0} f(x)$ *does NOT exist.*

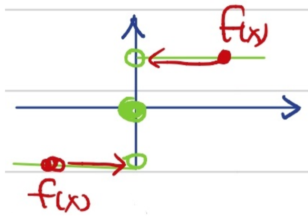


FIGURE 6. $\lim_{x\to 0} f(x)$ does NOT exist.

Example 2.4. *Let*

$$f(x) = \begin{cases} 1 - x^2, & x \leq 1, \\ \dfrac{1}{x-1}, & x > 1. \end{cases}$$

The left-hand limit *of f is* $\lim_{x\to 1^-} f(x) = 0$, *and the* right-hand limit *of f does NOT exist. (Some people say* $\lim_{x\to 1^+} f(x) = +\infty$ *in this case.) Since one of* $\lim_{x\to 1^-} f(x)$ *and* $\lim_{x\to 1^+} f(x)$ *doesn't exist, the limit* $\lim_{x\to 1} f(x)$ *does NOT exist.*

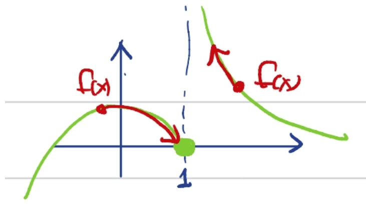


FIGURE 7. $\lim_{x\to 1} f(x)$ does NOT exist.

Example 2.5. *Let*

$$f(x) = \sin(\frac{\pi}{x}).$$

All the (one-side) limits $\lim_{x\to 0^-} f(x)$, $\lim_{x\to 0^+} f(x)$ *and* $\lim_{x\to 0} f(x)$ *do NOT exist.*

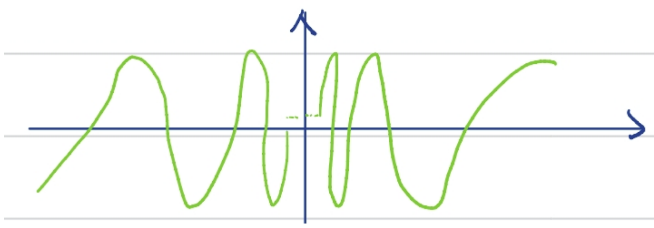


FIGURE 8. $\lim_{x\to 0} f(x)$ does NOT exist.

2.2. **Precise definitions of limit and one-side limits.** One may find more details in [3, Section 2.2] and [1, Section 2.3 & 2.4].

Definition 2.6 ([1, page 91 & page 101], [3, Definition 2.2.1, Definition 2.2.7, Definition 2.2.8]). *Let f be a function defined on $(c-p, c+p) \setminus \{c\}$, we say*

(1) $\lim\limits_{x \to c} f(x) = L$ *if* $\forall\, \varepsilon > 0$, $\exists\, \delta > 0$, *such that if* $0 < |x - c| < \delta$, *then* $|f(x) - L| < \varepsilon$;
(2) $\lim\limits_{x \to c^-} f(x) = L$ *if* $\forall\, \varepsilon > 0$, $\exists\, \delta > 0$, *such that if* $c - \delta < x < c$, *then* $|f(x) - L| < \varepsilon$;
(3) $\lim\limits_{x \to c^+} f(x) = L$ *if* $\forall\, \varepsilon > 0$, $\exists\, \delta > 0$, *such that if* $c < x < c + \delta$, *then* $|f(x) - L| < \varepsilon$.

Remark 2.7. *Note that in each case of Definition 2.6, we exclude the point $x = c$. This is the reason why the value of a function $f(x)$ at $x = c$ does NOT affect the limit of f at $x = c$.*

Example 2.8. *Prove that* $\lim\limits_{x \to 2} f(x) = 13$, *where* $f(x) = \begin{cases} 4x + 5, & x \neq 2 \\ 15, & x = 2. \end{cases}$

Proof. Given any $\varepsilon > 0$, we choose $\delta = \varepsilon/5 > 0$ (there many choices). If $0 < |x - 2| < \varepsilon/5$, then

$$|f(x) - 13| = |4x + 5 - 13| = 4|x - 2| < 4(\varepsilon/5) < \varepsilon.$$

This completes the proof. □

Proposition 2.9. *Let f be a function defined on $(c - p, c + p) \setminus \{c\}$. Then*

(1) $\lim\limits_{x \to c} f(x) = L$ *iff* $\lim\limits_{n \to \infty} f(x_n) = L$ *for any sequence* $(x_n)_{n=1}^{\infty}$ *with* $x_n \in (c - p, c + p) \setminus \{c\}$ *and* $\lim\limits_{n \to \infty} x_n = c$;
(2) $\lim\limits_{x \to c^-} f(x) = L$ *iff* $\lim\limits_{n \to \infty} f(x_n) = L$ *for any sequence* $(x_n)_{n=1}^{\infty}$ *with* $x_n \in (c - p, c)$ *and* $\lim\limits_{n \to \infty} x_n = c$;
(3) $\lim\limits_{x \to c^-} f(x) = L$ *iff* $\lim\limits_{n \to \infty} f(x_n) = L$ *for any sequence* $(x_n)_{n=1}^{\infty}$ *with* $x_n \in (c, c + p)$ *and* $\lim\limits_{n \to \infty} x_n = c$.

Proposition 2.10 ([1, Theorem 6 in Section 2.4] & [3, Theorem 2.3.1]). *Let $f(x)$ be a function defined near a point c. Then*

(1) $\lim\limits_{x \to c} f(x) = L$ *if and only if* $\lim\limits_{x \to c^-} f(x) = L$ *and* $\lim\limits_{x \to c^+} f(x) = L$;
(2) $\lim\limits_{x \to c} f(x)$ *may or may not exist. If* $\lim\limits_{x \to c} f(x)$ *exists, then* $\lim\limits_{x \to c} f(x)$ *is unique.*

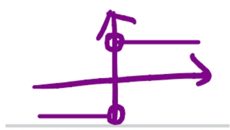


FIGURE 9. For this function f, the limit $\lim\limits_{x \to 0} f(x)$ doesn't exist.

If $\lim\limits_{x \to c} f(x)$ exists, we usually apply certain limit laws to compute it (see the next subsection). If it does not, the following are three common ways to prove it:

(1) Prove that $\lim\limits_{x \to c^-} f(x) \neq \lim\limits_{x \to c^+} f(x)$.
(2) Find a sequence $(x_n)_{n=1}^{\infty}$ with $\lim\limits_{n \to \infty} x_n = c$ such that $\lim\limits_{n \to \infty} f(x_n)$ does not exist.
(3) Find two sequences $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ with $\lim\limits_{n \to \infty} x_n = \lim\limits_{n \to \infty} y_n = c$ such that

$$\lim_{n \to \infty} f(x_n) \neq \lim_{n \to \infty} f(y_n).$$

Example 2.11. *Let*

$$f(x) = \begin{cases} 1, & x > 0, \\ 0, & x = 0, \\ -1, & x < 0, \end{cases} \qquad g(x) = \begin{cases} 1 - x^2, & x \le 1, \\ \dfrac{1}{x-1}, & x > 1, \end{cases} \qquad h(x) = \sin(\frac{\pi}{x}).$$

Show that none of the limits $\lim_{x\to 0} f(x)$, $\lim_{x\to 1} g(x)$, *and* $\lim_{x\to 0} h(x)$ *exists.*

2.3. **Limit laws.** To find the limit of a function, we use the following limit laws.

Theorem 2.12 ([1, Theorem 1 in Section 2.2], [3, Theorem 2.3.2])**.** *If* $\lim_{x\to c} f(x)$ *and* $\lim_{x\to c} g(x)$ *exist, and if k and c are fixed numbers, then we have the following:*

(1) $\lim_{x\to c} x = c$; 

(2) $\lim_{x\to c} |x| = |c|$; 

(3) $\lim_{x\to c} k = k$; 

(4) $\lim_{x\to c}(f(x) + g(x)) = \lim_{x\to c} f(x) + \lim_{x\to c} g(x)$;

(5) $\lim_{x\to c}(k \cdot f(x)) = k \cdot \lim_{x\to c} f(x)$;

(6) $\lim_{x\to c}(f(x) \cdot g(x)) = \lim_{x\to c} f(x) \cdot \lim_{x\to c} g(x)$.

Example 2.13. *Evaluate the following limits.*

(1) $\lim_{x\to 3} x^2 = \lim_{x\to 3} x \cdot \lim_{x\to 3} x = 3 \cdot 3 = 9.$

(2) $\lim_{x\to 3} x^2 - x = \lim_{x\to 3} x^2 + (-1) \cdot x = \lim_{x\to 3} x^2 - \lim_{x\to 3} x = 9 - 3 = 6.$

(3) $\lim_{x\to -2} |x|(x^3 + 5x) = \lim_{x\to -2} |x| \cdot \lim_{x\to -2} x^3 + 5x = |-2| \cdot (\lim_{x\to -2} x^3 + \lim_{x\to -2} 5x) = -36.$

Remark 2.14 (cf. [1, Theorem 1 in Section 2.2], [3, Section 2.3])**.** *Let* $k_1, \cdots, k_n, a_0, \cdots, a_n$ *be fixed numbers. Assume that* $\lim_{x\to c} f(x)$, $\lim_{x\to c} f_1(x), \cdots, \lim_{x\to c} f_n(x)$ *and* $\lim_{x\to c} g(x)$ *exist. The above computation techniques imply the following:*

(1) $\lim_{x\to c}(f(x) - g(x)) = \lim_{x\to c} f(x) - \lim_{x\to c} g(x)$;

(2) $\lim_{x\to c} (k_1 f_1(x) + \cdots k_n f_n(x)) = (k_1 \lim_{x\to c} f_1(x)) + \cdots + (k_n \lim_{x\to c} f_n(x))$;

(3) $\lim_{x\to c} (f_1(x) \cdots f_n(x)) = \lim_{x\to c} f_1(x) \cdot \cdots \cdot \lim_{x\to c} f_n(x)$;

(4) $\lim_{x\to c}(a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0) = a_n c^n + a_{n-1}c^{n-1} + \cdots a_0$;

(5) $\lim_{x\to c} \sqrt[n]{x} = \sqrt[n]{c}$ *for* $c > 0$ *and* $n \in \mathbb{N}$. *(If* $c < 0$*, even the definition of* $\sqrt[n]{c}$ *is not clear.)*

We discussed how the limits interplay with $\pm$, $\times$ and $\sqrt[n]{}$. Next, we consider the limit of a *quotient* of functions.

Remark 2.15 ([3, Section 2.3])**.** *Consider the limit* $\lim_{x\to c}\Big(\dfrac{f(x)}{g(x)}\Big)$*, provided* $\lim_{x\to c} f(x)$ *and* $\lim_{x\to c} g(x)$ *exist. There are three possible cases:*

case 1. *(*[3, Theorem 2.3.8]*) If* $\lim_{x\to c} g(x) \neq 0$*, then* $\lim_{x\to c}\Big(\dfrac{f(x)}{g(x)}\Big) = \dfrac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$.

(One can find a polynomial version in [1, Theorem 3 in Section 2.2].*)*

case 2. *(*[3, Theorem 2.3.10]*) If* $\lim_{x\to c} g(x) = 0$ *and* $\lim_{x\to c} f(x) \neq 0$*, then* $\lim_{x\to c}(\dfrac{f(x)}{g(x)})$ *does NOT exist.*

case 3. If $\lim_{x\to c} g(x) = \lim_{x\to c} f(x) = 0$*, then* anything *can happen.*

Basic technique for case 3: simplify $\dfrac{f(x)}{g(x)}$ *before taking the limit.*

Example 2.16. *Evaluate the following limits.*

(1) *(case 1)* $\lim_{x\to 2} \dfrac{1}{x^3-1} = \dfrac{1}{7}$.

(2) *(case 1)* $\lim_{x\to 2} \dfrac{3x-5}{x^2+1} = \dfrac{1}{5}$.

(3) *(case 2)* $\lim_{x\to 1} \dfrac{|x|}{x-1}$ *does NOT exist.*

(4) *(case 3)* $\lim_{x\to 1} \dfrac{|x|(x-1)}{x-1} = \lim_{x\to 1} |x| = 1$.

(5) *(case 3)* $\lim_{x\to 1} \dfrac{x-1}{(x-1)^2} = \lim_{x\to 1} \dfrac{1}{x-1}$ *does NOT exist.*

(6) *(case 3)* $\lim_{x\to 9} \dfrac{x-9}{\sqrt{x}-3} = \lim_{x\to 9} . \dfrac{(\sqrt{x}-3)(\sqrt{x}+3)}{(\sqrt{x}-3)} = \lim_{x\to 9} \sqrt{x}+3 = 6$.

Remark 2.17. *If neither* $\lim_{x\to c} f(x)$ *nor* $\lim_{x\to c} g(x)$ *exists, then, similar to case 3, the limit* $\lim_{x\to c} \left(\dfrac{f(x)}{g(x)}\right)$ *of quotient could be anything. The basic technique for this case is also simplifying before taking the limit. For example,*

$$\lim_{h\to 0} \frac{1+\frac{1}{h}}{2+\frac{1}{h}} = \lim_{h\to 0} \frac{h+1}{2h+1} = \frac{1}{1} = 1.$$

Remark 2.18. *All the properties in Theorem 2.12, Remark 2.14 and Remark 2.15 hold for one-side limits.*

Example 2.19. *Evaluate the following one-sides limits.*

(1) $\lim_{x\to 0^-} \dfrac{|x|}{x} \cdot \dfrac{\sqrt{x+1}}{x-1} = \lim_{x\to 0^-} \dfrac{-x}{x} \cdot \dfrac{\sqrt{x+1}}{x-1} = -1 \cdot \dfrac{\sqrt{1}}{-1} = 1$.

(2) $\lim_{x\to 0^+} \dfrac{x^{\frac{3}{2}}}{|x|} = \lim_{x\to 0^+} \sqrt{x} = \sqrt{0} = 0$.

Remark 2.20. *Search "limit calculator" when you compute limits with a computer and internet.*

2.4. **Pinching theorem and trigonometric limits.** The following theorem is referred to as the pinching theorem, squeeze theorem or sandwich theorem.

Theorem 2.21 ([1, Theorem 4 in Section 2.2], [3, Theorem 2.5.1])**.** *Let* f, g, h *be functions defined around* $x = c \in \mathbb{R}$*. Suppose there exists* $p > 0$ *such that for any* x *with the property* $0 < |x-c| < p$*, one has the inequality* $h(x) \le f(x) \le g(x)$*. If* $\lim_{x\to c} h(x) = \lim_{x\to c} g(x) = L$*, then* $\lim_{x\to c} f(x) = L$*.*

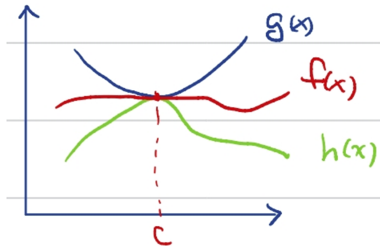


FIGURE 10. Pinching theorem

The following proposition can be proved by the pinching theorem.

Lemma 2.22. *Let f be a function defined near c. Then* $\lim_{x\to c} f(x) = L$ *if and only if* $\lim_{x\to c} |f(x) - L| = 0$.

Proof. Recall from Definition 2.6 that

- $\lim_{x\to c} f(x) = L$ means that $\forall\ \varepsilon > 0$, $\exists\ \delta > 0$, such that if $0 < |x - c| < \delta$, then
$$|f(x) - L| < \varepsilon.$$
- $\lim_{x\to c} |f(x) - L| = 0$ means that $\forall\ \varepsilon > 0$, $\exists\ \delta > 0$, such that if $0 < |x - c| < \delta$, then
$$\Big||f(x) - L| - 0\Big| < \varepsilon.$$

They mean exactly the same thing because $\Big||f(x) - L| - 0\Big| = |f(x) - L|$. □

Lemma 2.23. *Suppose that*
$$\lim_{x\to c} g(x) = d \qquad \textit{and} \qquad \lim_{y\to d} f(y) = L.$$
If there exists $p > 0$ such that $g(x) \neq d$ for any $0 < |x - c| < p$, then
$$\lim_{x\to c} f(g(x)) = L.$$

Proof. Given any $\epsilon > 0$, there exists $\delta > 0$ such that $|f(y) - L| < \epsilon$ when $0 < |y - d| < \delta$. For this $\delta > 0$, there exists $\delta' \in (0, p)$ such that $|g(x) - d| < \delta$ when $0 < |x - c| < \delta'$. Since $g(x) \neq d$ for any $0 < |x - c| < p$, we have $0 < |g(x) - d| < \delta$ whenever $0 < |x - c| < \delta'$, and thus
$$|f(g(x)) - L| < \epsilon, \qquad \text{when } 0 < |x - c| < \delta'.$$
This completes the proof. □

Now we apply the pinching theorem to find the limits of trigonometric functions.

Theorem 2.24 ([3, Equation (2.5.2), Equation (2.5.3)])**.** $\lim_{x\to 0} \sin x = 0$ *and* $\lim_{x\to 0} \cos x = 1$.

Proof. Consider a sector with radius 1 like the figure below. From the picture, we know
$$0 \leq |\sin x| \leq |x|, \quad \forall x \in [-\frac{\pi}{2}, \frac{\pi}{2}].$$

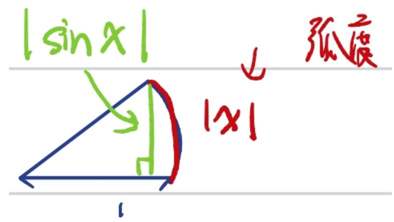


FIGURE 11. The sector with radius = 1, angle = x

Since $\lim_{x\to 0} 0 = \lim_{x\to 0} |x| = 0$, by the pinching theorem, we have $\lim_{x\to 0} |\sin x| = 0$ which implies $\lim_{x\to 0} \sin x = 0$. Furthermore, we have $\lim_{x\to 0} \cos x = \lim_{x\to 0} \sqrt{1-\sin^2 x} = 1$. □

Example 2.25. *Show that* $\lim_{x\to c} \sin x = \sin c$ *and* $\lim_{x\to c} \cos x = \cos c$.

- $\lim_{x\to c} \sin x = \lim_{h\to 0} \sin(c+h) = \lim_{h\to 0} \sin c \cos h + \cos c \sin h = \sin c$.
- $\lim_{x\to c} \cos x = \lim_{h\to 0} \cos(c+h) = \lim_{h\to 0} \cos c \cos h - \sin c \sin h = \cos c$.

Theorem 2.26 ([1, Theorem 7], [3, Equation (2.5.5)])**.** $\lim_{x\to 0} \frac{\sin x}{x} = 1$.

Proof. We discuss this limit via the two one-side limits:

- For $x \in [0, \frac{\pi}{2}]$, we consider a sector with radius 1, angle x and the two triangles defined as in the figure below.

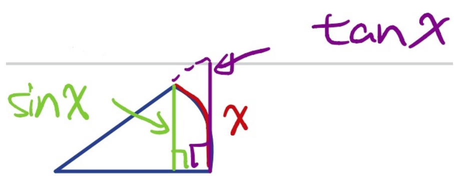
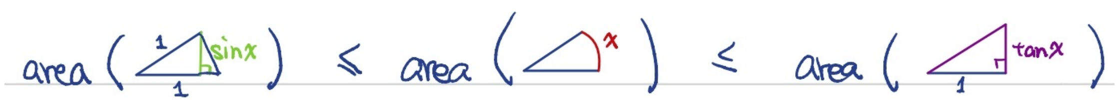


FIGURE 12. A sector (red) and two triangles (green/purple)

  Comparing the areas, we have



  Thus, $\frac{1}{2}\cdot 1 \cdot \sin x \le \pi \cdot \frac{x}{2\pi} \le \frac{1}{2} \cdot 1 \cdot \tan x \quad \Rightarrow \quad \cos x \le \frac{\sin x}{x} \le 1, \quad x \in [0, \frac{\pi}{2}]$. Since $\lim_{x\to 0^+} \cos x = \lim_{x\to 0^+} 1 = 1$, it follows from the pinching theorem that $\lim_{x\to 0^+} \frac{\sin x}{x} = 1$.
- For $x \in [-\frac{\pi}{2}, 0]$, we can use the same method to show $\lim_{x\to 0^-} \frac{\sin x}{x} = 1$.

Therefore, we have $\lim_{x\to 0} \frac{\sin x}{x} = 1$. □

Corollary 2.27 ([3, Equation (2.5.5)])**.** $\lim_{x\to 0} \frac{1-\cos x}{x} = 0$.

Proof. Note that

$$\frac{1-\cos x}{x} = \frac{1-\cos x}{x}\cdot\frac{1+\cos x}{1+\cos x} = \frac{1-\cos^2 x}{x(1+\cos x)} = \frac{\sin^2 x}{x(1+\cos x)} = \frac{\sin x}{x}\cdot\frac{\sin x}{1+\cos x}.$$

Since $\lim_{x\to 0}\frac{\sin x}{1+\cos x} = \frac{0}{1+1} = 0$, we have $\lim_{x\to 0}\frac{1-\cos x}{x} = \lim_{x\to 0}\frac{\sin x}{x}\cdot\lim_{x\to 0}\frac{\sin x}{1+\cos x} = 0.$ □

Corollary 2.28 ([3, Equation (2.5.6)]). *Let $a \neq 0$, and let $y = ax$.*

- $\lim_{x\to 0}\frac{\sin(ax)}{ax} = \lim_{y\to 0}\frac{\sin y}{y} = 1.$
- $\lim_{x\to 0}\frac{1-\cos(ax)}{ax} = \lim_{y\to 0}\frac{1-\cos y}{y} = 0.$

Example 2.29. *Compute the limits.*

- $\lim_{x\to 0}\frac{\sin(4x)}{3x} = \lim_{x\to 0}\frac{4}{3}\frac{\sin(4x)}{4x} = \frac{4}{3}.$
- $\lim_{x\to 0} x\cdot\cot(3x) = \lim_{x\to 0} x\cdot\frac{\cos 3x}{\sin 3x} = \lim_{x\to 0}\frac{1}{3}\cdot\frac{3x}{\sin 3x}\cdot\cos 3x = \frac{1}{3}.$
- $\lim_{x\to\pi/4}\frac{\sin(x-\pi/4)}{(x-\pi/4)^2} = \lim_{y\to 0}\frac{\sin y}{y}\cdot\frac{1}{y}$ *does NOT exist.* *(let $y = x - \pi/4$.)*
- $\lim_{x\to 0}\frac{x^2}{\sec x - 1} = 2$, *because*

$$\begin{aligned}\lim_{x\to 0}\frac{x^2}{\sec x-1} &= \lim_{x\to 0}\frac{x^2\cos x}{1-\cos x} = \lim_{x\to 0}\frac{(1+\cos x)x^2\cos x}{(1+\cos x)(1-\cos x)}\\ &= \lim_{x\to 0}\frac{x^2\cos x + x^2\cos^2 x}{\sin^2 x} = \lim_{x\to 0}\Big(\frac{x}{\sin x}\Big)^2(\cos x+\cos^2 x) = 2.\end{aligned}$$

2.5. **Limits involving infinity.**

Definition 2.30. *[cf. Definition 2.6] We say*

(1) $\lim_{x\to\infty} f(x) = L$ *if* $\forall\,\varepsilon > 0$, $\exists$ *a number* M, *such that if* $x > M$, *then* $|f(x) - L| < \varepsilon$;
(2) $\lim_{x\to-\infty} f(x) = L$ *if* $\forall\,\varepsilon > 0$, $\exists$ *a number* N, *such that if* $x < N$, *then* $|f(x) - L| < \varepsilon$;
(3) $\lim_{x\to c^+} f(x) = +\infty$ *if* $\forall\, M > 0$, $\exists\,\delta > 0$, *such that if* $c < x < c+\delta$, *then* $f(x) > M$;
(4) $\lim_{x\to c^+} f(x) = -\infty$ *if* $\forall\, N < 0$, $\exists\,\delta > 0$, *such that if* $c < x < c+\delta$, *then* $f(x) < N$;
(5) $\lim_{x\to c^-} f(x) = +\infty$ *if* $\forall\, M > 0$, $\exists\,\delta > 0$, *such that if* $c-\delta < x < c$, *then* $f(x) > M$;
(6) $\lim_{x\to c^-} f(x) = -\infty$ *if* $\forall\, N < 0$, $\exists\,\delta > 0$, *such that if* $c-\delta < x < c$, *then* $f(x) < N$.

Remark 2.31. *If* $\lim_{x\to\infty} f(x) = L$, *then the sequence* $(a_n = f(n))_{n=1}^{\infty}$ *also converges to* L.

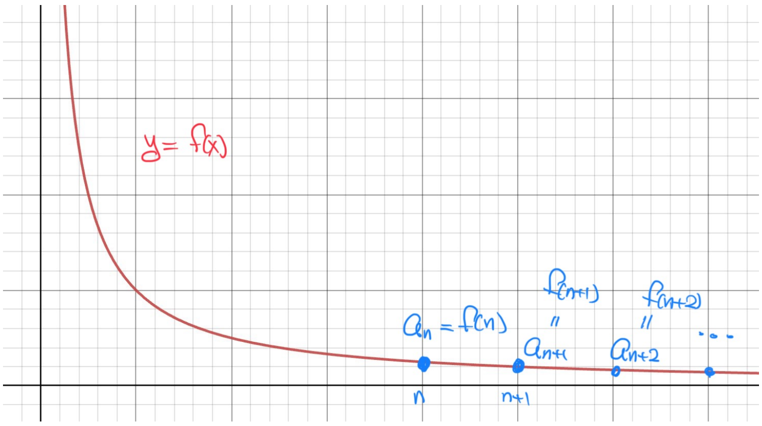


FIGURE 13. Function and sequence.

Remark 2.32 ([1, Theorem 8 in Section 2.5]). *The limit laws in Theorem 2.12, Remark 2.14 and Remark 2.15 are still true if we replace* $\lim_{x\to c}$ *by* $\lim_{x\to\infty}$ *or* $\lim_{x\to-\infty}$.

Example 2.33. *Compute the limits.*

- $\lim_{x\to\infty} \frac{1}{x} = \lim_{x\to-\infty} \frac{1}{x} = 0.$
- $\lim_{x\to\infty} \left(\frac{1}{x}\right)^2 = \left(\lim_{x\to\infty} \frac{1}{x}\right)^2 = 0.$
- $\lim_{x\to\infty} \frac{x^2+x}{2x^2+1} = \lim_{x\to\infty} \frac{1+(1/x)}{2+(1/x^2)} = \frac{1+0}{2+0} = \frac{1}{2}.$

Skip the formal definitions of asymptotes. Explain the geometry by the two examples.

Definition 2.34. *We say*

(1) *the line $x = c$ is called a* ***vertical asymptote*** *of a function f if*

$$f(x) \to \pm\infty \quad as \;\; x \to c^- \;\; (or \;\; c^+);$$

(2) *the line $y = L$ is called a* ***horizontal asymptote*** *of a function f if*

$$f(x) \to L \quad as \;\; x \to \infty \;\; (or \;\; -\infty).$$

Remark 2.35 (Geometric Meaning of Limits with Infinity). *Equations involving limits with infinity indicate the presence of asymptotes. Here are a couple of examples.*

(1) $f(x) = \frac{1}{x}$

FIGURE 14. The asymptote of function f

(2) $f(x) = \dfrac{\cos x}{x}, \quad x > 0$

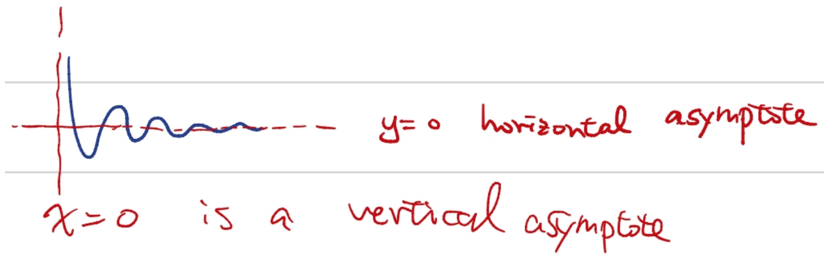


FIGURE 15. The asymptote of function f

2.6. **Continuity.**

Definition 2.36 ([1, Page 121, Page 123, Section 2.6], [3, Definition 2.4.1, Definition 2.4.5]). *Let f be a function, and c be a real number. The function f is **continuous at** c if f is defined in $(c-p, c+p)$ for some $p > 0$, and*

$$\lim_{x \to c} f(x) = f(c).$$

*We say f is **discontinuous at** c if f is not continuous at c.*

*The function f is **right-continuous at** c (or **continuous from the right**) if f is defined in $[c, c+p)$ for some $p > 0$, and*

$$\lim_{x \to c^+} f(x) = f(c).$$

*The function f is **left-continuous at** c (**continuous from the left**) if f is defined in $(c-p, c]$ for some $p > 0$, and*

$$\lim_{x \to c^-} f(x) = f(c).$$

*The function f is **continuous on the open interval** (a, b) if f is continuous at each point $c \in (a, b)$. The function f is **continuous on the closed interval** $[a, b]$ if (i) f is continuous on (a, b), (ii) f is right-continuous at a, and (iii) f is left-continuous at b. The concepts f is continuous on $(a, b]$, $[a, b)$, $(-\infty, b]$, $[a, \infty)$ or $(-\infty, \infty)$ can be defined similarly.*

Example 2.37. *The functions x and $|x|$ are continuous on $(-\infty, \infty)$. The function $f(x) = 0$ for $x < 0$ and $f(x) = 1$ for $x \geq 0$ is right-continuous at 0 and discontinuous at 0.*

Remark 2.38. *By Theorem 2.12, Remark 2.14 and Example 2.25, the functions*

- *any polynomials,*
- $\sin x$,
- $\cos x$,
- $|x|$

are continuous on $(-\infty, \infty)$, and the function $f(x) = \sqrt[n]{x}$ is continuous at any $c > 0$. (In fact, $f(x) = \sqrt[n]{x}$ is also right-continuous at $x = 0$.)

Remark 2.39. *A function f is continuous at c iff*

(1) *$f(c)$ exists,*

(2) $\lim_{x\to c} f(x)$ *exists, and*
(3) $f(c) = \lim_{x\to c} f(x)$.

iff for any sequence $(x_n)_{n=1}^{\infty}$ *with* $\lim_{n\to\infty} x_n = c$*, we have* $\lim_{n\to\infty} f(x_n) = f(c)$.

Example 2.40. *Determine whether the function is continuous.*

- $f(x) = \frac{1}{2}x + 1$ *is continuous on* $(-\infty, \infty)$.
- $g(x) = \begin{cases} \frac{1}{2}x + 1, & x \neq 1, \\ 3, & x = 1 \end{cases}$ *is discontinuous at* 1.
- $h(x) = \begin{cases} 1, & x < 1, \\ 2, & x \geq 1 \end{cases}$ *is discontinuous at* 1.

Theorem 2.41 ([1, Theorem 9 in Section 2.6], [3, Theorem 2.4.2])**.** *If* f*,* g *are continuous at* c *and* k *is a constant, then*

$$f + g, \quad f - g, \quad k \cdot f, \quad f \cdot g$$

are continuous at c*. Furthermore, if* $g(c) \neq 0$*, then* $\frac{f}{g}$ *is also continuous at* c.

Example 2.42. *Determine the continuity of the function*

$$f(x) = \begin{cases} 3|x| + \dfrac{x^3 - x}{x^2 - 5x + 6} & \text{if } x \neq 2, 3, \\ 0 & \text{if } x = 2, 3. \end{cases}$$

Solution. Since the function $3|x|$, $x^3 - x$ and $x^2 - 5x + 6$ are all continuous functions, and since $x^2 - 5x + 6 \neq 0$ for any $x \neq 2, 3$, it follows from Theorem 2.41 that f is continuous at any $c \neq 2, 3$. Moreover, f has *essential discontinuities* (i.e. the limit of f doesn't exist) at $x = 2, 3$, because neither $\lim_{x\to 2} f(x)$ not $\lim_{x\to 3} f(x)$ exists. □

Theorem 2.43 ([1, Theorem 10 in Section 2.6], [3, Theorem 2.4.4])**.** *If*

- g *is continuous at* c*, and*
- f *is continuous at* $g(c)$*,*

then $f \circ g$ *is continuous at* c.

Example 2.44. *Determine the continuity of the following functions.*

(1) $f(x) = \sqrt{\dfrac{x^2 + 1}{x - 3}}$ *is continuous on* $(3, \infty)$.
(2) $g(x) = \dfrac{1}{5 - \sqrt{x^2 + 16}}$ *is continuous at any* $c \neq \pm 3$.

Example 2.45. *Determine whether the function is continuous.*

(1) $f(x) = \sqrt{1 - x^2}$ *is continuous on* $[-1, 1]$.
(2) $g(x) = \dfrac{1}{\sqrt{1 - x^2}}$ *is continuous on* $(-1, 1)$.

(3) $h(x) = \begin{cases} 2x+1, & x \le 0, \\ 1, & 0 < x \le 1, \\ x^2+1, & 1 < x. \end{cases}$ *is continuous on* $(-\infty, 1]$*, and on* $(1, \infty)$*, but NOT on* $(-\infty, \infty)$*.*

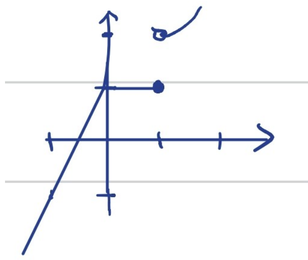


FIGURE 16. h is continuous from the left at 1

Application to limits.

Theorem 2.46 ([1, Theorem 11 in Section 2.6])**.** *If* $\lim_{x\to c} f(x) = b$ *and* g *is continuous at the point* b*, then*

$$\lim_{x\to c} g(f(x)) = g\Big(\lim_{x\to c} f(x)\Big) = g(b).$$

Note that c in the above theorem can be $\pm\infty$.

Example 2.47. *Compute the limit.*

$$\begin{aligned} \lim_{x\to\pi/2} \cos\Big(2x + \sin(\frac{3\pi}{2} + x)\Big) &= \cos\Big(\lim_{x\to\pi/2} 2x + \lim_{x\to\pi/2} \sin(\frac{3\pi}{2} + x)\Big) \\ &= \cos\Big(\pi + \sin(\lim_{x\to\pi/2}(\frac{3\pi}{2} + x))\Big) \\ &= \cos\big(\pi + \sin(2\pi)\big) = -1. \end{aligned}$$

Theorem 2.48 ([3, Theorem 11.3.12])**.** *Suppose that* $\lim_{n\to\infty} x_n = c$*. If a function* f *is continuous at* c*, then*

$$\lim_{n\to\infty} f(x_n) = f(\lim_{n\to\infty} x_n) = f(c).$$

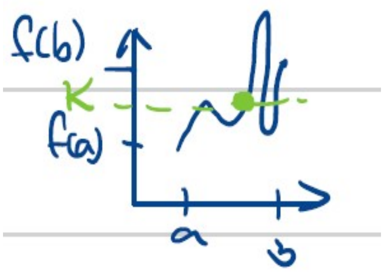
Example 2.49. *Compute the limits.*

(1) $\lim_{n\to\infty} \sin(\pi/n) = 0.$

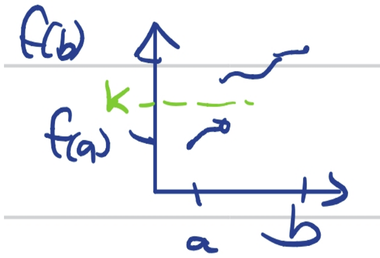
(2) $\lim_{n\to\infty} \sqrt{\frac{2n-1}{n}} = \sqrt{2}, \; \lim_{n\to\infty} \left|\frac{2n-1}{n}\right| = |2|.$

Application to solving equations.

Theorem 2.50 (Intermediate Value Theorem, [1, Theorem 12 in Section 2.6], [3, Theorem 2.6.1])**.** *Let* f *be a continuous function on* $[a, b]$*. If there is a number* k *such that either* $f(a) < k < f(b)$ *or* $f(a) > k > f(b)$*, then there exists* $c \in (a, b)$ *such that* $f(c) = k$*.*



(a) Continuous functions cannot skip any intermediate values



(b) No c with $f(c) = k$

Proof. Divide $[a, b]$ into $[a, \frac{a+b}{2}]$ and $[\frac{a+b}{2}, b]$. If $f(\frac{a+b}{2}) = k$, then the conclusion follows. If not, then k is either between $f(a)$ and $f(\frac{a+b}{2})$ or between $f(\frac{a+b}{2})$ and $f(b)$. Let

$$[a_1, b_1] = \begin{cases} [a, \frac{a+b}{2}], & \text{if } k \text{ is between } f(a) \text{ and } f(\frac{a+b}{2}); \\ [\frac{a+b}{2}, b], & \text{otherwise.} \end{cases}$$

Inductively, either we get the conclusion, or we obtain a sequence of intervals $[a_1, b_1], [a_2, b_2], \cdots$ such that (i) $b_n - a_n = \frac{b-a}{2^n}$, (ii) k is between $f(a_n)$ and $f(b_n)$, and (iii)

$$a \le a_1 \le a_2 \le \cdots \le a_n \le \cdots \le b_n \le \cdots \le b_2 \le b_1 \le b.$$

The sequences $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are bounded monotone, and thus they are convergent. Furthermore, since $b_n - a_n = \frac{b-a}{2^n}$, we have $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$, denoted this number by c. Since k is between $f(a_n)$ and $f(b_n)$, we have

$$(k - f(a_n))(k - f(b_n)) < 0,$$

and consequently

$$0 \le (k - f(c))^2 = \overline{\lim_{n\to\infty}}(k - f(a_n))(k - f(b_n)) \le \overline{\lim_{n\to\infty}}\, 0 = 0.$$

This shows that $(k - f(c))^2 = 0$, and the conclusion follows. □

Example 2.51. *Show that* $x^4 - x - 1 = 0$ *has a solution in* $[-1, 1]$.

Proof. The function $f(x) = x^4 - x - 1$ is continuous on $[-1, 1]$. Since $f(-1) = 1$ and $f(1) = -1$, we have $f(1) < 0 < f(-1)$. By the intermediate value theorem, there exists $c \in (-1, 1)$ such that $f(c) = 0$. The number c is a solution of the equation $x^4 - x - 1 = 0$. □

Example 2.52. *Show that* $x^3 - 4x + 2 = 0$ *has* 3 *distinct roots in* $[-3, 2]$.

Proof. Let $g(x) = x^3 - 4x + 2$. Note that $g(-3) = -13$, $g(-2) = 2$, $g(0) = 2$, $g(1) = -1$ and $g(2) = 2$. Thus, $g(x) = 0$ has a solution in $(-3, -2)$, a solution in $(0, 1)$ and a solution in $(1, 2)$. □

Application to extreme values.

Theorem 2.53 (Extreme Value Theorem, [1, Theorem 1 in Section 4.1], [3, Theorem 2.6.2])**.** *If f is continuous on $[a, b]$, then f takes an* **(absolute) maximum** *value M and an* **(absolute) minimum** *value m on $[a, b]$. That is, there exist $c_1, c_2 \in [a, b]$ such that $f(c_2) = m \le f(x) \le M = f(c_1)$, $\forall$ $x \in [a, b]$.*

In particular, a continuous function on a closed interval $[a, b]$ *is bounded.*

To prove Theorem 2.53, we need a lemma:

Lemma 2.54. *If* f *is a continuous function on* $[a, b]$*, then there exists* $M > 0$ *such that* $|f(x)| \leq M$ *for any* $x \in [a, b]$.

Proof. If the conclusion is not true, there exists $x_1, x_2, \cdots \in [a, b]$ such that $|f(x_n)| > n$ for all $n \in \mathbb{N}$. Since $x_n \in [a, b]$ $\forall n$, by Theorem 1.28, the sequence $(x_n)_{n=1}^{\infty}$ has a convergent subsequence $(x_{n_j})_{j=1}^{\infty}$. Let $c = \lim_{j\to\infty} x_{n_j}$. Since $x_{n_j} \in [a, b]$ $\forall j$, we have

$$a \leq c = \overline{\lim_{j\to\infty}}\, x_{n_j} \leq b.$$

Furthermore, since f is continuous on $[a, b]$ and $x_{n_j}, c \in [a, b]$, we have

$$f(c) = \lim_{j\to\infty} f(x_{n_j}).$$

This is a contradiction because $(f(x_{n_j}))_{j=1}^{\infty}$ is unbounded: $|f(x_{n_j})| > n_j \geq j$ for all $j \in \mathbb{N}$. □

Proof of Theorem 2.53. We prove the existence of c_1 here. The existence of c_2 can be proved similarly.

By Lemma 2.54, the set $f([a, b]) = \{f(x) \in \mathbb{R} \mid x \in [a, b]\}$ is bounded, and thus we have the real number

$$M := \sup f([a, b]) \in \mathbb{R}.$$

For any $n \in \mathbb{N}$, since $M - \frac{1}{n}$ is not an upper bound of $f([a, b])$ anymore, there exists $x_n \in [a, b]$ such that

$$M - \frac{1}{n} < f(x_n) \leq M. \tag{2}$$

Since $x_n \in [a, b]$, by Theorem 1.28, the sequence $(x_n)_{n=1}^{\infty}$ has a convergent subsequence $(x_{n_j})_{j=1}^{\infty}$. Let $c_1 = \lim_{j\to\infty} x_{n_j} \in [a, b]$. By Equation (2) and the pinching theorem, we have

$$\lim_{j\to\infty} f(x_{n_j}) = f(c_1) = M.$$

This completes the proof. □

Example 2.55. *Consider two functions* f, g *on* $[1, 5]$*:*

- $f(x) = x$ *is continuous on* $[1, 5]$*. And* f *attains a maximum value* 5 *and a minimum value* 1.
- $g(x) = \begin{cases} 3, & x = 1 \\ x, & 1 < x < 5 \\ 3, & x = 5 \end{cases}$ *is NOT continuous on* $[1, 5]$*, and* g *does not attain a maximum value nor a minimum value in* $[1, 5]$*. Note that the function* g *is NOT continuous on* $[1, 5]$*, so one cannot use the Extreme Value Theorem to say* g *has extremum in* $[1, 5]$.

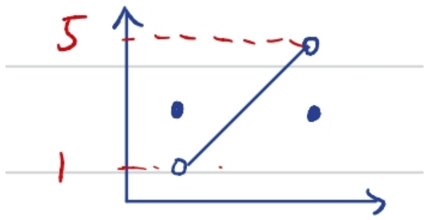


FIGURE 17. The function g has no maximum nor minimum value in $[1, 5]$

3. Derivatives

Remarks on organization. We move the discussions of exponential, logarithm and inverse trigonometric functions in [1, Chapter 3] to Section 6.

3.1. Derivatives and tangent lines.

Definition 3.1 ([1, Section 3.2], [3, Definition 3.1.1])**.** *A function f is **differentiable** at x if the limit*

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{t\to x}\frac{f(t)-f(x)}{t-x}\qquad \text{exists.}$$

*If this limit exists, it is called the **derivative** of f at x, denoted by $f'(x)$ or $\left.\frac{df}{dx}\right|_x$.*

Remark 3.2. *A physical meaning of derivative : $f'(x)$ = the rate of change of the function f at x. A geometric meaning of derivative : $f'(x)$ = slope of the tangent line of the graph of f at $(x, f(x))$.*

Example 3.3. *If $f(x)=\frac{1}{2}x+1$, then*

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{[\frac{1}{2}(x+h)+1]-[\frac{1}{2}x+1]}{h}=\lim_{h\to 0}\frac{\frac{1}{2}(x+h-x)}{h}=\frac{1}{2}.$$

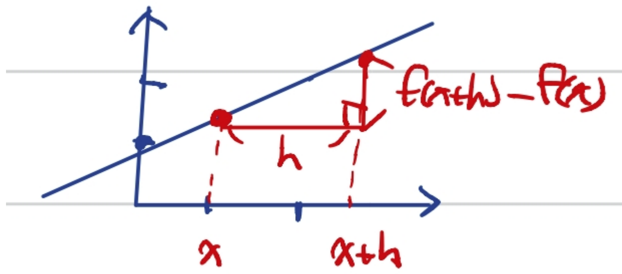


Figure 18. The derivative $f'(x)=\frac{1}{2}$ is the slope of $y=\frac{1}{2}x+1$

Example 3.4. *Find $f'(1)$, where $f(x)=x^2$.*

Solution. Note that

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{(x+h)^2-x^2}{h}=\lim_{h\to 0}\frac{(x+h+x)(x+h-x)}{h}=2x.$$

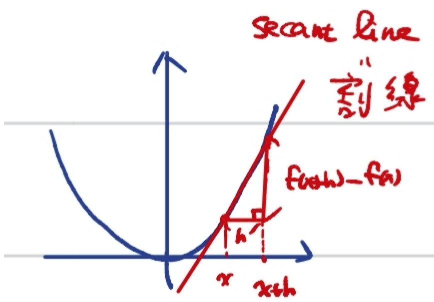


Figure 19. The derivative $f'(1)=2$ is the slope of $y=x^2$ at $x=1$

Thus, $f'(1) = \left(\frac{df}{dx}\right)_{x=1} = 2.$ □

Example 3.5. *Let $f(x) = \sqrt{x}$. Find the rate of change of $f(x)$ with respect to x at $x = 4$.*

Solution. $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} = \lim_{h\to 0} \frac{x+h-x}{h(\sqrt{x+h} + \sqrt{x})} = \frac{1}{2\sqrt{x}}.$

So the answer is $f'(4) = \left(\frac{df}{dx}\right)_{x=4} = \frac{1}{2\sqrt{4}} = \frac{1}{4}.$ □

Remark 3.6 ([3, Equation (3.1.2)]). *If f is differentiable at $x = c$, then the tangent line of the graph of f at $(c, f(c))$ is*

$$y - f(c) = f'(c)(x - c).$$

Example 3.7. *Consider following functions:*

(1) *$f(x) = \sqrt{x}$. The tangent line of the graph of f at $(4, 2)$ is*

$$y - 2 = f'(4) \cdot (x - 4) = \frac{1}{4}(x - 4).$$

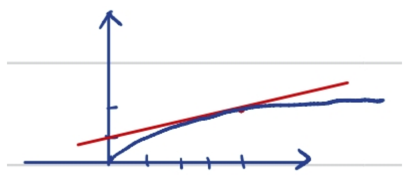


Figure 20. The tangent line of the graph of f at $(4, 2)$

(2) *$f(x) = \frac{1}{x}$. Since $f'(x) = \lim_{h\to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \lim_{h\to 0} \frac{x-(x+h)}{h(x+h)x} = -\frac{1}{x^2}$, we have $f'(1) = -1$ Thus, the tangent line of the graph of f at $(1, 1)$ is*

$$y - 1 = -(x - 1).$$

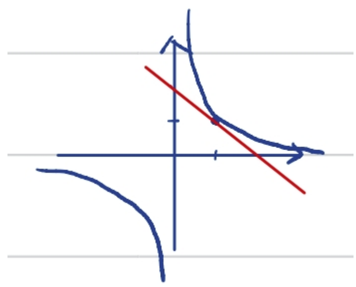


Figure 21. The tangent line of the graph of f at $(1, 1)$

Remark 3.8 ([1, Section 3.11] Skip). *Let f be a function. The tangent line L of f at $(a, f(a))$ also can be considered as the **linearization** of f at a. The idea is that $L(x)$ is close to $f(x)$ when x is in a neighborhood of a. The approximation*

$$f(x) \approx L(x) = f'(a)(x - a) + f(a)$$

of f by L is referred as the ***standard approximation*** *of f at a, and the point x = a is called the* ***center*** *of the approximation in* [1, Section 3.11].

The use of the notations dx and dy in [1, Section 3.11] *is not rigorous. A student is suggested to avoid these notations in the first course of Calculus.*

Example 3.9. *Show that the following functions are NOT differentiable at* $x = 0$*:*

$$f(x) = x^{\frac{1}{3}} \qquad \textit{and} \qquad g(x) = |x|.$$

Proof. For $f(x)$, the limit $\lim_{h\to 0} \frac{\sqrt[3]{0+h} - \sqrt[3]{0}}{h} = \lim_{h\to 0} \frac{\sqrt[3]{h}}{h} = \lim_{h\to 0} \frac{1}{h^{\frac{2}{3}}}$ doesn't exist. Hence function $f(x) = x^{\frac{1}{3}}$ is NOT differentiable at $x = 0$.

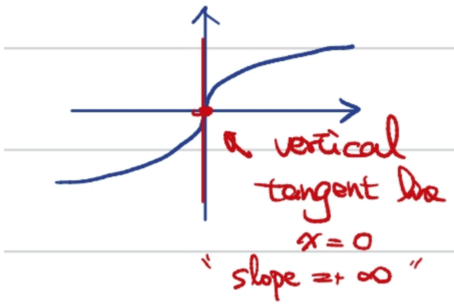


FIGURE 22. The slope = $+\infty$. The derivative doesn't exist.

For $g(x)$, the limit $\lim_{h\to 0} \frac{|h| - |0|}{h} = \lim_{h\to 0} \frac{|h|}{h}$ doesn't exist, because

$$\frac{|h|}{h} = \begin{cases} 1, & h > 0, \\ -1, & h < 0. \end{cases}$$

Thus, the function $g(x) = |x|$ is NOT differentiable at $x = 0$.

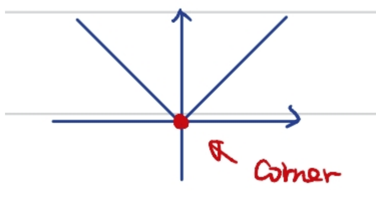


FIGURE 23. There is no tangent line through the point $(0, 0)$, and the derivative doesn't exist

□

Theorem 3.10 ([1, Theorem 1 in Section 3.2], [3, Theorem 3.1.3])**.** *If f is differentiable at x, then f is continuous at x.*

The converse statement of the above theorem is NOT true. See Example 3.9.

3.2. **Differentiation rules.** To compute derivatives, we need formulas similar to those in Theorem 2.12, Remark 2.14 and Remark 2.15.

Theorem 3.11 ([1, Section 3.3], [3, Section 3.2])**.** *Let f, $f_1, \cdots, f_n$ and g be functions which are differentiable at x, and $\alpha, \alpha_1, \cdots, \alpha_n$ be numbers.*

(1) $(f+g)'(x) = f'(x) + g'(x)$.
(2) $(f-g)'(x) = f'(x) - g'(x)$.
(3) $(\alpha f)'(x) = \alpha \cdot f'(x)$.
(4) $(\alpha_1 f_1 + \cdots + \alpha_n f_n)'(x) = \alpha_1 \cdot f_1'(x) + \cdots + \alpha_n \cdot f_n'(x)$.
(5) $(f \cdot g)'(x) = f'(x)g(x) + f(x)g'(x)$.
(6) If $g(x) \neq 0$, *then* $\left(\frac{1}{g}\right)'(x) = -\frac{g'(x)}{(g(x))^2}$.
(7) If $g(x) \neq 0$, *then* $\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$.

Proof. For (1) ∼ (3), one can prove them by definition directly. And one can show (4) by using induction together with the formulas (1) and (3). The formulas (5), (6) and (7) are shown below:

- For (5),

$$\begin{aligned}(f \cdot g)'(x) &= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\ &= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h)}{h} + \lim_{h\to 0} \frac{f(x)g(x+h) - f(x)g(x)}{h} \\ &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \cdot g(x+h) + \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} \cdot f(x) \\ &= f'(x) \cdot g(x) + f(x) \cdot g'(x).\end{aligned}$$

- For (6),

$$\begin{aligned}(\frac{1}{g})'(x) &= \lim_{h\to 0} \frac{\frac{1}{g(x+h)} - \frac{1}{g(x)}}{h} = \lim_{h\to 0} \frac{g(x) - g(x+h)}{hg(x+h)g(x)} \\ &= \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} \cdot \frac{-1}{g(x+h)g(x)} = -\frac{g'(x)}{(g(x))^2}.\end{aligned}$$

- The property (7) follows from (5) and (6).

□

Corollary 3.12 (Derivatives of Polynomials, [3, Equation (3.2.8)])**.** *One has the following formulas.*

(1) If $f(x)$ *is a constant function, then* $f'(x) = 0$.
(2) If $f(x) = x$, *then* $f'(x) = 1$.
(3) If $p(x) = x^n$, *then* $p'(x) = n \cdot x^{n-1}$.
(4) If $p(x) = a_n x^n + \cdots + a_1 x + a_0$, *then* $p'(x) = na_n x^{n-1} + \cdots + 2a_2 x + a_1$.

Example 3.13. *The derivative of* $(x^2-1)x^3(x+1)$ *is*

$$\begin{aligned}\left((x^2-1)x^3(x+1)\right)' &= (x^2-1)' \cdot x^3 \cdot (x+1) + (x^2-1) \cdot (x^3)' \cdot (x+1) + (x^2-1) \cdot x^3 \cdot (x+1)' \\ &= 2x \cdot x^3 \cdot (x+1) + (x^2-1) \cdot (3x^2) \cdot (x+1) + (x^2-1) \cdot x^3 \cdot 1.\end{aligned}$$

Example 3.14. *Find the derivative of function* $f(x) = \dfrac{1}{x^n}$.

Solution. $f'(x) = \left(\dfrac{1}{x^n}\right)' = \dfrac{-1}{(x^n)^2} \cdot (x^n)' = \dfrac{-1}{x^{2n}} \cdot (n \cdot x^{n-1}) = -n \cdot \dfrac{1}{x^{n+1}} = -n \cdot x^{-n-1}$. □

Example 3.15. *Find the derivative of function* $f(x) = \dfrac{6x^2 - 1}{x^4 + 5x + 1}$.

Solution.
$$f'(x) = \frac{(6x^2-1)'(x^4+5x+1) - (6x^2-1)(x^4+5x+1)'}{(x^4+5x+1)^2} = \frac{-12x^5 + 4x^3 + 30x^2 + 12x + 5}{(x^4+5x+1)^2}.$$ □

Example 3.16. *Find the tangent line of the graph of* $f(x) = \dfrac{3x}{1-2x}$ *at* $(2, -2)$.

Solution. $f'(x) = \dfrac{3 \cdot (1-2x) - 3x \cdot (-2)}{(1-2x)^2} = \dfrac{3}{(1-2x)^2}$.

$\Rightarrow$ $f'(2) = \dfrac{3}{(-3)^2} = \dfrac{1}{3}$.

$\Rightarrow$ The tangent line of the graph of f at $(2, -2)$ is given by the equation $y + 2 = \dfrac{1}{3}(x - 2)$. □

Example 3.17. *Find the point(s) on the graph of* $f(x) = \dfrac{4x}{x^2+4}$ *where the tangent line is horizontal.*

Solution. Solve "the points that $f'(x) = 0$," i.e. solve the equation

$$f'(x) = \frac{4 \cdot (x^2+4) - 4x \cdot 2x}{(x^2+4)^2} = \frac{-4x^2+16}{(x^2+4)^2} = 0$$

$\Rightarrow$ $x^2 - 4 = 0$ $\Rightarrow$ $x = \pm 2$ $\Rightarrow$ the desired points are $(2, 1)$ and $(-2, -1)$. □

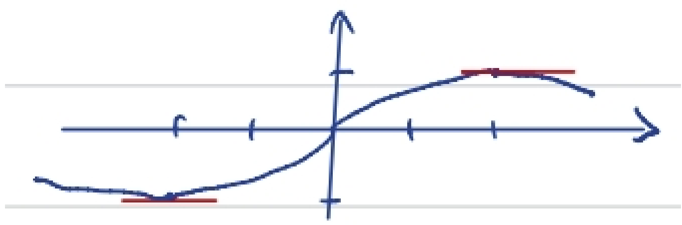


FIGURE 24. The horizontal tangent lines of f

Example 3.18. *Find the rate of change of the area of a circle with respect to the radius* r*. What is the rate when* $r = 2$*?*

Solution. Let $a(r) = \pi r^2$ be the function of area. The rate of change of the area is given by the derivative $a'(r) = 2\pi r$, and thus the answer is $a'(2) = 4\pi$. □

One can find more applications of this type in [3, Section 4.9, Section 4.10].

Definition 3.19 (Higher Order Derivatives, [3, Section 3.3])**.** *We write the first order derivative as* $f'(x) = \dfrac{df}{dx}$*, and the second order derivative as* $f''(x) = \dfrac{d^2 f}{dx^2}$*. The n-th order derivative of* f *is denoted by* $f^{(n)}(x) = \dfrac{d^n f}{dx^n}$ *which means differentiate the function* f n *times.*

Note that the precise meaning of the notation $\frac{df}{dx}$ is NOT quotient.

Example 3.20. *Consider function* $f(x) = \frac{1}{x} = x^{-1}$

$$\Rightarrow \quad \frac{df(x)}{dx} = -x^{-2}, \; \frac{d^2 f(x)}{dx^2} = 2x^{-3}\,, \; \frac{d^3 f(x)}{dx^3} = -6x^{-4}, \cdots.$$

In general, we have

$$\frac{d^n f(x)}{dx^n} = (-1)^n \cdot n! \cdot x^{-n-1}.$$

Let f be a function defined on $[a,b]$. We say that f is **differentiable on** $[a,b]$ if (i) f is differentiable at any point in (a,b), (ii) $\lim_{x\to a^+} \frac{f(x)-f(a)}{x-a}$ exists, and (iii) $\lim_{x\to b^-} \frac{f(x)-f(b)}{x-b}$ exists. Differentiable functions on (a,b), $(a,b]$ or $[a,b)$ are defined similarly.

Notations: Let $I = (a,b), [a,b), (a,b]$, or $[a,b]$ be an interval, where an endpoint a or b may be $\pm\infty$ if it lies on the open side.

(1) $C^0(I) = C(I) = \{$all the continuous functions on $I\}$.
(2) $C^1(I) = \{f \in C^0(I) \mid$ f is differemtiable on $I, f' \in C^0(I)\}$.
(3) Inductively, for $k \geq 1$, $C^k(I) = \{f \in C^1(I) \mid f' \in C^{k-1}(I)\}$.
(4) $C^\infty(I) = \bigcap_{k=1}^\infty C^k(I)$.

3.3. **Chain rule.**

Theorem 3.21 (Chain Rule, [1, Theorem 2 in Section 3.6], [3, Theorem 3.5.6])**.** *If g is differentiable at a and f is differentiable at g(a), then $f \circ g$ is differentiable at a, and*

$$(f \circ g)'(a) = f'(g(a)) \cdot g'(a).$$

Equivalently, let $y = g(x)$ and $z = (f \circ g)(x)$. One has

$$\frac{dz}{dx}\Big|_{x=a} = \Big(\frac{dz}{dy}\Big|_{y=g(a)}\Big) \cdot \Big(\frac{dy}{dx}\Big|_{x=a}\Big).$$

Proof. Consider the function

$$\varphi(t) := \begin{cases} \dfrac{f(t) - f(g(a))}{t - g(a)}, & \text{if } t \neq g(a), \\ f'(g(a)), & \text{if } t = g(a). \end{cases}$$

By the construction, the function φ is continuous at $g(a)$. Moreover, g is continuous at a because it is differentiable at a. Hence, the composition $\varphi \circ g$ is continuous at a. We will use this fact below in the line $(*)$.

Note that for all $x \neq a$,

$$\frac{f(g(x)) - f(g(a))}{x - a} = \varphi(g(x)) \cdot \frac{g(x) - g(a)}{x - a}.$$

(This equality also holds when $g(x) = g(a)$, since in that case both sides of the equation are equal to 0.) Hence,

$$\begin{aligned}(f \circ g)'(a) &= \lim_{x\to a} \frac{f(g(x)) - f(g(a))}{x-a} \\ &= \lim_{x\to a} \varphi(g(x)) \cdot \lim_{x\to a} \frac{g(x)-g(a)}{x-a} \\ &= \varphi(g(a)) \cdot g'(a) \qquad (*) \\ &= f'(g(a)) \cdot g'(a),\end{aligned}$$

as claimed. □

Example 3.22. *Find the derivative of the following functions:*

(1) $(x^2-1)^{100}$. *(2)* $(x+\frac{1}{x})^{-3}$. *(3)* $((x+1)^5+1)^4$.

Solution. By Theorem 3.21, we have:

(1) $\frac{d}{dx}\left((x^2-1)^{100}\right) = 100 \cdot (x^2-1)^{99} \cdot 2x$.

(2) $\frac{d}{dx}\left((x+\frac{1}{x})^{-3}\right) = -3 \cdot (x+\frac{1}{x})^{-4} \cdot (1-\frac{1}{x^2})$.

(3) Let $y = (x+1)^5 + 1$ and $u = x+1$.

$\Rightarrow \quad \frac{df}{dx} = \frac{df}{dy} \cdot \frac{dy}{dx} = \frac{df}{dy} \cdot \frac{dy}{du} \cdot \frac{du}{dx}$.

$\Rightarrow \quad \frac{df(x)}{dx} = 4y^3 \cdot 5u^4 \cdot 1 = 20 \cdot ((x+1)^5+1)^3 \cdot (x+1)^4$.

□

Example 3.23. *Let* $y = 3u+1$, $u = x^{-2}$, *and* $x = 1-s$. *Find* $\frac{dy}{ds}$.

Solution. $\frac{dy}{ds} = \frac{dy}{dx} \cdot \frac{dx}{ds} = \frac{dy}{du} \cdot \frac{du}{dx} \cdot \frac{dx}{ds} = 3 \cdot (-2x^{-3})(-1) = 6(1-s)^{-3}$. □

3.4. Derivatives of trigonometric functions.

Theorem 3.24 ([1, Section 3.5], [3, Equation (3.6.1), (3.6.2)])**.** *The functions* $\sin x$ *and* $\cos x$ *are differentiable at any* $x \in \mathbb{R}$. *Furthermore, we have*

$$\frac{d}{dx}\sin x = \cos x \qquad \textit{and} \qquad \frac{d}{dx}\cos x = -\sin x.$$

Proof. It follows from the facts

$$\lim_{h\to 0}\frac{\cos h - 1}{h} = 0 \qquad \text{and} \qquad \lim_{h\to 0}\frac{\sin h}{h} = 1$$

that

$$(\sin x)' = \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h} = \lim_{h\to 0}\left(\sin x \cdot \frac{\cos h - 1}{h} + \cos x \cdot \frac{\sin h}{h}\right) = \cos x,$$

$$(\cos x)' = \lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h} = \lim_{h\to 0}\left(\cos x \cdot \frac{\cos h - 1}{h} - \sin x \cdot \frac{\sin h}{h}\right) = -\sin x.$$

□

Corollary 3.25. *One has the following formulas:*

- $\frac{d}{dx}(\tan x) = \sec^2 x;$
- $\frac{d}{dx}(\cot x) = -\csc^2 x;$
- $\frac{d}{dx}(\sec x) = \sec x \cdot \tan x;$
- $\frac{d}{dx}(\csc x) = -\csc x \cdot \cot x.$

Example 3.26. *Find the tangent line of $y = \cos x$ at $(\frac{\pi}{3}, \frac{1}{2})$.*

Solution. Since

$$\left(\frac{d}{dx}(\cos x)\right)_{x=\frac{\pi}{3}} = -\sin(\frac{\pi}{3}) = -\frac{\sqrt{3}}{2},$$

the tangent line of f at $x = \frac{\pi}{3}$ is given by the equation $(y - \frac{1}{2}) = -\frac{\sqrt{3}}{2}(x - \frac{\pi}{3})$. □

Example 3.27. *Find the derivatives of the following functions.*

(1) $\cos(2x)$. *(2)* $\sec(x^2 + 1)$. *(3)* $\sin x^\circ$.

Solution. We compute as follows.

(1) Let $y = 2x$. Then $\frac{d}{dx}(\cos(2x)) = \frac{d\cos y}{dy} \cdot \frac{dy}{dx} = -\sin y \cdot 2 = -2\sin(2x)$.

(2) $$\frac{d}{dx}(\sec(x^2+1)) = \frac{d}{dx}\left(\frac{1}{\cos(x^2+1)}\right) = -\frac{(\cos(x^2+1))'}{\cos^2(x^2+1)}$$
$$= -\frac{-\sin(x^2+1)\cdot(2x)}{\cos^2(x^2+1)} = 2x \cdot \frac{\sin(x^2+1)}{\cos^2(x^2+1)}.$$

(3) $$\frac{d}{dx}(\sin x^\circ) = \frac{d}{dx}\left(\sin(\frac{2\pi}{360}\cdot x)\right) = \frac{\pi}{180}\cdot\cos(\frac{\pi}{180}\cdot x) = \frac{\pi}{180}\cos x^\circ.$$

□

3.5. **Implicit differentiation.** The basic idea of implicit differentiation is taking derivatives of the both sides of the equation. We will show this technique by examples in this section. See [3, Section 3.7] for more examples.

Example 3.28. *Find the tangent line of $x^2 + y^2 = 1$ at $(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$.*

Solution. By taking the derivatives of the both sides of the equation $x^2 + y^2 = 1$, we have:

$$\frac{d}{dx}(x^2+y^2) = \frac{d}{dx}(1) = 0.$$

$$\Rightarrow \quad \frac{d}{dx}(x^2+y^2) = \frac{dx^2}{dx} + \frac{dy^2}{dx} = 2x + \frac{dy^2}{dy}\cdot\frac{dy}{dx} = 2x + 2y\cdot\frac{dy}{dx} = 0.$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{x}{y}, \text{ for } y \neq 0. \quad \Rightarrow \quad \left(\frac{dy}{dx}\right)_{(x,y)=(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}})} = -\frac{\frac{1}{\sqrt{2}}}{-\frac{1}{\sqrt{2}}} = 1.$$

$\Rightarrow$ tangent line is $y + \frac{1}{\sqrt{2}} = x - \frac{1}{\sqrt{2}}$. □

Example 3.29. *Find the tangent line of* $2x^3 + 2y^3 = 9xy$ *at* $(1, 2)$.

Solution. By taking the derivatives of the both sides of the equation $2x^3 + 2y^3 = 9xy$, we have

$$6x^2 + 6y^2 \cdot \frac{dy}{dx} = 9y + 9x \cdot \frac{dy}{dx}.$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{9y - 6x^2}{6y^2 - 9x}. \quad \Rightarrow \quad \left(\frac{dy}{dx}\right)_{(x,y)=(1,2)} = \frac{4}{5}.$$

$$\Rightarrow \quad \text{the tangent line is } (y - 2) = \frac{4}{5}(x - 1).$$ □

Example 3.30. *Let* $\cos(x - y) = xy$. *Find* $\frac{dy}{dx}$.

Solution. By taking the derivatives of the both sides of the equation $\cos(x - y) = xy$, we have

$$-\sin(x - y) \cdot (1 - \frac{dy}{dx}) = \frac{d}{dx}(\cos(x - y)) = \frac{d}{dx}(xy) = y + x \cdot \frac{dy}{dx}.$$

$$\Rightarrow \quad \frac{dy}{dx} \cdot (\sin(x - y) - x) = y + \sin(x - y). \quad \Rightarrow \quad \frac{dy}{dx} = \frac{y + \sin(x - y)}{\sin(x - y) - x}.$$ □

Example 3.31. *Find the derivative of following functions.*

(1) $x^{\frac{1}{n}}$, *for* $x > 0$. *(2)* $x^{\frac{m}{n}}$, *for* $x > 0$. *(3)* $\sqrt{\frac{x}{1 + x^2}}$, *for* $x > 0$.

Solution. We compute as follows.

- For (1), let $y = x^{\frac{1}{n}} \quad \Rightarrow \quad y^n = x \quad \Rightarrow \quad \frac{dy^n}{dx} = \frac{dy^n}{dy} \cdot \frac{dy}{dx} = ny^{n-1} \cdot \frac{dy}{dx} = \frac{dx}{dx} = 1.$
  Thus, $\frac{dy}{dx} = \frac{1}{ny^{n-1}} = \frac{1}{nx^{\frac{n-1}{n}}} = \frac{1}{n} \cdot x^{\frac{1}{n}-1}.$
- For (2), $\frac{d}{dx}(x^{\frac{m}{n}}) = m(x^{\frac{1}{n}})^{m-1} \cdot \frac{1}{n}x^{\frac{1}{n}-1} = \frac{m}{n} \cdot x^{\frac{m}{n}-1}.$
- For (3), $$\frac{d}{dx}\left(\sqrt{\frac{x}{1 + x^2}}\right) = \frac{1}{2}\left(\frac{x}{1 + x^2}\right)^{-\frac{1}{2}} \cdot \frac{d}{dx}\left(\frac{x}{1 + x^2}\right) = \frac{1}{2}\sqrt{\frac{1 + x^2}{x}} \cdot \frac{(1 + x^2) - x(2x)}{(1 + x^2)^2}$$ $$= \frac{1}{2} \cdot \frac{1 - x^2}{\sqrt{x} \cdot (1 + x^2)^{\frac{3}{2}}}.$$

□

4. Applications of derivatives

Remarks on organization. We combine put the extreme value theorem in the section of mean values theorem as a preparation for the proof of mean value theorem. We move the discussions of local extreme values and first derivative tests in [1, Section 4.3] to the section of extreme values, and move antiderivative to the chapter of integration.

4.1. Mean value theorem.

Lemma 4.1. *Let f be a function which is differentiable on (a, b). Suppose a function f takes a maximum value or a minimum value at $c \neq a, b$. Then $f'(c) = 0$.*

Proof. Assume f takes a maximum value M at c. The case of minimum value is similar. Since $f(c+h) - f(c) \leq 0$ for any $h \in (a-c, b-c)$, we have

$$\begin{cases} \dfrac{f(c+h) - f(c)}{h} \leq 0, & \text{if } h > 0, h \in (a-c, b-c), \\ \dfrac{f(c+h) - f(c)}{h} \geq 0, & \text{if } h < 0, h \in (a-c, b-c). \end{cases}$$

Therefore, the right limit is non-positive, and the left limit is non-negative. It follows from the differentiability of f at c that

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h} = 0.$$

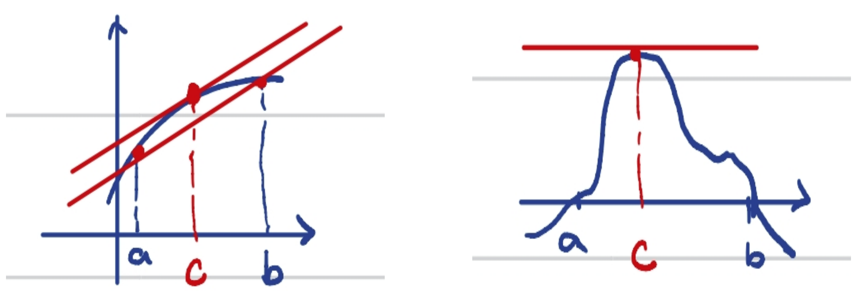
□

Theorem 4.2 (Mean Value Theorem, [1, Theorem 4 in Section 4.2], [3, Theorem 4.1.1])**.** *If f is differentiable on (a, b) and continuous on $[a, b]$, then there exists $c \in (a, b)$, such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

In particular (Rolle's theorem, [1, Theorem 3 in Section 4.2], [3, Theorem 4.1.3]*), if $f(a) = f(b) = 0$, then there exists $c \in (a, b)$, such that*

$$f'(c) = 0.$$



(a) Mean Value Theorem (b) Rolle's Theorem

Figure 25. Theorem 4.1.

Proof. We divide the proof into two steps.

- **Step1 (Rolle's theorem):** Suppose $g(a) = g(b) = 0$, and g is differentiable on (a, b) and continuous on $[a, b]$. By the extreme value theorem, there exist $c_0, c_1 \in [a, b]$ such that g takes a minimum value m at c_0 and takes a maximum value M at c_1. If $m = M = 0$, then $g \equiv 0$, and thus $g' \equiv 0$ in (a, b). If $m \neq M$, then one of them is not equal 0. Assume $M \neq 0$ which implies that $c_1 \neq a, b$. Thus, by Lemma 4.1, we have $g'(c_1) = 0$.
- **Step2 (Mean value theorem):** Let
$$g(x) = f(x) - \left(\frac{f(b) - f(a)}{b - a}(x - a) + f(a)\right).$$
Then we have $g(a) = g(b) = 0$, g is differentiable on (a, b) and continuous on $[a, b]$. Thus, by Roll's theorem, there exists $c \in (a, b)$, such that
$$g'(c) = f'(c) - \frac{f(b) - f(a)}{b - a} = 0$$
which implies $f'(c) = \dfrac{f(b) - f(a)}{b - a}$. □

The following corollary is a more general version of the mean value theorem.

Corollary 4.3 (Generalized Mean Value Theorem/ Cauchy's Mean Value Theorem)**.** *Let $f, g \in C[a, b]$. Suppose that (i) f, g are differentiable on (a, b), and (ii) $g'(x) \neq 0$ for any $x \in (a, b)$. Then there exists $c \in (a, b)$ such that*
$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$

Proof. First note that by the mean value theorem, $g(b) - g(a) \neq 0$. Consider the function
$$F(x) := f(x) - f(a) - \frac{f(b) - f(a)}{g(b) - g(a)}(g(x) - g(a)).$$
It is clear that F is differentiable on (a, b) and continuous on $[a, b]$. Thus, there exists $c \in (a, b)$ such that
$$F'(c) = \frac{F(b) - F(a)}{b - a} = 0.$$
Since
$$F'(c) = f'(c) - \frac{f(b) - f(a)}{g(b) - g(a)}g'(c),$$
we conclude that
$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)},$$
as desired. □

Example 4.4. *Let $f(x) = \sqrt{1 - x}$, $-1 \leq x \leq 1$. By the mean value theorem, there exists $c \in (-1, 1)$ such that*
$$f'(c) = \frac{f(1) - f(-1)}{1 - (-1)} = -\frac{\sqrt{2}}{2}.$$
In this example, we can find c by solving the equation:
$$f'(c) = \frac{1}{2}(1 - c)^{-\frac{1}{2}}(-1) = -\frac{\sqrt{2}}{2} \quad \Rightarrow \quad c = \frac{1}{2}.$$

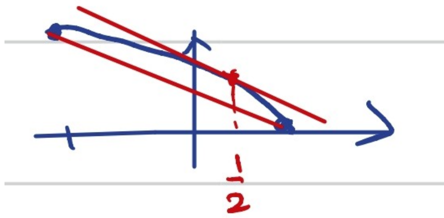


FIGURE 26. Slope of f at $x = \frac{1}{2}$ is $\frac{f(1) - f(-1)}{1 - (-1)}$

Example 4.5. *Suppose f is differentiable on $(-\infty, \infty)$. Assume $f(1) = 2$ and $2 \le f'(x) \le 3$. Show that $8 \le f(4) \le 11$.*

Proof. By the mean value theorem, there exists $c \in (1, 4)$ such that $f'(c) = \frac{f(4) - f(1)}{4 - 1} = \frac{f(4) - 2}{3}$. Thus, we have $2 \le \frac{f(4) - 2}{3} \le 3$ which implies $8 \le f(4) \le 11$. □

Example 4.6. *Show that $p(x) = 2x^3 + 5x - 1$ has exactly one real root.*

Proof. Since any polynomial is differentiable and continuous, we can apply the intermediate theorem and mean value theorem to $p(x)$. Since $p(0) = -1$ and $p(1) = 6$, by the intermediate value theorem, there exists $c \in (0, 1)$ such that $p(c) = 0$. Thus we have a root for the equation $p(x) = 0$.

Suppose there is another point $d \neq c$ satisfying $p(d) = 0$. By the mean value theorem, there exists k between c and d such that

$$p'(k) = \frac{p(c) - p(d)}{c - d} = 0.$$

But since $p'(k) = 6k^2 + 5 > 0$ for any $k \in \mathbb{R}$, we then have $0 = p'(k) > 0$ which is a contradiction. Therefore, the polynomial $p(x)$ must have exactly one real root. □

4.2. Monotone functions.

Definition 4.7 ([3, Definition 4.2.1]). *A function f is said to be*

- ***increasing*** *on (a, b) if $x_1 < x_2$ and $x_1, x_2 \in (a, b)$, then $f(x_1) \le f(x_2)$,*
- ***strictly increasing*** *on (a, b) if $x_1 < x_2$ and $x_1, x_2 \in (a, b)$, then $f(x_1) < f(x_2)$,*
- ***decreasing*** *on (a, b) if $x_1 < x_2$ and $x_1, x_2 \in (a, b)$, then $f(x_1) \ge f(x_2)$,*
- ***strictly decreasing*** *on (a, b) if $x_1 < x_2$ and $x_1, x_2 \in (a, b)$, then $f(x_1) > f(x_2)$,*
- ***constant*** *on (a, b) if $f(x_1) = f(x_2)$, $\forall x_1, x_2 \in (a, b)$.*

Remark 4.8. *Note that our terminologies here are slightly different from* [3, Definition 4.2.1]. *The notions* strictly increasing *and* strictly decreasing *in Definition 4.7 are respectively* increasing *and* decreasing *in* [3, Definition 4.2.1], *and* increasing *and* decreasing *in Definition 4.7 might respectively be called* non-decreasing *and* non-increasing *in* [3].

Example 4.9. *Consider following functions:*

- *$f(x) = x^2$ is decreasing on $(-\infty, 0]$ and increasing on $[0, \infty)$.*

- $f(x) = \begin{cases} 1, & x < 0 \\ x, & x \geq 0 \end{cases}$ *is constant on* $(-\infty, 0)$ *and increasing on* $[0, \infty)$.
- $f(x) = x^3$ *is increasing on* $(-\infty, \infty)$.
- $f(x) = \begin{cases} 1, & x \text{ rational} \\ 0, & x \text{ irrational} \end{cases}$ *is neither.*

Theorem 4.10 ([1, Corollary 3 in Section 4.3], [3, Theorem 4.2.2, Theorem 4.2.3])**.** *Suppose f is differentiable on (a, b).*

- *If $f'(x) \geq 0$, $\forall x \in (a, b)$, then f is increasing on (a, b).*
- *If $f'(x) > 0$, $\forall x \in (a, b)$, then f is strictly increasing on (a, b).*
- *If $f'(x) \leq 0$, $\forall x \in (a, b)$, then f is decreasing on (a, b).*
- *If $f'(x) < 0$, $\forall x \in (a, b)$, then f is strictly decreasing on (a, b).*
- *If $f'(x) = 0$, $\forall x \in (a, b)$, then f is constant on (a, b).*

Proof. By mean value theorem. □

Example 4.11. *Consider following functions:*

- $f(x) = x^3 - 3x \quad \Rightarrow \quad f'(x) = 3x^2 - 3 = 3(x-1)(x+1) \quad \Rightarrow$

x	$(-\infty, -1)$	$(-1, 1)$	$(1, \infty)$
$f'(x)$	$+$	$-$	$+$
$f(x)$	$\nearrow$	$\searrow$	$\nearrow$

Table 1. Increasing and decreasing interval of f

- $g(x) = x - 2\sin x \quad \forall x \in [0, 2\pi] \quad \Rightarrow \quad g'(x) = 1 - 2\cos x \quad \forall x \in [0, 2\pi] \quad \Rightarrow$

x	$(0, \frac{\pi}{3})$	$(\frac{\pi}{3}, \frac{5\pi}{3})$	$(\frac{5\pi}{3}, 2\pi)$
$g'(x)$	$-$	$+$	$-$
$g(x)$	$\searrow$	$\nearrow$	$\searrow$

Table 2. Increasing and decreasing interval of g

For the case of constant functions, we have the following theorem.

Theorem 4.12 ([3, Theorem 4.2.4])**.** *Suppose f, g are differentiable on (a, b).*

- $f'(x) = 0 \quad \forall x \in (a, b) \quad \Leftrightarrow \quad f(x) = c$ *for some constant c.*
- $f'(x) = g'(x) \quad \forall x \in (a, b) \quad \Leftrightarrow \quad f(x) = g(x) + c$ *for some constant c.*

4.3. **Extreme values.**

Definition 4.13 (Local (Relative) Extreme Values, [1, Page 239, Section 4.1], [3, Definition 4.3.1])**.** *We say that a function f has a **local maximum** value at c if there exits $\delta > 0$ such that*

$$f(c) \geq f(x) \quad \forall x \in (c - \delta, c + \delta).$$

We say that a function f has a ***local minimum*** *value at c if there exists $\delta > 0$ such that*

$$f(c) \le f(x) \quad \forall x \in (c-\delta, c+\delta).$$

Remark 4.14. *If f has an absolute maximum (resp. minimum) value at c, then f has a local maximum (resp. minimum) value at c.*

Theorem 4.15 ([1, Theorem 2 in Section 4.1], [3, Theorem 4.3.2])**.** *If f has a local maximum/minimum value at c, then either $f'(c) = 0$ or $f'(c)$ doesn't exist.*

Proof. Similar as the proof of Lemma 4.1. □

Definition 4.16 ([1, Page 241, Section 4.1], [3, Definition 4.3.3])**.** *A point c is called a* **critical point** *for f if either $f'(c) = 0$ or $f'(c)$ doesn't exist.*

Example 4.17. *Find the critical points of the following functions.*

- $f(x) = x^2 \quad \Rightarrow \quad f'(x) = 2x \quad \forall x \in \mathbb{R}.$
  Critical points of f: $x = 0$.

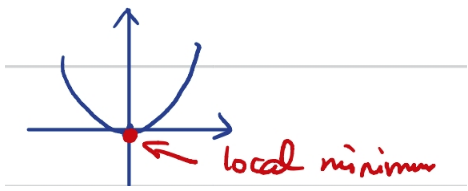


FIGURE 27. Critical point of x^2

- $f(x) = |x| \quad \Rightarrow \quad f'(x) = \begin{cases} 1, & x > 0, \\ doesn't\ exist, & x = 0, \\ -1, & x < 0. \end{cases}$
  Critical points of f: $x = 0$.

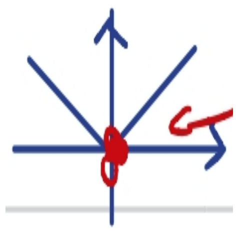


FIGURE 28. Critical point of $|x|$

Now we want to find the local maximum and local minimum of a function f. Since we already know that a local maximum/minimum is a critical point, we only need to check which critical points (and endpoints) are the local maximum/minimum. We introduce the following two methods.

Theorem 4.18 (First-order Derivative Test, [1, Page 255, Section 4.3], [3, Theorem 4.3.4])**.** *Suppose c is a critical point of f and f is continuous at c. If there exists $\delta > 0$ such that*

- *$f'(x) > 0$, $\forall x \in (c-\delta, c)$, and $f'(x) < 0$, $\forall x \in (c, c+\delta)$, then f has a local maximum at c.*

- $f'(x) < 0$, $\forall x \in (c-\delta, c)$, *and* $f'(x) > 0$, $\forall x \in (c, c+\delta)$, *then* f *has a local minimum at* c.
- *either* $f'(x) > 0$ *or* $f'(x) < 0$ $\quad \forall x \in (c-\delta, c+\delta)$, *then* f *does NOT have a local extreme at* c.

Theorem 4.19 (Second-order Derivative Test, [1, Section 4.4, Theorem 5], [3, Theorem 4.3.5])**.** *Suppose that* $f''(c)$ *exists,* f' *exists near* c, *and* $f'(c) = 0$.

- *If* $f''(c) > 0$, *then* f *has a local minimum at* c.
- *If* $f''(c) < 0$, *then* f *has a local maximum at* c.
- *If* $f''(c) = 0$, *then anything may happen.*

Example 4.20. *Consider following functions.*

(1) $f(x) = x^2 \quad \Rightarrow \quad f'(x) = 2x \quad \Rightarrow \quad f''(x) = 2 > 0.$
By second order derivative test, a local minimum occurs at $x = 0$.

x	$(-\infty, 0)$	$x = 0$	$(0, \infty)$
$f(x)$	$\searrow$	0	$\nearrow$
$f'(x)$	$-$	0	$+$
$f''(x)$	$+$	$+$	$+$

(2) $f(x) = |x| \quad \Rightarrow \quad f'(x) = \begin{cases} 1, & x > 0 \\ doesn't\ exist, & x = 0 \\ -1, & x < 0. \end{cases} \qquad$ *(*$\Rightarrow f''(0)$ *doesn't exist.)*
By first order derivative test, $f(x)$ *has a local minimum at* $x = 0$.
NOTE : *Here second order derivative test doesn't work, because* $f''(0)$ *doesn't exist.*

x	$(-\infty, 0)$	$x = 0$	$(0, \infty)$
$f(x)$	$\searrow$	0	$\nearrow$
$f'(x)$	-1	*doesn't exist*	1
$f''(x)$		*doesn't exist*	

(3) $f(x) = x^3 \quad \Rightarrow \quad f'(x) = 3x^2 \quad \Rightarrow \quad f''(x) = 6x.$
We have a critical point $x = 0$, *and* $f'(0) = f''(0) = 0$.
By first order derivative test, f *does not have a local extreme at* $x = 0$.
NOTE : *The second order derivative test is not useful here because* $f''(0) = 0$.

x	$(-\infty, 0)$	$x = 0$	$(0, \infty)$
$f(x)$	$\nearrow$	0	$\nearrow$
$f'(x)$	$+$	0	$+$
$f''(x)$	$-$	0	$+$

(4) $f(x) = x^4 \quad \Rightarrow \quad f'(x) = 4x^3 \quad \Rightarrow \quad f''(x) = 12x^2.$
We have a critical point $x = 0$, *and* $f'(0) = f''(0) = 0$.
By first order derivative test, f *has a local minimum at* $x = 0$.
NOTE : *The second order derivative is not useful here because* $f''(0) = 0$.

x	$(-\infty, 0)$	$x = 0$	$(0, \infty)$
$f(x)$	$\searrow$	0	$\nearrow$
$f'(x)$	$-$	0	$+$
$f''(x)$	$+$	0	$+$

(5) $f(x) = x^4 - 2x^3 \quad \Rightarrow \quad f'(x) = 4x^3 - 6x^2 = 2x^2(2x - 3)$
$\Rightarrow \quad f''(x) = 12x^2 - 12x = 12x(x - 1).$
The critical points are $x = 0, \frac{3}{2}$, *and* $f''(0) = 0$, $f''(\frac{3}{2}) > 0$.
By first order derivative test, f *does not have a local extreme at* $x = 0$.
By second order derivative test, f *has a local minimum at* $x = \frac{3}{2}$.

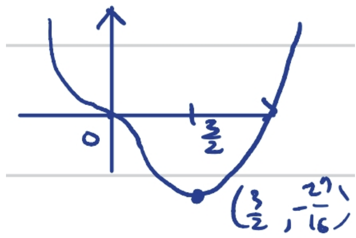


FIGURE 29. Local extreme of $x^4 - 2x^3$

(6) $f(x) = 2x^{\frac{5}{3}} + 5x^{\frac{2}{3}} \quad \Rightarrow \quad f'(x) = \frac{10}{3}x^{\frac{2}{3}} + \frac{10}{3}x^{-\frac{1}{3}} = \frac{10}{3\sqrt[3]{x}}(x + 1)$
$\Rightarrow \quad f''(x) = \frac{20}{9}x^{-\frac{1}{3}} - \frac{10}{9}x^{-\frac{4}{3}} = \frac{10}{9}x^{-\frac{4}{3}}(2x - 1).$
The critical points are $x = 0, -1$ *because* $f'(0)$ *doesn't exist and* $f'(-1) = 0$.
By first order derivative test, f *has a local minimum at* $x = 0$.
By second order derivative test, f *has a local maximum at* $x = -1$.

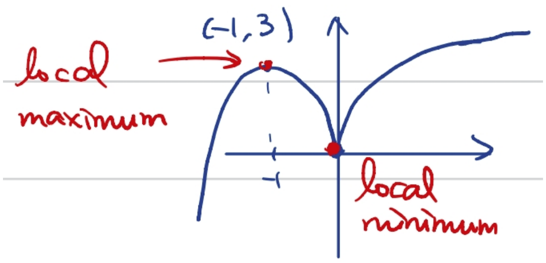


FIGURE 30. Local extremes of $2x^{\frac{5}{3}} + 5x^{\frac{2}{3}}$

Definition 4.21 ([3, Definition 4.4.1])**.** *We say that*

- a, b *are endpoints of the interval* $[a, b]$
- a *is an endpoint of the interval* $[a, b)$, *and* b *is not.*
- b *is an endpoint of the interval* $(a, b]$, *and* a *is not.*

Let c *be an endpoint of the domain of* f. *We say* f *has an* **endpoint maximum** *(or local maximum) at* c *if there exists* $\delta > 0$ *such that*

$$f(c) \geq f(x), \quad \forall\, x \in (c - \delta, c + \delta) \cap \text{domain}(f),$$

and f has an **endpoint minimum** *(or local minimum) at c if there exists $\delta > 0$ such that*

$$f(c) \leq f(x), \quad \forall\, x \in (c-\delta, c+\delta) \cap \text{domain}(f).$$

Recall from Theorem 2.53 that (i) a function f is said to have an **absolute maximum** (or **global maximum**) at d if

$$f(d) \geq f(x), \quad \forall x \in \text{domain}(f),$$

and (ii) a function f is said to have an **absolute minimum** (or **global minimum**) at d if

$$f(d) \leq f(x), \quad \forall x \in \text{domain}(f).$$

Remark 4.22. *The candidates at which all kinds of extreme values happen are* endpoints *and* critical points.

Example 4.23. *Find the critical points of* $f(x) = 1 + 4x^2 - \frac{1}{2}x^4$, $x \in [-1, 3]$, *and classify all the extreme values.*

Solution. To find the extreme value(s) of f, we find all of the endpoints and critical points:

- the endpoints are $-1, 3$, and
- since $f'(x) = 8x - 2x^3 = 0$, the critical points are $x = 0, 2$.

Computing the values $f(-1), f(3), f(0)$, and $f(2)$, we conclude that

(i) the global maximum is $f(2) = 9$, and
(ii) the global minimum is $f(3) = -\frac{7}{2}$.

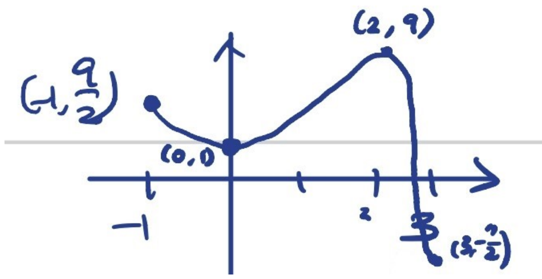


FIGURE 31. Critical points and endpoints of f

□

Example 4.24. *Find the critical points of*

$$f(x) = \frac{1}{4}(x^3 - \frac{3}{2}x^2 - 6x + 2), \qquad x \in [-2, \infty),$$

and classify all the extreme values.

Solution. Similar as the previous example.

- local minimum at $x = -2, 2$
- local maximum at $x = -1$
- global minimum at $x = 2$

- global maximum doesn't exist

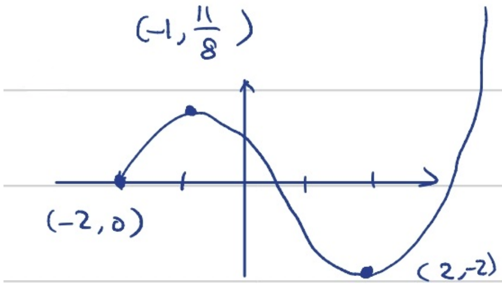


FIGURE 32. Critical points and endpoints of f

□

Example 4.25. *A window in the shape as follows is to have perimeter* 10*m. Choose* x, y *so that the window admits the most light (largest area).*

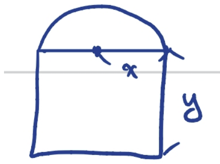


FIGURE 33. Shape of window parameterized by x, y

Solution. The area of window is $A(x, y(x)) = \frac{1}{2}\pi x^2 + 2xy$. Since the perimeter is $\pi x + 2y + 2x = 10$, we have $y = \frac{1}{2}[10 - (2+\pi)x]$. Thus,

$$A(x) = A(x, y(x)) = \frac{1}{2}\pi x^2 + x[10 - (2+\pi)x],$$

and $A'(x) = 10 - (4+\pi)x$. As a result, the critical point is $x = \frac{10}{4+\pi}$. Since the area makes sense only when $x \in \left[0, \frac{10}{2+\pi}\right]$, the endpoints are $0, \frac{10}{2+\pi}$. By computing $A(0), A(\frac{10}{4+\pi})$ and $A(\frac{10}{2+\pi})$, we conclude that $A(x)$ has a global maximum at $x = \frac{10}{4+\pi}$.

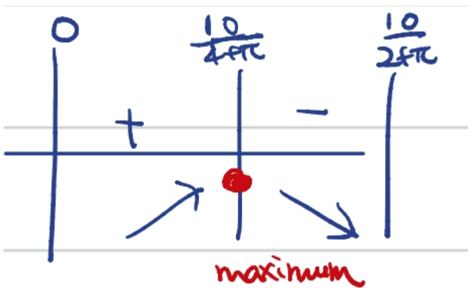


FIGURE 34. Extreme values of function $A(x)$

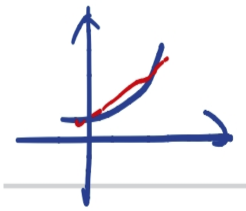
□

4.4. **Concavity and curve sketching.**

Definition 4.26 ([1, Page 259-260, Section 4.4], [3, Definition 4.6.1, Definition 4.6.2])**.** *The graph of a differentiable function f is said to be*

- ***concave up*** *if $f'(x)$ is increasing,*
- ***concave down*** *if $f'(x)$ is decreasing.*

A point $(c, f(c))$ is called a ***point of inflection*** *if $\exists\, \delta > 0$ such that f is concave up/down on $(c-\delta, c)$, and f is concave down/up on $(c, c+\delta)$, i.e., f has the opposite concavity on $(c-\delta, c)$ and $(c, c+\delta)$.*


(a) Concave up

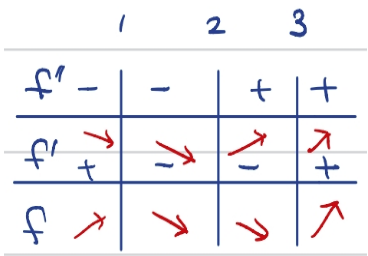

(b) Concave down

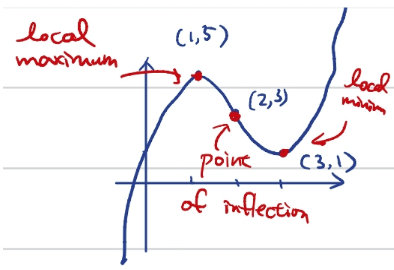
Theorem 4.27 ([3, Theorem 4.6.3, Theorem 4.6.4])**.** *If $(c, f(c))$ is a point of inflection, then $f''(c) = 0$ or $f''(c)$ doesn't exist. Suppose f is twice differentiable. Then*

- $f''(x) > 0,\ \forall x \in (a, b) \quad \Rightarrow \quad$ *the graph of f is concave up on (a, b);*
- $f''(x) < 0,\ \forall x \in (a, b) \quad \Rightarrow \quad$ *the graph of f is concave down on (a, b).*

Example 4.28. *Analyze the graphs of the following functions.*

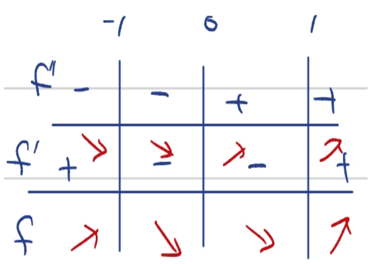
- $f(x) = x^3 - 6x^2 + 9x + 1 \quad \Rightarrow \quad f'(x) = 3x^2 - 12x + 9 = 3(x-1)(x-3)$
  $\Rightarrow \quad f''(x) = 6x - 12 = 6(x-2)$.

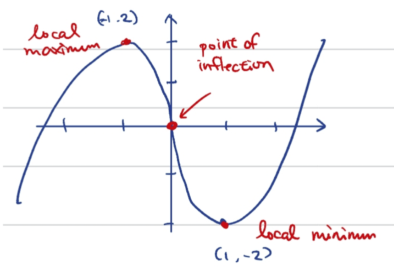

(c) Analysis of f



(d) Graph of f

- $f(x) = 3x^{\frac{5}{3}} - 5x \quad \Rightarrow \quad f'(x) = 5x^{\frac{2}{3}} - 5 \quad \Rightarrow \quad f''(x) = \dfrac{10}{3} \cdot \dfrac{1}{\sqrt[3]{x}}$.


(e) Analysis of f



(f) Graph of f

4.5. **L'Hôpital's rule.**

Theorem 4.29 (L'Hôpital's rule of the $0/0$ type, [1, Theorem 6 in Section 4.5], [3, Section 11.5])**.** *Suppose that there exists $\delta > 0$ such that f and g are differentiable on $(c-\delta, c+\delta)$, and that*

$$\lim_{x\to c} f(x) = 0, \qquad \lim_{x\to c} g(x) = 0.$$

If $g'(x) \neq 0$ for all x sufficiently close to c (with $x \neq c$), and if the limit

$$\lim_{x\to c} \frac{f'(x)}{g'(x)} = L$$

exists (finite), then one has

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)} = L.$$

Proof. Fix $x \neq c$ close to c. Since f and g are differentiable near c, they are continuous on the closed interval between x and c and differentiable on the open interval between them. By Cauchy's Mean Value Theorem (Corollary 4.3), there exists a point ξ between x and c such that

$$\frac{f(x)-f(c)}{g(x)-g(c)} = \frac{f'(\xi)}{g'(\xi)}.$$

Because $f(c) = g(c) = 0$, this simplifies to

$$\frac{f(x)}{g(x)} = \frac{f'(\xi)}{g'(\xi)}.$$

As $x \to c$, the corresponding $\xi = \xi(x)$ lies between x and c, so $\xi(x) \to c$. Therefore,

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(\xi(x))}{g'(\xi(x))} = \lim_{t\to c} \frac{f'(t)}{g'(t)} = L.$$

Hence,

$$\lim_{x\to c} \frac{f(x)}{g(x)} = L,$$

as desired. □

In addition to the $0/0$ type, there is also L'Hôpital's rule for the ∞/∞ type. To prove it, we will need a lemma:

Lemma 4.30. *Let $A, B \in \mathbb{R}$, and $(a_n)_{n=1}^{\infty}, (b_n)_{n=1}^{\infty}$ be sequences such that $b_n \to \pm\infty$. Then the sequence $\left(\frac{a_n}{b_n}\right)_{n=1}^{\infty}$ converges iff the sequence $\left(\frac{a_n+A}{b_n+B}\right)_{n=1}^{\infty}$ converges. Furthermore,*

$$\overline{\lim_{n\to\infty}} \frac{a_n}{b_n} = \overline{\lim_{n\to\infty}} \frac{a_n+A}{b_n+B}, \qquad \underline{\lim_{n\to\infty}} \frac{a_n}{b_n} = \underline{\lim_{n\to\infty}} \frac{a_n+A}{b_n+B}$$

Sketch of proof. The key observation is that $\overline{\lim_{n\to\infty}} \frac{a_n+A}{b_n+B} = \overline{\lim_{n\to\infty}} \frac{\frac{a_n}{b_n} + \frac{A}{b_n}}{1 + \frac{B}{b_n}} = \overline{\lim_{n\to\infty}} \frac{a_n}{b_n}$. □

Theorem 4.31 (L'Hôpital's rule of the ∞/∞ type, [1, Theorem 6 in Section 4.5], [3, Section 11.6])**.** *Let f, g be differentiable on an open interval (a, b) and suppose $g'(x) \neq 0$ for all $x \in (a, b)$. Assume*

$$\lim_{x\to a^+} f(x) = \pm\infty, \qquad \lim_{x\to a^+} g(x) = \pm\infty,$$

and that the limit

$$\lim_{x\to a^+}\frac{f'(x)}{g'(x)}=L$$

exists. Then

$$\lim_{x\to a^+}\frac{f(x)}{g(x)}=L.$$

(The cases $x\to b^-$ is similar.)

Proof. It suffices to show that for any sequence $x_n\in(a,b)$ with $\lim_{n\to\infty}x_n=a$, we have $\lim_{n\to\infty}\frac{f(x_n)}{g(x_n)}=L$. Given any $\epsilon>0$, since $\lim_{x\to a^+}\frac{f'(x)}{g'(x)}=L$, there exists $\delta_\epsilon\in(0,b-a)$ such that $\left|\frac{f'(x)}{g'(x)}-L\right|<\epsilon$ for any $x\in(a,a+\delta_\epsilon)$. Let $x_\epsilon=a+\frac{\delta_\epsilon}{2}\in(a,a+\delta_\epsilon)$. Since $\lim_{n\to\infty}x_n=a$, there exists N_ϵ such that $x_n\in(a,x_\epsilon)$ for all $n\geq N_\epsilon$. For $n\geq N_\epsilon$, since f,g are differentiable on (x_n,x_ϵ) and continuous on $[x_n,x_\epsilon]$, by Corollary 4.3, there exists $c\in(x_n,x_\epsilon)\subset(a,a+\delta_\epsilon)$ such that

$$\frac{f(x_\epsilon)-f(x_n)}{g(x_\epsilon)-g(x_n)}=\frac{f'(c)}{g'(c)}.$$

This implies that

$$\left|\frac{f(x_\epsilon)-f(x_n)}{g(x_\epsilon)-g(x_n)}-L\right|<\epsilon$$

for any $n\geq N_\epsilon$. By Lemma 4.30,

$$\left|\overline{\lim_{n\to\infty}}\frac{f(x_n)}{g(x_n)}-L\right|=\overline{\lim_{n\to\infty}}\left|\frac{f(x_\epsilon)-f(x_n)}{g(x_\epsilon)-g(x_n)}-L\right|\leq\epsilon$$

for any $\epsilon>0$. This shows that $\overline{\lim_{n\to\infty}}\frac{f(x_n)}{g(x_n)}=L$. Similarly, we also have $\underline{\lim_{n\to\infty}}\frac{f(x_n)}{g(x_n)}=L$, and thus $\lim_{n\to\infty}\frac{f(x_n)}{g(x_n)}=L$. This completes the proof. □

See Definition 2.30 for the meaning of $\lim_{x\to c}f(x)=\pm\infty$ and $\lim_{x\to c}g(x)=\pm\infty$.

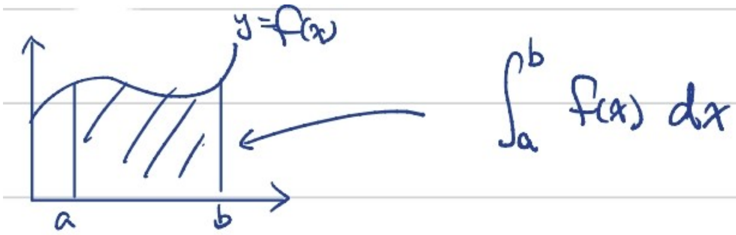
Example 4.32. *Use the L'Hôpital's rule to compute the following limits.*

- $\lim_{x\to\pi/2}\frac{\cos x}{\pi-2x}=\lim_{x\to\pi/2}\frac{-\sin x}{-2}=\frac{1}{2}$. *(0/0 type)*
- $\lim_{x\to1}\frac{x^4+2x^3-2x^2-1}{x^3-1}=\lim_{x\to1}\frac{4x^3+6x^2-4x}{3x^2}=\frac{6}{3}=2$. *(0/0 type)*
- $\lim_{x\to(\pi/2)^-}\frac{\sec x}{1+\tan x}=\lim_{x\to(\pi/2)^-}\frac{\sec x\tan x}{\sec^2 x}=\lim_{x\to(\pi/2)^-}\sin x=1$. *($\infty/\infty$ type)*
- $\lim_{x\to(\pi/2)^-}\left(\frac{\pi}{2}-x\right)\tan x=\lim_{x\to(\pi/2)^-}\frac{(\frac{\pi}{2}-x)\sin x}{\cos x}=\lim_{x\to(\pi/2)^-}\frac{-\sin x+(\frac{\pi}{2}-x)\cos x}{-\sin x}=1$. *($0\cdot\infty$ type $\to0/0$ type)*
- $\lim_{x\to0}\left(\frac{1}{\sin x}-\frac{1}{x}\right)=\lim_{x\to0}\frac{x-\sin x}{x\sin x}=\lim_{x\to0}\frac{1-\cos x}{\sin x+x\cos x}=\lim_{x\to0}\frac{\sin x}{\cos x+\cos x-x\sin x}=0$. *($\infty-\infty$ type $\to0/0$ type)*

5. Integration

Remarks on organization. We move the mean value theorems for integrals in [1, Section 5.4] to Section 5.4, move integration by parts in [1, Section 8.2] to Section 5.3 and move area between curves to Section A.

The **integral** of a function f from a to b, denoted by $\int_a^b f(x)\,dx$, is the following area.



Example 5.1. *If $f(x) = c$ is a constant, then $\int_a^b c\,dx = c(b-a)$.*

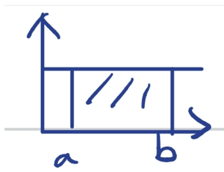


Figure 35. Integration of $f(x) = c$

Example 5.2. *If $g(x) = x$, then $\int_0^b x\,dx = \frac{1}{2}b^2$.*

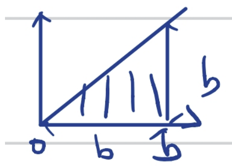


Figure 36. Integration of $g(x) = x$

5.1. Approximation of integrals.

Definition 5.3 ([3, (5.2.1)])**.** *A **partition** P of $[a,b]$ is a finite sequence $P = \{x_0, x_1 \cdots x_n\}$, where*

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b.$$

*The number $\|P\| := max\{|x_1 - x_0|, |x_2 - x_1|, \cdots, |x_n - x_{n-1}|\}$ is called the **norm** of P.*

Example 5.4. *$P = \{0, \frac{1}{2}, \frac{2}{3}, 1\}$ is a partition of $[0,1]$, and $\|P\| = max\{\frac{1}{2}, \frac{1}{6}, \frac{1}{3}\} = \frac{1}{2}$.*

Definition 5.5 ([1, Page 331, Section 5.2], [3, (5.2.6)])**.** *A **Riemann sum** for f on the interval $[a,b]$ with respect to a partition P is*

$$R_f(P) = \sum_{i=1}^{n} f(c_i) \cdot |x_i - x_{i-1}|, \quad \text{where } c_i \in [x_{i-1}, x_i].$$

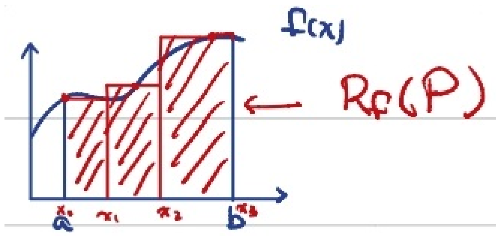


FIGURE 37. Riemann sum for f w.r.t. P

Remark 5.6 ([3, (5.2.2)]). *Theoretically, people consider upper Riemann sums and lower Riemann sums. One can find the descriptions of upper/lower Riemann sums on the following websites:*

- *upper Riemann sum*
- *lower Riemann sum*

Definition 5.7 ([1, Page 334, Section 5.3], [3, (5.2.7)]). *A function f is said to be* ***(Riemann) integrable*** *on $[a, b]$ if the limit $\lim_{\|P\|\to 0} R_f(P)$ exists and doesn't depend on the choices of Riemann sums. In this case, the number*

$$\int_a^b f(x)\,dx := \lim_{\|P\|\to 0} R_f(P)$$

is called the ***(definite) integral*** *of f from a to b.*

Theorem 5.8 ([1, Theorem 1, Section 5.3]). *If f is continuous on $[a, b]$, then f is integrable on $[a, b]$.*

The proof of the above theorem might require some properties on uniform continuity, so we postpone the proof to the part of Advanced Calculus.

Remark 5.9. *The values of f at* finite *points do not affect the integrability and the integral $\int_a^b f(x)\,dx$.*

Example 5.10. *Let $f(x) = 1,\ \forall x \in [0, 1]$. If $P = \{0, \frac{1}{3}, \frac{1}{2}, 1\}$, then*

$$R_f(P) = 1 \cdot (\frac{1}{3} - 0) + 1 \cdot (\frac{1}{2} - \frac{1}{3}) + 1 \cdot (1 - \frac{1}{2}) = 1.$$

In fact, for any choice of Riemann sum, one has $R_f(P) = 1$, and thus $\int_0^1 1\,dx = 1$.

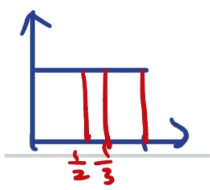


FIGURE 38. The value of $R_f(P)$

Example 5.11. *Let* $f(x) = x$, $\forall x \in [0,1]$. *Suppose* $P_n = \{0, \frac{1}{n}, \frac{2}{n}, \cdots, \frac{n}{n} = 1\}$, *and* $c_i = \frac{i}{n} \in [\frac{i-1}{n}, \frac{i}{n}]$. *The Riemann sum* $R_f(P_n)$ *is*

$$R_f(P_n) = \sum_{i=1}^{n} f(c_i) \cdot (\frac{i}{n} - \frac{i-1}{n}) = \sum_{i=1}^{n} \frac{i}{n} \cdot \frac{1}{n} = \frac{n+1}{2n} \to \frac{1}{2}, \quad as\ n \to \infty.$$

Thus $\int_0^1 x\,dx = \frac{1}{2}$.

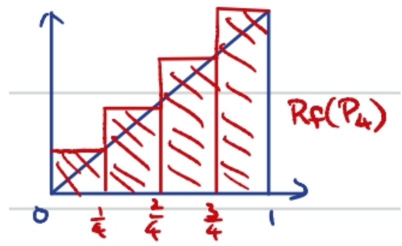


FIGURE 39. The value of $R_f(P)$

Example 5.12 ([1, Example 1, Section 5.3])**.** *The function*

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is rational,} \\ 0, & \text{if } x \text{ is irrational,} \end{cases}$$

is NOT (Riemann) integrable.

Theorem 5.13 ([1, Theorem 2, Section 5.3], [3, Section 5.3, Section 5.4 & Section 5.8])**.** *Suppose f and g are integrable over* $[a,b]$.

(1) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$.
(2) $\int_a^a f(x)\,dx = 0$.
(3) $\int_a^b \alpha f(x) + \beta g(x)\,dx = \alpha \cdot \int_a^b f(x)\,dx + \beta \cdot \int_a^b g(x)\,dx$, *for any* $\alpha, \beta \in \mathbb{R}$.
(4) $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$.
(5) *If* $m \le f(x) \le M$, $\forall x \in [a,b]$, *then* $m \cdot (b-a) \le \int_a^b f(x)\,dx \le M \cdot (b-a)$.
(6) *If* $f(x) \ge g(x)$, $\forall x \in [a,b]$, *then* $\int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$.

5.2. **Fundamental theorem of calculus.**

Theorem 5.14 (Fundamental Theorem of Calculus, [1, Section 5.4, Theorem 4], [3, Theorems 5.3.5 & 5.4.2])**.** *Suppose f is a continuous function on* $[a,b]$.

(1) *The function* $F(x) := \int_a^x f(t)\,dt$ *is continuous on* $[a,b]$ *and differentiable on* (a,b). *Furthermore,*

$$F'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x), \quad \forall x \in (a,b).$$

(2) *Suppose G is another function which is continuous on $[a,b]$ and differentiable on (a,b). If $G'(x) = f(x)$, $\forall x \in (a,b)$, then*

$$\int_a^b f(x)\,dx = \int_a^b G'(x)\,dx = G(b) - G(a).$$

*Such a function G is called an **antiderivative** of f.*

Proof. For the first assertion, note that

$$\frac{F(x+h) - F(x)}{h} = \frac{1}{h}\left(\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt\right) = \frac{1}{h}\int_x^{x+h} f(t)\,dt.$$

Since f is continuous on $[x, x+h]$, there exists $x_{m,h}, x_{M,h} \in [x, x+h]$ such that (i) $m_h := f(x_{m,h})$ is the minimum value of f on $[x, x+h]$, and (ii) $M_h := f(x_{M,h})$ is the maximum value of f on $[x, x+h]$. (We implicitly assume $h > 0$. One can prove the claim for the case of $h < 0$ by a similar argument.) Thus,

$$m_h = \frac{1}{h} \cdot m_h \cdot h \leq \frac{1}{h}\int_x^{x+h} f(t)\,dt \leq \frac{1}{h} \cdot M_h \cdot h = M_h.$$

Furthermore, the continuity of f implies that $\lim_{h\to 0} m_h = \lim_{h\to 0} M_h = f(x)$, and thus

$$F'(x) = \lim_{h\to 0} \frac{F(x+h) - F(x)}{h} = f(x).$$

This proves that $F(x)$ is differentiable on (a,b) and $F'(x) = f(x)$ for all $x \in (a,b)$.

Since F is differentiable on (a,b), it is continuous on (a,b). To prove that F is continuous on $[a,b]$, it remains to show that

$$\lim_{x\to a^+} F(x) = F(a) \qquad \text{and} \qquad \lim_{x\to b^-} F(x) = F(b).$$

We prove the second equation here, and the other one is similar. Since f is continuous on $[a,b]$, there exist m and M such that

$$m \leq f(x) \leq M, \qquad \forall x \in [a,b].$$

Thus,

$$(b-x)\,m \leq \int_x^b f(t)\,dt \leq (b-x)\,M, \qquad \forall x \in [a,b].$$

By the pinching theorem, we have

$$\lim_{x\to b^-} \int_x^b f(t)\,dt = 0.$$

Therefore,

$$\lim_{x\to b^-} (F(x) - F(b)) = \lim_{x\to b^-} \left(\int_a^x f(t)\,dt - \int_a^b f(t)\,dt\right) = -\lim_{x\to b^-} \int_x^b f(t)\,dt = 0$$

which shows the continuity of F at b.

For the second assertion, since $F'(x) = f(x) = G'(x)$ for all $x \in (a,b)$, it follows from Theorem 4.12 that there exists $c \in \mathbb{R}$ such that $G(x) = F(x) + c$ for all $x \in (a,b)$. Furthermore, by the continuity of F and G on $[a,b]$, we have

$$G(a) = \lim_{x\to a^+} G(x) = \lim_{x\to a^+} (F(x) + c) = F(a) + c,$$
$$G(b) = \lim_{x\to b^-} G(x) = \lim_{x\to b^-} (F(x) + c) = F(b) + c.$$

Thus,

$$G(x) = F(x) + c, \qquad \forall x \in [a, b].$$

Consequently,

$$\int_a^b f(x)\,dx = F(b) - F(a) = (F(b) + c) - (F(a) + c) = G(b) - G(a),$$

as desired. □

Example 5.15. *Calculate.*

(1) *If* $F(x) := \int_0^x \sin(\pi t)\,dt$, *then* $F'(\frac{3}{4}) = \sin(\frac{3}{4}\pi) = \frac{\sqrt{2}}{2}$.

(2) $\int_1^4 x^2\,dx = \frac{1}{3}x^3\Big|_1^4 = \frac{4^3}{3} - \frac{1^3}{3} = \frac{64-1}{3} = 21.$

(3) $\int_0^{\frac{\pi}{2}} \sin x\,dx = -\cos x\Big|_0^{\frac{\pi}{2}} = -\cos\frac{\pi}{2} - (-\cos 0) = 0 - (-1) = 1.$

(4) $\int_0^1 2x - 6x^4 + 5\,dx = (x^2 - \frac{6}{5}x^5 + 5x)\Big|_0^1 = 1 - \frac{6}{5} + 5 = \frac{24}{5}.$

(5) $\int_0^1 x^{\frac{3}{2}} - x^{\frac{1}{2}}\,dx = (\frac{2}{5}x^{\frac{5}{2}} - \frac{2}{3}x^{\frac{3}{2}})\Big|_0^1 = \frac{2}{5} - \frac{2}{3} = -\frac{4}{15}.$

(6) $\int_{-1}^1 |x|\,dx = \int_{-1}^0 (-x)\,dx + \int_0^1 x\,dx = 1.$

(7) $\int_0^{\frac{\pi}{4}} \sec x \cdot (2\tan x - 5\sec x)\,dx = 2\int_0^{\frac{\pi}{4}} \sec x \tan x\,dx - 5\int_0^{\frac{\pi}{4}} \sec^2 x\,dx$
$= 2\sec x\Big|_0^{\frac{\pi}{4}} - 5\tan x\Big|_0^{\frac{\pi}{4}} = 2(\sqrt{2} - 1) - 5(1 - 0) = 2\sqrt{2} - 7.$

Remark 5.16 (Indefinite Integral, [1, Section 5.5], [3, Section 5.6])**.** *Some people write*

$$\int f(x)\,dx = F(x) + C$$

if $F'(x) = f(x)$. *For example,* $\int x\,dx = \frac{1}{2}x^2 + C$.

Example 5.17. *Find the area of the region bounded by the x-axis and the graph of* $f(x) = 4 - x^2$.

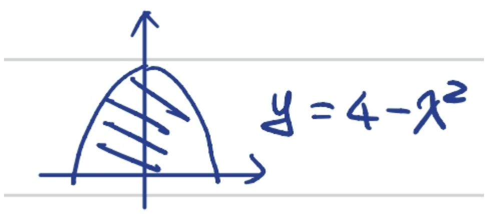


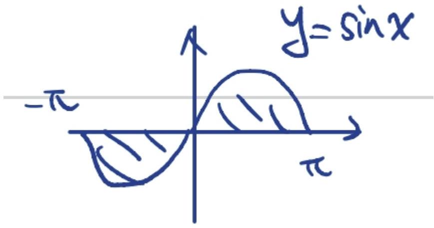
FIGURE 40. Region bounded by the x-axis and the graph of $f(x) = 4 - x^2$

Solution. Since the function $f(x)$ and x-axis intersect at the points $(-2, 0)$ and $(2, 0)$, the area is

$$\int_{-2}^2 4 - x^2\,dx = 4x - \frac{1}{3}x^3\Big|_{-2}^2 = 16 - \frac{1}{3}(8 + 8) = \frac{32}{3}.$$

□

Example 5.18. *Find the area of the following region.*



Solution. The curve $y = \sin x$ and the x-axis intersects at the point $(-\pi, 0)$, $(0, 0)$ and $(\pi, 0)$ in interval $[-\pi, \pi]$. Thus, the area is

$$\int_0^{\pi} \sin x \, dx - \int_{-\pi}^{0} \sin x \, dx = -\cos x \Big|_0^{\pi} + \cos x \Big|_{-\pi}^{0} = 1 + 1 + 1 + 1 = 4.$$

□

5.3. **Integration by substitution and integration by parts.**

Theorem 5.19 (Integration by Substitution, a.k.a. u-substitution, [1, Theorem 6 in Section 5.5 & Theorem 7 in Section 5.6], [3, Theorem 5.7.1 & Equation 5.7.2])**.** *If f is continuous, $F' = f$ and u is differentiable, then*

$$\int f(u(x)) \cdot u'(x) \, dx = F(u(x)) + C,$$

and

$$\int_a^b f(u(x)) \cdot u'(x) \, dx = \int_{u(a)}^{u(b)} f(u) du.$$

Proof. Use the chain rule of derivative. □

Convenient notation:

$$\int f(u(x))u'(x) \, dx = \int f(u) du, \qquad du = u' dx.$$

Example 5.20. *Calculate the integral* $\displaystyle\int_0^1 \frac{1}{(3 + 5x)^2} \, dx$.

Solution. Let $u = 3 + 5x$. Then we have $du = 5dx$.

$$\int_0^1 \frac{1}{(3 + 5x)^2} \, dx = \int_0^1 \frac{1}{(3 + 5x)^2} \cdot \frac{1}{5} 5dx = \int_3^8 \frac{1}{5u^2} \, du = -\frac{1}{5u} \Big|_3^8 = -\frac{1}{40} + \frac{1}{15} = \frac{1}{24}.$$

□

Example 5.21. *Calculate the integral* $\displaystyle\int_0^1 x^2 \sqrt{4 + x^3} \, dx$.

Solution. Let $u = 4 + x^3$. Then we have $du = 3x^2 dx$ which implies $\frac{1}{3} du = x^2 dx$.

$$\int_0^1 x^2 \sqrt{4 + x^3} \, dx = \int_4^5 \sqrt{u} \, \frac{1}{3} du = \frac{1}{3} \cdot \frac{2}{3} u^{\frac{3}{2}} \Big|_4^5 = \frac{2}{9}(5\sqrt{5} - 8).$$

□

Example 5.22. *Calculate the integral* $\displaystyle\int_0^2 (x^2 - 1)(x^3 - 3x + 2)^3 \, dx$.

Solution. Let $u = x^3 - 3x + 2$. Then we have $du = (3x^2 - 3)dx$ which implies $\frac{1}{3}du = (x^2 - 1)dx$.

$$\int_0^2 (x^2 - 1)(x^3 - 3x + 2)^3\,dx = \int_2^4 u^3\,\frac{1}{3}du = \frac{1}{3}\cdot\frac{1}{4}u^4\Big|_2^4 = \frac{1}{12}(256 - 16) = 20.$$

□

Example 5.23. *Calculate the integral* $\int_0^{\frac{1}{2}} \cos^3(\pi x)\cdot\sin(\pi x)\,dx.$

Solution. Let $u = \cos(\pi x)$. Then we have $du = -\pi\sin(\pi x)dx$ which implies $-\frac{1}{\pi}du = \sin(\pi x)dx$.

$$\int_0^{\frac{1}{2}} \cos^3(\pi x)\cdot\sin(\pi x)\,dx = \int_1^0 u^3\,\frac{1}{-\pi}du = -\frac{1}{\pi}\cdot\frac{1}{4}u^4\Big|_1^0 = \frac{1}{4\pi}.$$

□

Theorem 5.24 (Integration by Parts, [1, Section 8.2], [3, Section 8.2])**.** *Suppose u, v are functions with continuous derivatives. Then*

$$\int u(x)v'(x)\,dx = u(x)v(x) - \int v(x)u'(x)\,dx,$$

and

$$\int_a^b u(x)v'(x)\,dx = u(x)v(x)\Big|_a^b - \int_a^b v(x)u'(x)\,dx$$

Proof. Use the product formula of derivative: $(uv)' = u'\cdot v + u\cdot v'$. □

Using the convenient notations $du = u'(x)dx$ and $dv = v'(x)dx$, one writes $\int u\,dv = uv - \int v\,du$.

Example 5.25. *Calculate the integral* $\int_0^{\pi} x\sin x\,dx.$

Solution. Let $u(x) = x$ and $v(x) = -\cos x$. Then we have $u'(x) = 1$ and $v'(x) = \sin x$. Thus,

$$\int_0^{\pi} x\sin x\,dx = -x\cos x\Big|_0^{\pi} - \int_0^{\pi} -\cos x\cdot 1\,dx = -\pi\cos\pi - (-\sin x)\Big|_0^{\pi} = \pi.$$

□

Example 5.26. *Calculate the integral* $\int_0^1 x\sqrt{x+1}\,dx.$

Solution. Let $u(x) = x$ and $v(x) = \frac{2}{3}(x+1)^{\frac{3}{2}}$. Then $u'(x) = 1$ and $v'(x) = \sqrt{x+1}$. Thus,

$$\int_0^1 x\sqrt{x+1}\,dx = x\cdot\frac{2}{3}(x+1)^{\frac{3}{2}}\Big|_0^1 - \int_0^1 \frac{2}{3}(x+1)^{\frac{3}{2}}\,dx = \frac{4}{3}\sqrt{2} - \frac{16}{15}\sqrt{2} + \frac{4}{15}.$$

□

5.4. **More properties of integrals.**

Theorem 5.27 (First Mean Value Theorem for Integrals, [1, Theorem 3 in Section 5.4], [3, Theorem 5.9.1])**.** *If f is continuous on $[a, b]$, then there is $c \in [a, b]$ such that*

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

*Such $f(c)$ is called the **average value** (or **mean value**) of f on $[a,b]$*

Proof. Let $F(x) = \int_a^x f(t)\,dt$. By the mean value theorem, $\exists c \in (a,b)$ such that

$$F'(c) = \frac{F(b)-F(a)}{b-a} = f(c) = \frac{1}{b-a} \cdot \int_a^b f(x)\,dx.$$

□

Example 5.28. $\displaystyle\int_0^2 x+1\,dx = 4 = f(1)\cdot(2-0).$

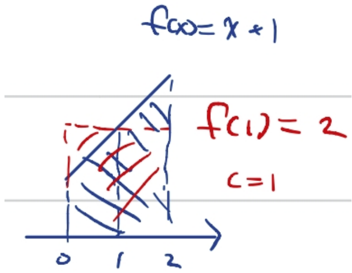


FIGURE 41. Average of $x+1$ on $[0,2]$ is $f(1)=2$

Skip.

Theorem 5.29 (Second Mean Value Theorem for Integrals, [3, Theorem 5.9.3])**.** *If f and g are continuous on $[a,b]$ and $g \geq 0$ on $[a,b]$, then $\exists c \in [a,b]$ such that*

$$\int_a^b f(x)g(x)\,dx = f(c)\cdot\int_a^b g(x)\,dx.$$

*Such $f(c)$ is called the **g-weight average** of f on $[a,b]$.*

Proof. Since f is continuous on $[a,b]$, by extreme value theorem, f takes on a minimum value m and a maximum value M. Since $g \geq 0$,

$$m\cdot g(x) \leq f(x)\cdot g(x) \leq M\cdot g(x), \quad \forall x \in [a,b]$$

$\Rightarrow \quad m\int_a^b g(x)\,dx \leq \int_a^b f(x)g(x)\,dx \leq M\int_a^b g(x)\,dx$

If $g \equiv 0$, then the proof is clearly.

If $g \neq 0$, then we have

$$m \leq \frac{\int_a^b f(x)g(x)\,dx}{\int_a^b g(x)\,dx} \leq M$$

Hence by Intermediate value theorem, $\exists c$ such that $f(c) = \dfrac{\int_a^b f(x)g(x)\,dx}{\int_a^b g(x)\,dx}$

$\Rightarrow \quad \int_a^b f(x)g(x)\,dx = f(c)\cdot\int_a^b g(x)\,dx$ □

Example 5.30. *Let $f(x) = (x^2+1)^2$ and $g(x) = 2x$*

$\int_0^1 f(x)g(x)\,dx = \int_1^2 u^2\,du = \frac{7}{3} = f(\sqrt{\sqrt{\frac{7}{3}}-1})\int_0^1 2x\,dx \quad\Rightarrow\quad$ *g-weight average of f on $[0,1]$: $\frac{7}{3}$*

Theorem 5.31 ([3, Section 5.8]). *Suppose f and g are integrable functions.*

(1) $\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx.$

(2) *If f is continuous on $[a,b]$, then* $\dfrac{d}{dx}\displaystyle\int_a^{u(x)} f(t)\,dt = f(u(x))\cdot u'(x).$

(3) *If f is an* **odd** *function on $[-a,a]$,i.e. $f(-x) = -f(x)$, $\forall x \in [-a,a]$, then*

$$\int_{-a}^{a} f(x)\,dx = 0.$$

(4) *If f is an* **even** *function on $[-a,a]$, i.e. $f(-x) = f(x)$, $\forall x \in [-a,a]$, then*

$$\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx.$$

(5) *If f is continuous on $[a,b]$ and if $\displaystyle\int_a^b |f(x)|\,dx = 0$, then $f(x) = 0$ $\forall x \in [a,b]$.*

One can see [1, Theorem 8, Section 5.6] for the properties (3) and (4).

Example 5.32. *For the property (2), we compute.*

- $\dfrac{d}{dx}\left(\displaystyle\int_0^{x^3} \frac{1}{1+t}\,dt\right) = \dfrac{1}{1+x^3}\cdot 3x^2.$
- $\dfrac{d}{dx}\left(\displaystyle\int_x^{2x} \frac{1}{1+t^2}\,dt\right) = \dfrac{2}{1+(2x)^2} - \dfrac{1}{1+x^2}.$

Example 5.33. *For the properties (3) and (4), we compute.*

- $\displaystyle\int_{-1}^{1} x\,dx = 0.$
- $\displaystyle\int_{-1}^{1} x^2\,dx = 2\int_0^1 x^2\,dx = \frac{2}{3}.$

Remark 5.34. *For functions with finite discontinuities, one can compute the integration as follows. Let*

$$f(x) = \begin{cases} f_1(x), & x \in [a_0, a_1), \\ \quad\vdots & \\ f_n(x), & x \in [a_{n-1}, a_n]. \end{cases}$$

Then

$$\int_{a_0}^{a_n} f(x)\,dx = \int_{a_0}^{a_1} f_1(x)\,dx + \cdots \int_{a_{n-1}}^{a_n} f_n(x)\,dx.$$

Example 5.35. *Integrate.*

- *If* $f(x) = \begin{cases} x, & x > 0, \\ 2, & x = 0, \\ 1, & x < 0, \end{cases}$ *then* $\displaystyle\int_{-1}^{1} f(x)\,dx = \int_{-1}^{0} 1\,dx + \int_0^1 x\,dx = \frac{3}{2}.$
- *If* $g(x) = \begin{cases} 0, & x \text{ is rational}, \\ 1, & x \text{ is irrational}, \end{cases}$ *then g is NOT Riemann-integrable.*

6. Transcendental functions

6.1. **Inverse functions.** Recall that a **function** $f : A \to B$ has 3 ingredients:

- Domain = A is the set of possible inputs.
- Codomain = B is the set of possible outputs.
- How it works.

Definition 6.1. *Suppose we are given a function $f : A \to B$.*

- *We say f is **one-to-one** (or **1-1**) if it satisfies the condition: if $x_1, x_2 \in A$ and $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$.*
- *The **range** (or **image**) of f is the set* $\operatorname{range}(f) = \operatorname{im}(f) = \{f(x) \in B \mid x \in A\}$.
- *We say f is **onto** if for any $y \in B$, there exists $x \in A$ such that $f(x) = y$. That is, range$(f) = B$.*

Example 6.2. *Let*

- *$f : (0, \infty) \to (0, \infty)$ be the function defined by $f(x) = \dfrac{1}{x}$.*
- *$g : (-\infty, 0) \cup (0, \infty) \to \mathbb{R}$ be the function defined by $g(x) = \dfrac{1}{x}$.*

The function f is 1-1 and onto. The function g is 1-1 but NOT onto, because for $0 \in \mathbb{R}$, one has $g(x) \neq 0$, $\forall x \in (-\infty, 0) \cup (0, \infty)$.

Definition 6.3. *A function $f : A \to B$ is **invertible** if there exists $g : B \to A$ such that*

- *$g(f(x)) = x, \ \forall x \in A$,*
- *$f(g(y)) = y, \ \forall y \in B$.*

*Such a function $g : B \to A$ is called the **inverse** of f, denoted by f^{-1}.*

Theorem 6.4. *A function $f : A \to B$ is invertible if and only if f is 1-1 and onto.*

Remark 6.5. *If $f : A \to B$ is 1-1, then its corestriction $f : A \to \operatorname{im}(f)$ is 1-1 and onto, and hence invertible. See* [3, Theorem 7.1.2].

Example 6.6. *Determine which functions are invertible, and compute the inverses.*

- *$f(x) = x^3$ is 1-1 and onto on $\mathbb{R}$.*
  $y = f(f^{-1}(y)) = (f^{-1}(y))^3 \quad \Rightarrow \quad f^{-1}(y) = \sqrt[3]{y}$.
- *$f(x) = 3x - 5$ is 1-1 and onto on $\mathbb{R}$.*
  $y = f(f^{-1}(y)) = 3f^{-1}(y) - 5 \quad \Rightarrow \quad f^{-1}(y) = \dfrac{1}{3}y + \dfrac{5}{3}$.
- *Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$.*
  The function $f(x)$ is NOT 1-1 because $f(x) = f(-x) = x^2$ for any $x > 0 \Rightarrow x \neq -x$. It is NOT onto, either, because $x^2 \geq 0$ for any $x \in \mathbb{R}$. In particular, $f : \mathbb{R} \to \mathbb{R}$ is NOT invertible.

Example 6.7. *Show that $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^5 + 2x^3 + 3x - 4$ is 1-1.*

Proof. The directest method is checking $f(x_1) = f(x_2)$ imply $x_1 = x_2$, for any $x_1, x_2 \in \mathbb{R}$. But it is not so obvious in this case, so we introduce a different method.

Since the derivative $f'(x) = 5x^4 + 6x^2 + 3 > 0$, $\forall x \in \mathbb{R}$, we have f is strictly increasing on $\mathbb{R}$. (Recall Theorem 4.10.) In other words,

$$x_1 < x_2 \quad \Rightarrow \quad f(x_1) < f(x_2) \quad \forall x_1, x_2 \in \mathbb{R}.$$

Thus, $x_1 \neq x_2 \;\Rightarrow\; x_1 < x_2$ or $x_2 < x_1 \;\Rightarrow\; f(x_1) \neq f(x_2)$, so f is 1-1 on $\mathbb{R}$. □

Theorem 6.8 ([1, Theorem 3 in Section 3.8], [3, Theorem 7.1.8])**.** *Let f be a 1-1 differentiable function, and let $f(a) = b$. If $f'(a) \neq 0$, then f^{-1} is differentiable at b and*

$$(f^{-1})'(b) = \frac{1}{f'(a)}.$$

Proof. We omit the proof of the differentiability of f^{-1}. See [3, Theorem B.3.2] for the proof of differentiability.

Differentiating $x = f^{-1}(f(x))$, we have

$$1 = (f^{-1})'(f(x)) \cdot f'(x) \quad \Rightarrow \quad (f^{-1})'(f(x)) = \frac{1}{f'(x)}.$$

□

Example 6.9. *Let $f(x) = x^3 + \frac{1}{2}x$. Find the value of $(f^{-1})'(9)$.*

Solution. Since $f'(x) = 3x^2 + \frac{1}{2} > 0$, $\forall x \in \mathbb{R}$, the function f is 1-1. It is easy to check $f(2) = 9$. Thus,

$$x = (f^{-1})(x^3 + \frac{1}{2}x) \cdot (3x^2 + \frac{1}{2}) \quad \Rightarrow \quad (f^{-1})'(9) = \frac{1}{3 \cdot 2^2 + \frac{1}{2}} = \frac{2}{25}.$$

□

6.2. **The logarithm function.** Recall that the derivative of $\frac{x^{n+1}}{n+1}$, $n \neq -1$, is equal to x^n. Thus,

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C, \qquad \forall n \neq -1.$$

One may have the question: Is there a function f such that $\frac{df(x)}{dx} = \frac{1}{x}$? The answer is obviously yes by the fundamental theorem of Calculus.

Definition 6.10 ([1, Page 441, Section 7.1], [3, Section 7.2])**.** *The function*

$$\ln x := \int_1^x \frac{1}{t} \, dt, \qquad x > 0,$$

*is called the **(natural) logarithm function**.*

Note that many people write "$\log x$" for "$\ln x$."

The following theorem is immediate from the properties of integration.

Theorem 6.11 ([3, Section 7.3])**.** *One has the following properties.*

(1) *The function $\ln x$ is a differentiable function from $(0, \infty)$ to $\mathbb{R}$.*

(2) $\frac{d}{dx}(\ln |x|) = \frac{1}{x}$, *for all $x \neq 0$. In particular, the function $\ln x$ strictly increases on $(0, \infty)$.*

$$(3)\ \ln x \begin{cases} > 0, & x > 1, \\ = 0, & x = 1, \\ < 0, & 0 < x < 1. \end{cases}$$

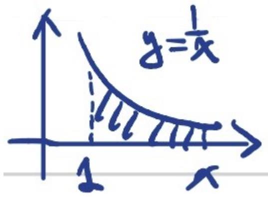


FIGURE 42. $\ln x = \int_1^x \frac{1}{t}\,dt$ is the area of the region

Theorem 6.12 ([1, Page 443, Section 7.1], [3, Section 7.2]). *The function* $\ln x$ *satisfies the following equations for any* $a, b > 0$.

(1) $\ln(ab) = \ln a + \ln b$.
(2) $\ln(\frac{1}{b}) = -\ln b$.
(3) $\ln(\frac{a}{b}) = \ln a - \ln b$.
(4) $\ln(a^{\frac{p}{q}}) = \frac{p}{q} \cdot \ln a$.

Proof. Since

$$\frac{d}{dx}(\ln(xb)) = \frac{1}{xb} \cdot b = \frac{1}{x} = \frac{d}{dx}(\ln x), \qquad \forall x > 0,$$

we have $\ln(xb) = \ln x + c$ for some constant $c \in \mathbb{R}$. By taking $x = 1$, we have $\ln b = 0 + c = c$, and thus $\ln(xb) = \ln x + \ln b$. This shows the first assertion. The other assertions follow from the first one and basic algebraic techniques. □

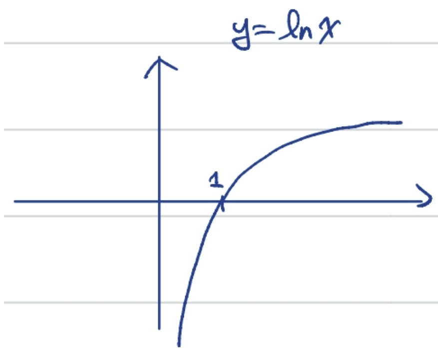


FIGURE 43. Graph of $\ln x$

The above theorem (fourth claim) implies the range of function ln is $(-\infty, \infty)$. Therefore, the function $\ln : (0, \infty) \to (-\infty, \infty)$ is invertible, and there exists a unique number $e \in (0, \infty)$ such that $\ln(e) = 1$.

Definition 6.13 ([3, Section 7.2]). *The number e is the unique number with the property*

$$\ln(e) = 1 = \int_1^e \frac{1}{t}\,dt.$$

Remark 6.14. *In fact, the number $e = 2.718...$ is a transcendental number and, in particular, an irrational number.*

n	1	2	3	4
$\ln n$	0	0.69	1.1	1.39

TABLE 3. Approximation of $\ln x$

Example 6.15. *Skip. Let $f(x) = x\ln x$ which is defined on $(0,\infty)$. Then $f'(x) = \ln x + 1$. The critical point of f is $x = \frac{1}{e}$, and f is decreasing on $(0, \frac{1}{e})$ and increasing on $(\frac{1}{e}, \infty)$.*

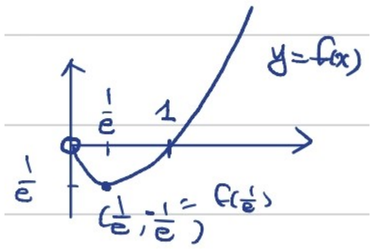


FIGURE 44. Graph of $x\ln x$

Example 6.16. *Skip. Let $f(x) = \ln(\frac{x^4}{x-1}) = 4\ln x - \ln(x-1)$ which is defined on $(1,\infty)$. Then $f'(x) = \frac{4}{x} - \frac{1}{x-1} = \frac{3x-4}{x(x-1)}$.The critical point of f is $x = \frac{4}{3}$, and f is decreasing on $(1, \frac{4}{3})$ and increasing on $(\frac{4}{3}, \infty)$. Since $f''(x) = -\frac{(x-2)(3x-2)}{x^2(x-1)^2}$, the function f has a point of inflection at $x = 2$.*

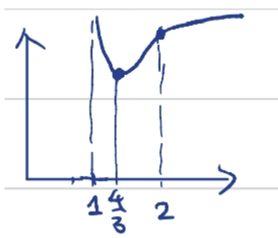


FIGURE 45. Graph of $\ln(\frac{x^4}{x-1})$

Remark 6.17 ([3, Equation (7.3.2)])**.** *Since $\frac{d}{dx}(\ln|x|) = \frac{1}{x}$, $\forall x \neq 0$, one has*

$$\int \frac{1}{x}\,dx = \ln|x| + C.$$

Example 6.18. *Calculate.*

(1) Skip. $\int \frac{x^2}{1-4x^3}\,dx$. Let $u = 1-4x^3$. Then $du = -12x^2dx$, and thus

$$\int \frac{x^2}{1-4x^3}\,dx = \int \frac{1}{-12u}\,du = -\frac{1}{12}\ln|u| + c = -\frac{1}{12}\ln|1-4x^3| + C.$$

(2) $\int_1^2 \frac{6x^2+2}{x^3+x+1}\,dx$. *Let* $u = x^3 + x + 1$. *Then*

$$\int_1^2 \frac{6x^2+2}{x^3+x+1}\,dx = \int_{u(1)}^{u(2)} \frac{2}{u}\,du = 2\ln|u|\Big|_{u=3}^{11} = 2\ln(\frac{11}{3}).$$

(3) Skip. $\int \frac{\ln x}{x}\,dx$. *Let* $u = \ln x$. *Then* $du = \frac{1}{x}dx$, *and*

$$\int \frac{\ln x}{x}\,dx = \int u\,du = \frac{1}{2}u^2 + c = \frac{1}{2}(\ln x)^2 + C.$$

(4) $\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx$. *Let* $u = \cos x$. *Then* $du = -\sin x\,dx$, *and*

$$\int \tan x\,dx = -\int \frac{1}{u}\,du = -\ln|u| + C = \ln|\sec x| + C.$$

(5) Similarly, $\int \cot x\,dx = \ln|\sin x| + C$.

Skip to exponential map.

Theorem 6.19 ([1, Page 363, Section 5.5], [3, Equation (7.3.3)]). *One has the following formulas.*

- $\int \sin x\,dx = -\cos x + C.$
- $\int \cos x\,dx = \sin x + C.$
- $\int \tan x\,dx = \ln|\sec x| + C.$
- $\int \csc x\,dx = \ln|\csc x - \cot x| + C.$
- $\int \sec x\,dx = \ln|\sec x + \tan x| + C.$
- $\int \cot x\,dx = \ln|\sin x| + C.$

Remark 6.20. *Be careful about the interval of integration. If the integrand is not continuous on the interval, these formulas may not apply directly.*

Example 6.21. *Consider the integral* $\int_0^a \tan x\,dx$.

- *If* $a = \frac{\pi}{4}$, $\tan x$ *is continuous on* $[0, a] = [0, \frac{\pi}{4}]$, *and* $\int_0^{\frac{\pi}{4}} \tan x\,dx = -\ln|\cos x|\Big|_0^{\frac{\pi}{4}} = \frac{1}{2}\ln 2$.
- *If* $a = \frac{\pi}{2}$, $\tan x$ *is continuous on* $[0, a) = [0, \frac{\pi}{2})$, *but* $\int_0^{\frac{\pi}{2}} \tan x\,dx = -\ln|\cos x|\Big|_0^{\frac{\pi}{2}} = \infty$ *which means* $\tan x$ *is NOT integrable on* $[0, \frac{\pi}{2})$.
- *If* $a = \pi$, $\tan x$ *is NOT continuous at* $x = \frac{\pi}{2}$, *and* $\int_0^{\pi} \tan x\,dx = \int_0^{\frac{\pi}{2}} \tan x\,dx + \int_{\frac{\pi}{2}}^{\pi} \tan x\,dx$ *does NOT exist. (Because* $\int_0^{\frac{\pi}{2}} \tan x\,dx$ *doesn't exist.) Note that*

$$\int_0^{\pi} \tan x\,dx \neq -\ln|\cos x|\Big|_0^{\pi} = -\ln 1 + \ln 1 = 0.$$

6.3. **Exponential function.** Since the function $\ln : (0, \infty) \to (-\infty, \infty)$ is 1-1 and onto, there exists an inverse function of ln which is called the (natural) exponential function.

Definition 6.22. *The* ***(natural) exponential function***

$$\exp : (-\infty, \infty) \to (0, \infty)$$

is defined to be the inverse of $\ln : (0, \infty) \to (-\infty, \infty)$*. That is, it is the function satisfying the equations*

$$\ln(\exp(x)) = x, \qquad \forall\, x \in (-\infty, \infty),$$
$$\exp(\ln(y)) = y, \qquad \forall\, y \in (0, \infty).$$

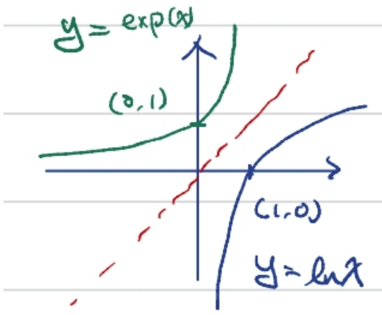


FIGURE 46. Exponential and logarithm function

Theorem 6.23 ([1, Theorem 1 in Section 7.1], [3, Section 7.4])**.** *The exponential function satisfies the following properties.*

(1) $\exp(0) = 1$ *and* $\exp(1) = e$. *(Because* $\ln(1) = 0$ *and* $\ln(e) = 1$*.)*
(2) $\exp(x) > 0, \quad \forall\, x \in \mathbb{R}$.
(3) $\lim_{x \to -\infty} \exp(x) = 0$.
(4) $\exp$ *is a strictly increasing function.*
(5) $\exp(a + b) = \exp(a) \cdot \exp(b)$.
(Since $\ln(\exp(a + b)) = a + b = \ln(\exp(a)) + \ln(\exp(b)) = \ln(\exp(a) \cdot \exp(b))$*, we have that* $\exp(a + b) = \exp(a) \cdot \exp(b)$*.)*
(6) $\exp(n) = \exp(1) \cdots \exp(1) = e^n$.
(7) $\exp(\frac{1}{n}) = e^{\frac{1}{n}}$. *(Because* $\exp(\frac{1}{n}) \cdots \exp(\frac{1}{n}) = \exp(1)$*.)*
(8) $\exp(\frac{p}{q}) = e^{\frac{p}{q}}$.

Remark 6.24. *By the above properties, one can see that* $\exp(x)$ *is the only continuous function such that* $\exp(x) = e^x$*,* $\forall x \in \mathbb{Q}$*, and thus its inverse function* $\ln = \log_e$*. For* $x \notin \mathbb{Q}$*, the number* e^x *is not defined yet, and we just* define *it to be* $\exp(x)$*.*

Theorem 6.25 ([1, Page 445-446, Section 7.1], [3, Theorem 7.4.9])**.** *The function* $\exp(x) = e^x$ *is differentiable at any* $x \in \mathbb{R}$*, and*

$$\frac{d}{dx}(e^x) = e^x.$$

Therefore,

$$\int e^x \, dx = e^x + C.$$

Proof. Since $x = \ln(e^x)$, we have

$$1 = \frac{d}{dx}(\ln(e^x)) = \frac{1}{e^x} \cdot \frac{d}{dx}(e^x)$$

which implies the desired equation $\frac{d}{dx}(e^x) = e^x$. □

Example 6.26. *Calculate.*

(1) $\frac{d}{dx}(e^{ax}) = a \cdot e^{ax}$.
(2) $\frac{d}{dx}(e^{\sqrt{x}}) = e^{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}}$.
(3) $\frac{d}{dx}(xe^{-x}) = e^{-x} - xe^{-x}$.

Example 6.27. *Calculate.*

(1) $\int_0^1 9e^{3x}\,dx = 3e^{3x}\Big|_0^1 = 3e^3 - 3$.
(2) $\int_0^1 \frac{e^{3x}}{e^{3x}+1}\,dx \quad (u = e^{3x},\ du = 3e^{3x}dx)$
$= \int_{u(0)}^{u(1)} \frac{1}{3(u+1)}\,du = \frac{1}{3}\ln(u+1)\Big|_1^{e^3} = \frac{1}{3}\ln(\frac{e^3+1}{2})$.
(3) Skip. $\int_0^1 e^x(e^x+1)^{\frac{1}{5}}\,dx \quad (u = e^x + 1,\ du = e^x dx)$
$= \int_{u(0)}^{u(1)} u^{\frac{1}{5}}\,du = \frac{5}{6}u^{\frac{6}{5}}\Big|_2^{e+1} = \frac{5}{6}((e+1)^{\frac{6}{5}} - 2^{\frac{6}{5}})$.
(4) $\int_0^1 xe^x\,dx = xe^x\Big|_0^1 - \int_0^1 e^x\,dx = e - e^x\Big|_0^1 = e - e + 1 = 1.$ *(Integration by parts.)*
(5) Skip. $\int_0^1 x^2e^x\,dx = x^2e^x\Big|_0^1 - \int_0^1 2xe^x\,dx = e - 2\int_0^1 xe^x\,dx = e - 2.$ *(Integration by parts.)*

Theorem 6.28 ([3, Theorem 7.6.1])**.** *If*

$$f'(t) = kf(t), \qquad \forall\, t \in (a, b),$$

then there exists $C \in \mathbb{R}$ such that $f(t) = Ce^{kt}$, $\forall\, t \in (a, b)$.

Proof. By assumption, we have $f'(t) - kf(t) = 0$, and thus

$$(e^{-kt}f(t))' = e^{-kt}f'(t) - e^{-kt}kf(t) = 0$$

which implies $f(t) = Ce^{kt}$. □

Theorem 6.29. *One has that $\lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n = e$. (Maybe skip to the next general theorem)*

Proof. Recall that, in our definition, e is the positive number with the property

$$\ln e = \int_1^e \frac{1}{t}\,dt = 1.$$

Given any positive integer n, since

$$\frac{n}{n+1} = \frac{1}{1+\frac{1}{n}} \le \frac{1}{t} \le 1, \qquad \forall\, t \in [1, 1 + \frac{1}{n}],$$

we have

$$\frac{1}{n+1} \leq \ln\left(1+\frac{1}{n}\right) = \int_1^{1+\frac{1}{n}} \frac{1}{t}\, dt \leq \frac{1}{n}.$$

Therefore,

$$e^{\frac{1}{n+1}} \leq 1+\frac{1}{n} \leq e^{\frac{1}{n}},$$

which implies that

$$\left(1+\frac{1}{n}\right)^n \leq e \leq \left(1+\frac{1}{n}\right)^{n+1}.$$

Dividing the right inequality by $1+1/n$, we have

$$\frac{e}{1+\frac{1}{n}} \leq \left(1+\frac{1}{n}\right)^n \leq e.$$

Since $\lim_{n\to\infty} \frac{e}{1+\frac{1}{n}} = e$, the desired equation follows from the pinching theorem. □

Theorem 6.30 ([3, Section 11.4])**.** *One has that* $\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x$.

Proof. One can use a proof similar to Theorem 6.29. Here, we prove it by a different argument.

For $x = 0$, the result is clear. For $x \neq 0$,

$$\ln\left(1+\frac{x}{n}\right)^n = n\ln\left(1+\frac{x}{n}\right) = x\left(\frac{\ln(1+x/n) - \ln 1}{x/n}\right)$$

which converges to $x \cdot (\ln t)'\Big|_{t=1} = x$. Therefore,

$$e^x = e^{\lim_{n\to\infty}\ln(1+\frac{x}{n})^n} = \lim_{n\to\infty} e^{\ln(1+\frac{x}{n})^n} = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n,$$

as desired. □

6.4. Arbitrary powers.

In high school, we discussed numbers with rational powers such as

$$10^5, 2^{\frac{1}{3}}, \pi^{-\frac{3}{2}}, e^{-\frac{1}{2}} \cdots,$$

but we didn't define what are numbers with irrational powers such as

$$10^{\sqrt{2}}, 2^{\pi}, \pi^{-\sqrt{3}}, e^{-\pi} \cdots.$$

In the previous section, we say $e^x = \exp(x)$ for $x \notin \mathbb{Q}$. In general, we define them as follows.

Definition 6.31 ([1, Page 446, Section 7.1], [3, (7.5.1)])**.** *Let* $a > 0$, $x \in \mathbb{R}$ *which is not necessarily rational. We define*

$$a^x := \exp(x \cdot \ln a) = e^{x\ln a}$$

which is the unique continuous function coinciding with a^x *at any rational numbers* $x = \frac{p}{q} \in \mathbb{Q}$.

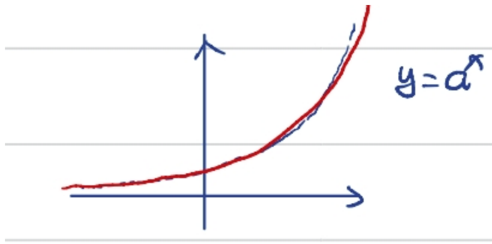


FIGURE 47. Graph of $y = a^x$

Theorem 6.32 ([3, (7.5.2)]). *For any $x, y \in \mathbb{R}$, and $a > 0$,*

(1) $a^{x+y} = a^x \cdot a^y$,
(2) $a^{x-y} = \frac{a^x}{a^y}$,
(3) $(a^x)^y = a^{xy}$.

Proof. For (1),

$$a^{x+y} = \exp((x + y) \cdot \ln a) = \exp(x \ln a) \cdot \exp(y \cdot \ln a) = a^x \cdot a^y.$$

The other two are similar. □

Theorem 6.33 ([3, (7.5.3) & (7.5.4)]). *Let $r \in \mathbb{R}$, $x > 0$, we have*

$$\frac{d}{dx}(x^r) = rx^{r-1}.$$

Thus, for $r \neq -1$,

$$\int x^r \, dx = \frac{1}{r+1} x^{r+1} + C.$$

Proof. $\frac{d}{dx}(x^r) = \frac{d}{dx}(e^{r \ln x}) = e^{r \ln x} \cdot \frac{r}{x} = r \cdot x^r \cdot x^{-1} = rx^{r-1}$. □

Example 6.34. *Skip. Calculate.*

- *The derivative of x^π is $\frac{d}{dx}(x^\pi) = \pi x^{\pi-1}$.*
- *The integral $\int_0^1 \frac{x^3}{(2x^4 + 1)^\pi} dx$. Let $u = 2x^4 + 1$. Then $du = 8x^3 dx$, and*

$$\int_1^3 \frac{1}{8u^\pi} du = \frac{1}{8} \cdot \frac{u^{-\pi+1}}{-\pi + 1} \Big|_1^3 = \frac{3^{1-\pi} - 1}{8(1 - \pi)}.$$

Theorem 6.35 ([1, Page 447, Section 7.1], [3, (7.5.5) & (7.5.6)]). *For $a > 0$, the function a^x is differentiable on $\mathbb{R}$, and*

$$\frac{d}{dx}(a^x) = a^x \cdot \ln a.$$

Thus,

$$\int a^x \, dx = \frac{1}{\ln a} \cdot a^x + C.$$

Proof. $\frac{d}{dx}(a^x) = \frac{d}{dx}(e^{x \ln a}) = e^{x \ln a} \cdot \ln a = a^x \cdot \ln a$. □

Example 6.36. *Calculate the derivatives of following functions.*

- $\frac{d}{dx}(3^{x^2}) = \frac{d}{dx}(e^{\ln(3^{x^2})}) = \frac{d}{dx}(e^{x^2 \ln 3}) = e^{x^2 \ln 3} \cdot 2x \ln 3 = 2 \ln 3 \cdot x \cdot 3^{x^2}$.

- $\frac{d}{dx}((x^2+1)^{3x}) = \frac{d}{dx}(e^{3x\cdot\ln(x^2+1)}) = e^{3x\cdot\ln(x^2+1)}\cdot(3\ln(x^2+1)+3x\cdot\frac{2x}{x^2+1})$
  $= (x^2+1)^{3x}(3\ln(x^2+1)+\frac{6x^2}{x^2+1})$.

Example 6.37. *Evaluate the integrals.*

- $\int_0^1 x\cdot 5^{-x^2}\,dx \qquad (u=-x^2,\ du=-2xdx)$
  $= \int_0^{-1} 5^u\cdot\frac{-1}{2}\,du = -\frac{1}{2}\cdot\frac{5^u}{\ln 5}\Big|_0^{-1} = \frac{1}{2\ln 5}-\frac{1}{10\ln 5}.$
- *Skip.* $\int_1^2 3^{2x-1}\,dx \qquad (u=2x-1,\ du=-2xdx)$
  $= \int_1^3 3^u\cdot\frac{1}{2}\,du = \frac{1}{2\ln 3}\cdot 3^u\Big|_1^3 = \frac{12}{\ln 3}.$

Recall that for any positive number $a\neq 1$, the **logarithm with base** a, $\log_a : (0,\infty)\to\mathbb{R}$, is the inverse function of $\mathbb{R}\to(0,\infty)$, $x\mapsto a^x$.

Remark 6.38 ([1, Page 448, Section 7.1], [3, (7.5.7) & (7.5.8)])**.** *Let $a>0$, $a\neq 1$. Then*

$$\log_a(a^x) = x, \qquad \forall x\in\mathbb{R},$$
$$\log_a(x) = \frac{\ln x}{\ln a}, \qquad \forall x>0,$$

and thus

$$\frac{d}{dx}(\log_a x) = \frac{1}{\ln a}\cdot\frac{1}{x}.$$

Example 6.39. *Calculate.*

- $\frac{d}{dx}(\log_5|x|) = \frac{d}{dx}(\frac{\ln|x|}{\ln 5}) = \frac{1}{x\cdot\ln 5}.$
- $\frac{d}{dx}(\log_2(3x^2+1)) = \frac{d}{dx}(\frac{\ln(3x^2+1)}{\ln 2}) = \frac{1}{\ln 2}\cdot\frac{6x}{3x^2+1}.$
- $\int\frac{1}{x\ln 10}\,dx = \frac{\ln|x|}{\ln 10}+C = \log_{10}|x|+C.$

Proposition 6.40 ([3, Section 11.4])**.** *One has the following:*

(1) $\lim_{n\to\infty} a^{1/n} = 1$, *for any $a>0$. (By continuity of a^x and $1/n\to 0$.)*
(2) $\lim_{n\to\infty}\frac{1}{n^a} = 0$, *for any $a>0$. (Use $\frac{1}{n^a} = e^{a\ln(1/n)}$.)*
(3) $\lim_{n\to\infty}\frac{\ln n}{n} = 0$. *(Apply L'Hôpital's rule for the last two.)*
(4) $\lim_{n\to\infty} n^{1/n} = 1$.

6.5. **Inverse trigonometric functions.** In this section, we will discuss inverse trigonometric functions and their properties. See [1, Section 3.9] or [3, Section 7.7].

Definition 6.41. *The restriction of sine function*

$$\sin : [-\frac{\pi}{2},\frac{\pi}{2}]\to[-1,1]$$

*is 1-1 and onto, and its inverse is called **arc sine** and denoted $\sin^{-1}$ or arcsin. In other words,*

$$x = \sin^{-1}(\sin x) \quad \text{and} \quad y = \sin(\sin^{-1} y), \qquad \forall\, x\in[-\frac{\pi}{2},\frac{\pi}{2}],\ \forall\, y\in[-1,1].$$

FIGURE 48. Graph of arcsin

Theorem 6.42 ([3, (7.7.3)]). *The function* $\arcsin x$ *is differentiable on* $(-1,1)$ *and its derivative is*

$$\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}}.$$

Thus,

$$\int \frac{1}{\sqrt{1-x^2}}\, dx = \arcsin x + C.$$

Proof. Let $y = \sin x$. Since $x = \arcsin(\sin x) = \sin^{-1}(y)$, one has

$$1 = \frac{d \arcsin y}{dy} \cdot \frac{dy}{dx} = \frac{d \arcsin y}{dy} \cdot \cos x,$$

and thus

$$\frac{d \arcsin y}{dy} = \frac{1}{\cos x} = \frac{1}{\sqrt{1-\sin^2 x}} = \frac{1}{\sqrt{1-y^2}}, \quad \forall\, y \in (-1,1).$$

□

Example 6.43. *Calculate.*

- $\arcsin(1) = \frac{\pi}{2}$, $\arcsin(\frac{1}{2}) = \frac{\pi}{6}$, $\arcsin(0) = 0$.
- $\frac{d}{dx}(\arcsin(3x^2)) = \frac{1}{\sqrt{1-(3x^2)^2}} \cdot \frac{d}{dx}(3x^2) = \frac{6x}{\sqrt{1-9x^4}}$.
- $\int_0^{\sqrt{3}} \frac{1}{\sqrt{4-x^2}}\, dx = \int_0^{\sqrt{3}} \frac{1}{2} \cdot \frac{1}{\sqrt{1-(\frac{x}{2})^2}}\, dx$. *Let* $u = \frac{x}{2}$. *Then* $du = \frac{1}{2}dx$, *and*

$$\int_0^{\sqrt{3}} \frac{1}{\sqrt{4-x^2}}\, dx = \int_0^{\frac{\sqrt{3}}{2}} \frac{1}{\sqrt{1-u^2}}\, du = \arcsin u \Big|_0^{\frac{\sqrt{3}}{2}} = \frac{\pi}{3}.$$

Definition 6.44. *The restriction of tangent function*

$$\tan : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$$

is 1-1 and onto, and its inverse function is called ***arc tangent*** *and denoted by* $\tan^{-1}$ *or* arctan. *In other words,*

$$x = \tan^{-1}(\tan x) \quad \textit{and} \quad y = \tan(\tan^{-1} y), \qquad \forall\, x \in (-\frac{\pi}{2}, \frac{\pi}{2}),\ \forall\, y \in \mathbb{R}.$$

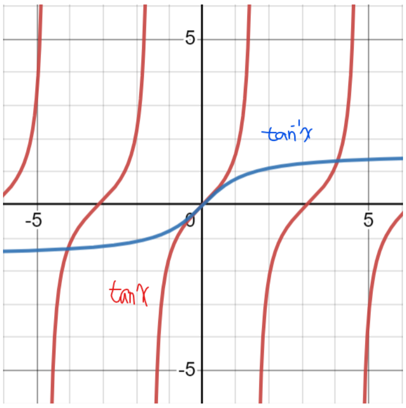


FIGURE 49. Graph of arctan

Theorem 6.45 ([3, (7.7.8)]). *The function* arctan x *is differentiable on* $\mathbb{R}$ *and its derivative is*

$$\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}.$$

Thus,

$$\int \frac{1}{1+x^2}\,dx = \arctan x + C.$$

Proof. Let $y = \tan x$. Since $x = \arctan(\tan x) = \tan^{-1}(y)$,

$$1 = \frac{d\arctan y}{dy} \cdot \frac{dy}{dx} = \frac{d\arctan y}{dy} \cdot \sec^2 x,$$

and thus

$$\frac{d\arctan y}{dy} = \cos^2 x = \frac{1}{1+\tan^2 x} = \frac{1}{1+y^2}, \quad \forall\, y \in \mathbb{R}.$$

□

Example 6.46. *Calculate.*

- $\arctan(1) = \frac{\pi}{4}$, $\arctan(0) = 0$.
- $\frac{d}{dx}(\arctan(\frac{x}{3})) = \frac{1}{1+(\frac{x}{3})^2} \cdot \frac{d}{dx}(\frac{x}{3}) = \frac{1}{3+\frac{x^2}{3}} = \frac{3}{9+x^2}$.
- $\int_0^2 \frac{1}{4+x^2}\,dx = \int_0^2 \frac{1}{4} \cdot \frac{1}{1+(\frac{x}{2})^2}\,dx$. *Let* $u = \frac{x}{2}$. *Then* $du = \frac{1}{2}dx$, *and*

$$\int_0^2 \frac{1}{4+x^2}\,dx = \frac{1}{2}\int_0^1 \frac{1}{1+u^2}\,du = \frac{1}{2}\arctan u\Big|_0^1 = \frac{1}{2}(\frac{\pi}{4}-0) = \frac{\pi}{8}.$$

Similarly, we have

- $\cos^{-1} = \arccos : [-1,1] \to [0,\pi]$.
- $\sec^{-1} = \operatorname{arcsec} : (-\infty,-1] \cup [1,\infty) \to [0,\frac{\pi}{2}) \cup (\frac{\pi}{2},\pi]$.
- $\cot^{-1} = \operatorname{arccot} : \mathbb{R} \to (0,\pi)$.
- $\csc^{-1} = \operatorname{arccsc} : (-\infty,-1] \cup [1,\infty) \to [-\frac{\pi}{2},0) \cup (0,\frac{\pi}{2}]$.

Theorem 6.47 ([3, (7.7.3), (7.7.8), (7.7.12)]). *One has the following formulas.*

- $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$.
- $\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$.
- $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$.
- $\frac{d}{dx}(\csc^{-1} x) = \frac{-1}{|x|\sqrt{x^2-1}}$.
- $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}$.
- $\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$.

We will use the integral formulas

$$\int \frac{1}{1+x^2}\,dx = \tan^{-1} x + C$$

and

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + C.$$

Proposition 6.48. Maybe skip. *One has the following identities.*

- $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$.
- $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$.
- $\sec^{-1} x + \csc^{-1} x = \frac{\pi}{2}$.

Proof. Since

$$\frac{d}{dx}(\sin^{-1} x + \cos^{-1} x) = \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}} = 0,$$

the function $\sin^{-1} x + \cos^{-1} x$ is a constant. This number can be found by evaluating $x = 0$, and one has $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$. The other two equations can be proved similarly. □

6.6. Hyperbolic sine and cosine.

Definition 6.49 ([3, Section 7.8])**.** *The* ***hyperbolic sine*** *is defined by*

$$\sinh x := \frac{e^x - e^{-x}}{2}.$$

The ***hyperbolic cosine*** *is defined by*

$$\cosh x := \frac{e^x + e^{-x}}{2}.$$

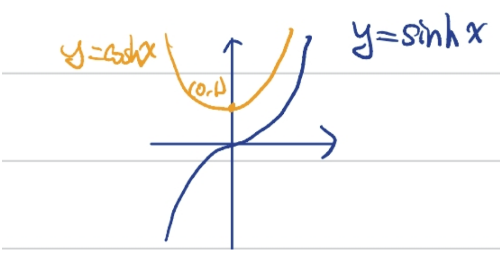


FIGURE 50. Graphs of sinh x and cosh x

It is straightforward to show the following:

- $\frac{d}{dx}(\sinh x) = \cosh x$.

- $\dfrac{d}{dx}(\cosh x) = \sinh x$.
- $\cosh^2 x - \sinh^2 x = 1$.
- $\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$.
- $\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$.

Example 6.50. *Integrate.*

$$\int_0^{\ln 2} x \cosh x \, dx = x \sinh x \Big|_0^{\ln 2} - \int_0^{\ln 2} \sinh x \, dx = x \sinh x \Big|_0^{\ln 2} - \cosh x \Big|_0^{\ln 2} = \frac{3}{4} \ln 2 - \frac{1}{4}.$$

7. Techniques of integration

Recall the following formulas:

(1) $\int x^r\,dx = \frac{1}{r+1}x^{r+1} + C \quad \text{if } r \neq -1.$

(2) $\int x^r\,dx = \ln|x| + C \qquad \text{if } r = -1.$

(3) $\int e^x\,dx = e^x + C.$

(4) $\int \sin x\,dx = -\cos x + C.$

(5) $\int \cos x\,dx = \sin x + C.$

(6) $\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + C.$

(7) $\int \frac{1}{1+x^2}\,dx = \tan^{-1} x + C.$

(8) (Integration by substitution) $\int f(u(x))\cdot u'(x)\,dx = f(u(x)) + C.$

(9) (Integration by parts) $\int u(x)v'(x)\,dx = u(x)v(x) - \int v(x)u'(x)\,dx.$

Example 7.1. *Integrate.*

(1) $\int xe^x\,dx = xe^x - \int e^x\,dx = xe^x - e^x + C.$

(2) $\int x\ln x\,dx = \frac{x^2}{2}\ln x - \int \frac{x^2}{2}\cdot\frac{1}{x}\,dx = \frac{x^2}{2}\ln x - \frac{x^2}{4} + C.$

(3) $\int e^x\cos x\,dx = e^x\cos x - \int e^x(-\sin x)\,dx = e^x\cos x + e^x\sin x - \int e^x\cos x\,dx$

$\Rightarrow \quad \int e^x\cos x\,dx = \frac{e^x}{2}(\cos x + \sin x) + C.$

(4) $\int x^5\cos(x^3)\,dx = \int (\frac{x^3}{3})\cdot(3x^2\cos(x^3))\,dx = \frac{x^3}{3}\sin(x^3) - \int x^2\sin(x^3)\,dx$

$= \frac{x^3}{3}\sin(x^3) + \frac{1}{3}\cos(x^3) + C.$

(5) $\int \ln x\,dx = x\ln x - \int x\cdot\frac{1}{x}\,dx = x\ln x - x + C.$

(6) $\int \sin^{-1} x\,dx = x\sin^{-1} x - \int x\cdot\frac{1}{\sqrt{1-x^2}}\,dx = x\sin^{-1} x + \sqrt{1-x^2} + C.$

(7) $\int \tan^{-1} x\,dx = x\tan^{-1} x - \frac{1}{2}\ln(1+x^2) + C.$

(8) $\int_1^2 x^3\ln x\,dx = \frac{1}{4}x^4\ln x\Big|_1^2 - \int_1^2 \frac{1}{4}x^4\cdot\frac{1}{x}\,dx = 4\ln 2 - \frac{15}{16}.$

7.1. **Products of trigonometric functions.** Recall the following trigonometric formulas:

(1) $\sin(-\theta) = -\sin\theta.$

(2) $\cos(-\theta) = \cos\theta.$

(3) $\sin^2\theta + \cos^2\theta = 1.$

(4) $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta.$

(5) $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$.
(6) $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$.
(7) $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$.
(8) $\sin\alpha\cos\beta = \frac{1}{2}(\sin(\alpha+\beta) + \sin(\alpha-\beta))$.
(9) $\cos\alpha\sin\beta = \frac{1}{2}(\sin(\alpha+\beta) - \sin(\alpha-\beta))$.
(10) $\cos\alpha\cos\beta = \frac{1}{2}(\cos(\alpha+\beta) + \cos(\alpha-\beta))$.
(11) $\sin\alpha\sin\beta = \frac{1}{2}(\cos(\alpha-\beta) - \cos(\alpha+\beta))$.
(12) $\sin^2\theta = \frac{1}{2} - \frac{1}{2}\cos 2\theta$.
(13) $\cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos 2\theta$.

Example 7.2. *Calculate.*

(1) *Skip.* $\int \sin^2 x\cos^2 x\,dx = \frac{1}{4}\int \sin^2(2x)\,dx = \frac{1}{4}\int \frac{1}{2}(1-\cos 4x)\,dx = \frac{1}{8}(x - \frac{1}{4}\sin(4x)) + C.$

(2)
$$\int \sin(5x)\sin(3x)\,dx = \int \frac{1}{2}(\cos(5x-3x) - \cos(5x+3x))\,dx$$
$$= \int \frac{1}{2}(\cos(2x) - \cos(8x))\,dx = \frac{1}{4}\sin(2x) - \frac{1}{16}\sin(8x) + C.$$

Now we study the integral of the form (see [1, Section 8.3] or [3, Section 8.3].)

$$\int \sin^p x\cos^q x\,dx.$$

Case 1: p or q is odd. Without loss of generality, assume $p = 2k+1$ is odd, $k \in \mathbb{Z}_{\geq 0}$. Then

$$\int \sin^p x\cos^q x\,dx = \int \sin^{2k+1} x\cos^q x\,dx = \int \sin x \cdot (1-\cos^2 x)^k \cdot \cos^q x\,dx.$$

Let $u = \cos x$. Then $du = -\sin x dx$, and by the substitution method,

$$\int \sin^p x\cos^q x\,dx = -\int u^q \cdot (1-u^2)^k\,du.$$

Example 7.3. *Integrate.*

(1) $\int \sin^2 x\cos^5 x\,dx = \int \sin^2 x(1-\sin^2 x)^2 \cdot \cos x\,dx = \int u^2(1-u^2)^2\,du.$

(2) $\int \sin^5 x\,dx = \int (1-\cos^2 x)^2 \cdot \sin x\,dx = -\int (1-u^2)^2\,du.$

Case 2: both p and q are even. Then use the product-to-sum formulas to reduce the powers.

Example 7.4. *Integrate.*

$$\begin{aligned}
\int \sin^2 x \cos^6 x\, dx &= \int (\sin x \cos x)^2 \cdot \cos^4 x\, dx \\
&= \int (\frac{\sin(2x)}{2})^2 \cdot (\frac{\cos(2x)+1}{2})^2\, dx \\
&= \frac{1}{16}\int \sin^2(2x)\cdot\cos^2(2x) + 2\sin^2(2x)\cos(2x) + \sin^2(2x)\, dx \\
&= \frac{1}{16}\int (\frac{\sin 4x}{2})^2 + \sin 2x \sin 4x + \frac{1-\cos 4x}{2}\, dx \\
&= \frac{1}{64}\int \frac{1-\cos 8x}{2}\, dx + \frac{1}{16}\int \frac{1}{2}(\cos(2x-4x) - \cos(2x+4x))\, dx + \frac{1}{32}\int 1-\cos 4x\, dx \\
&= \frac{1}{128}(x - \frac{1}{8}\sin(8x)) + \frac{1}{32}(-\frac{1}{2}\sin(-2x) - \frac{1}{6}\sin(6x)) + \frac{1}{32}(x - \frac{1}{4}\sin(4x)) + C.
\end{aligned}$$

7.2. **Trigonometric substitution.** Recall the trigonometric formulas:

(1) $\cos^2 x + \sin^2 x = 1$
(2) $1 + \tan^2 x = \sec^2 x$

In this section, we will use these formulas to integrate the functions

$$\sqrt{a^2-x^2}, \qquad \sqrt{a^2+x^2}, \qquad \sqrt{x^2-a^2},$$

for $a > 0$. The principles are

(1) for $\sqrt{a^2-x^2}$, let $x = a\sin u \Rightarrow \sqrt{a^2-x^2} = \sqrt{a^2(1-\sin^2 u)} = a\cos u$;
(2) for $\sqrt{a^2+x^2}$, let $x = a\tan u \Rightarrow \sqrt{a^2+x^2} = \sqrt{a^2(1+\tan u)} = a\sec u$;
(3) for $\sqrt{x^2-a^2}$, let $x = a\sec u \Rightarrow \sqrt{x^2-a^2} = \sqrt{a^2(\sec^2 u - 1)} = a\tan u$.

See [1, Section 8.4] or [3, Section 8.4].

Example 7.5 (type $\sqrt{a^2-x^2}$)**.** *Integrate* $\int_{-a}^{a}\sqrt{a^2-x^2}\,dx$*. Let* $x = a\sin u$*. Then* $dx = a\cos u\,du$*, and*

$$\begin{aligned}
\int_{-a}^{a}\sqrt{a^2-x^2}\,dx &= \int_{u(-a)}^{u(a)}\sqrt{a^2-a^2\sin^2 u}\cdot a\cos u\,du \\
&= \int_{\sin^{-1}(-1)}^{\sin^{-1}(1)} a^2\cos^2 u\,du = a^2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{1+\cos 2u}{2}\,du = \frac{1}{2}\pi a^2.
\end{aligned}$$

Example 7.6 (type $\sqrt{a^2-x^2}$)**.** *Maybe skip. Integrate* $\int \frac{1}{(a^2-x^2)^{\frac{3}{2}}}\,dx$*. Let* $x = a\sin u$*. Then we have* $dx = a\cos u\,du$*, and*

$$\begin{aligned}
\int \frac{1}{(a^2-x^2)^{\frac{3}{2}}}\,dx &= \int \frac{1}{(a^2\cos^2 u)^{\frac{3}{2}}}\cdot a\cos u\,du = \frac{1}{a^2}\int \sec^2 u\,du = \frac{1}{a^2}\tan u + C \\
&= \frac{1}{a^2}\cdot\frac{\sin u}{\sqrt{1-\sin^2 u}} + C = \frac{1}{a^2}\frac{\frac{x}{a}}{\sqrt{1-\frac{x^2}{a^2}}} + C \\
&= \frac{x}{a^2\sqrt{a^2-x^2}} + C.
\end{aligned}$$

Example 7.7 (type $\sqrt{a^2+x^2}$). *Integrate* $\int \sqrt{a^2+x^2}\,dx$. *Let* $x = a\tan u$. *Then* $dx = a\sec^2 u\,du$, *and*

$$\int \sqrt{a^2+x^2}\,dx = a^2 \int \sqrt{1+\tan^2 u}\cdot \sec^2 u\,du = a^2 \int \sec^3 u\,du.$$

Applying integration by parts to $\int \sec^3 u\,du$, *one has*

$$\int \sec u \cdot \sec^2 u\,du = \sec u \tan u - \int \sec^3 u - \sec u\,du.$$

Thus,

$$a^2 \int \sec^3 u\,du = \frac{a^2}{2}(\sec u\tan u + \int \sec u\,du) = \frac{a^2}{2}(\sec u \tan u + \ln|\sec u + \tan u|) + C,$$

and

$$\begin{aligned}\int \sqrt{a^2+x^2}\,dx &= \frac{a^2}{2}(\sec u\tan u + \ln|\sec u + \tan u|) + C\\ &= \frac{a^2}{2}\Big(\sqrt{1+(\frac{x}{a})^2}\cdot\frac{x}{a} + \ln|\sqrt{1+(\frac{x}{a})^2} + \frac{x}{a}|\Big) + C\\ &= \frac{1}{2}x\sqrt{a^2+x^2} + \frac{a^2}{2}\ln|x + \sqrt{a^2+x^2}| + C.\end{aligned}$$

Example 7.8 (type $\sqrt{x^2-a^2}$). *Integrate* $\int \frac{1}{\sqrt{x^2-1}}\,dx$. *Let* $x = \sec u$. *Then* $dx = \tan u \sec u\,du$, *and*

$$\begin{aligned}\int \frac{1}{\sqrt{x^2-1}}\,dx &= \int \frac{1}{\tan u}\cdot \tan u \sec u\,du\\ &= \int \sec u\,du = \ln|\sec u + \tan u| + C = \ln|x + \sqrt{x^2-1}| + C.\end{aligned}$$

Example 7.9 (type $\sqrt{x^2-a^2}$). *Maybe skip. Integrate* $\int \frac{1}{x^2\sqrt{x^2-4}}\,dx$. *Let* $x = 2\sec u$. *Then we have* $dx = 2\tan u\sec u\,du$, *and*

$$\begin{aligned}\int \frac{1}{x^2\sqrt{x^2-4}}\,dx &= \int \frac{2\tan u\cdot\sec u}{4\sec^2 u\cdot 2\tan u}\,du = \int \frac{1}{4}\cos u\,du = \frac{1}{4}\sin u + C\\ &= \frac{1}{4}\sqrt{\frac{x^2-4}{x^2}} + C = \frac{\sqrt{x^2-4}}{4x} + C.\end{aligned}$$

Example 7.10 (type $\sqrt{a^2+x^2}$). *Integrate* $\int \frac{1}{\sqrt{x^2+2x+5}}\,dx$. *Let* $x + 1 = 2\tan u$. *Then* $dx = 2\sec^2 u\,du$, *and*

$$\begin{aligned}\int \frac{1}{\sqrt{x^2+2x+5}}\,dx &= \int \frac{1}{\sqrt{(x+1)^2+4}}\,dx = \int \frac{2\sec^2 u}{2\sec u}\,du\\ &= \int \sec u\,du = \ln|\sec u + \tan u| + C.\end{aligned}$$

Since $x+1 = 2\tan u$, $\tan u = \dfrac{x+1}{2}$ *and* $\sec u = \sqrt{1+(\dfrac{x+1}{2})^2} = \dfrac{1}{2}\sqrt{x^2+2x+5}$, *we have*

$$\begin{aligned}\int \frac{1}{\sqrt{x^2+2x+5}}\,dx &= \ln|\frac{1}{2}\sqrt{x^2+2x+5}+\frac{x+1}{2}|+C\\ &= \ln|\sqrt{x^2+2x+5}+x+1|+C',\end{aligned}$$

where $C' = C + \ln\dfrac{1}{2}$.

Example 7.11 (type $\sqrt{a^2+x^2}$)**.** *Maybe skip. Integrate* $\displaystyle\int \frac{1}{(x^2+a^2)^2}\,dx$. *Let* $x = a\tan u$. *Then* $dx = a\sec^2 u\,du$, *and*

$$\begin{aligned}\int \frac{1}{(x^2+a^2)^2}\,dx = \int \frac{a\sec^2 u}{(a^2\sec^2 u)^2}\,du = \frac{1}{a^3}\int \cos^2 u\,du\\ = \frac{1}{a^3}\int \frac{\cos(2u)+1}{2}\,du = \frac{1}{2a^3}(u+\sin u\cos u)+C.\end{aligned}$$

Since $\tan u = \dfrac{x}{a}$, *one has* $\sin u = \dfrac{x}{\sqrt{a^2+x^2}}$ *and* $\cos u = \dfrac{a}{\sqrt{a^2+x^2}}$. *Thus,*

$$\int \frac{1}{(x^2+a^2)^2}\,dx = \frac{1}{2a^3}(\tan^{-1}\frac{x}{a}+\frac{x}{\sqrt{a^2+x^2}}\cdot\frac{a}{\sqrt{a^2+x^2}+c}) = \frac{x}{2a^2(a^2+x^2)}+\frac{1}{2a^3}\cdot\tan^{-1}\frac{x}{a}+C.$$

7.3. **Rational function.** Let $P(x)$, $Q(x)$ be polynomials. In this section, we study the integration

$$\int \frac{P(x)}{Q(x)}\,dx.$$

See [1, Section 8.5] or [3, Section 8.5].

Step 1. If $\deg P(x) \geq \deg Q(x)$, then applying the division algorithm, one has a polynomial $q(x)$ and $r(x)$, such that

$$\frac{P(x)}{Q(x)} = q(x)+\frac{r(x)}{Q(x)} \qquad \text{and} \qquad \deg r(x) < \deg Q(x).$$

Example 7.12. *Let* $P(x) = x^3+x^2+x+1$ *and* $Q(x) = x-1$. *Since* $P(x) = (x^2+2x+3)Q(x)+4$,

$$\int \frac{P(x)}{Q(x)}\,dx = \int (x^2+2x+3)+\frac{4}{x-1}\,dx = \frac{x^3}{3}+x^2+3x+4\ln|x-1|+C.$$

Step 2. By Step 1, we can assume $\deg P(x) < \deg Q(x)$. To compute the integral, we factorize $Q(x)$ and decompose $\dfrac{P(x)}{Q(x)}$.

Example 7.13. *Let* $P(x) = 1$ *and* $Q(x) = x^2-2x-3 = (x-3)(x+1)$. *By solving* A *and* B *in*

$$\frac{P(x)}{Q(x)} = \frac{1}{(x-3)(x+1)} = \frac{A}{x-3}+\frac{B}{x+1} = \frac{Ax+A+Bx-3B}{(x-3)(x+1)},$$

we have $A+B=0$ *and* $A-3B=1$ *which implies* $A=\dfrac{1}{4}$ *and* $B=-\dfrac{1}{4}$. *Thus,*

$$\int \frac{1}{x^2-2x-3}\,dx = \int \frac{1}{4}\cdot\frac{1}{x-3}-\frac{1}{4}\cdot\frac{1}{x+1}\,dx = \frac{1}{4}\ln|x-3|-\frac{1}{4}\ln|x+1|+C = \frac{1}{4}\ln|\frac{x-3}{x+1}|+C.$$

It's possible just do a few examples below.

Remark 7.14. *One can factorize a real polynomial*

$$Q(x) = a(x-b_1)^{r_1}\cdots(x-b_i)^{r_i}\cdot(x^2+c_{i+1}x+d_{i+1})^{r_{i+1}}\cdots(x^2+c_jx+d_j)^{r_j} \tag{3}$$

where $a, b_k, c_k, d_k \in \mathbb{R}$, *and* $x^2+c_kx+d_k$ *are irreducible. With the factorization* (3), *one has the decomposition*

$$\frac{P(x)}{Q(x)} = \frac{1}{a}\left(\frac{f_1(x)}{(x-b_1)^{r_1}}+\cdots+\frac{f_i(x)}{(x-b_i)^{r_i}}+\frac{f_{i+1}(x)}{(x^2+c_{i+1}x+d_{i+1})^{r_{i+1}}}+\cdots+\frac{f_j(x)}{(x^2+c_jx+d_j)^{r_j}}\right),$$

where $\deg f_k(x) < r_k$ *for* $k \le i$, *and* $\deg f_k(x) < 2r_k$ *for* $k \ge i+1$.

Example 7.15. $\dfrac{x+2}{x^3-1} = \dfrac{1}{x-1} - \dfrac{x+1}{x^2+x+1}$.

Thus, we are interested in the integrals of the forms

$$\int \frac{f(x)}{(x-b)^r}\,dx \qquad \text{and} \qquad \int \frac{f(x)}{(x^2+cx+d)^r}\,dx.$$

Case 1. We decompose

$$\frac{f(x)}{(x-b)^r} = \frac{A_1}{(x-b)^1}+\frac{A_2}{(x-b)^2}+\cdots+\frac{A_r}{(x-b)^r}$$

where $A_k \in \mathbb{R}$.

Example 7.16. $\displaystyle\int \frac{x+1}{(x-1)^2}\,dx = \int \frac{1}{x-1}+\frac{2}{(x-1)^2}\,dx = \ln|x-1| + \frac{2}{-1}(x-1)^{-1}+C.$

Case 2. We decompose

$$\frac{f(x)}{(x^2+cx+d)^r} = \frac{A_1x+B_1}{(x^2+cx+d)^1}+\frac{A_2x+B_2}{(x^2+cx+d)^2}+\cdots+\frac{A_rx+B_r}{(x^2+cx+d)^r}$$

where $A_k, B_k \in \mathbb{R}$

Example 7.17. $\displaystyle\int \frac{x^3}{(x^2+1)^2}\,dx = \int \frac{x}{x^2+1}\,dx - \int \frac{x}{(x^2+1)^2}\,dx = \frac{1}{2}\ln|x^2+1| + \frac{1}{2}(x^2+1)^{-1}+C.$

To integrate $\displaystyle\int \frac{Ax+B}{(x^2+cx+d)^r}\,dx$, one might get the following four cases.

Case 2-1: $r=1$ and $A=0$.

$$\int \frac{B}{x^2+cx+d}\,dx = \int \frac{B}{(x+\frac{c}{2})^2+(d-\frac{c^2}{4})}\,dx = \frac{B}{d-\frac{c^2}{4}}\cdot\int \frac{1}{1+(\frac{2x+c}{\sqrt{4d-c^2}})^2}\,dx$$

Note that since x^2+cx+d is irreducible, $4d-c^2>0$. Let $u = \dfrac{2x+c}{\sqrt{4d-c^2}}$. Then $du = \dfrac{2}{\sqrt{4d-c^2}}\,dx$, and

$$\begin{aligned}\int \frac{B}{x^2+cx+d}\,dx &= \frac{B}{d-\frac{c^2}{4}}\cdot\frac{\sqrt{4d-c^2}}{2}\int \frac{1}{u^2+1}\,du \\ &= \frac{2B}{\sqrt{4d-c^2}}\cdot\tan^{-1}u + C' = \frac{2B}{\sqrt{4d-c^2}}\cdot\tan^{-1}\left(\frac{2x+c}{\sqrt{4d-c^2}}\right)+C'.\end{aligned}$$

Example 7.18. *Integrate.*

$$\int \frac{1}{x^2+x+1}\,dx = \int \frac{1}{(x+\frac{1}{2})^2+\frac{3}{4}}\,dx = \frac{4}{3}\cdot\int \frac{1}{(\frac{2x+1}{\sqrt{3}})^2+1}\,dx = \frac{4}{3}\cdot\frac{\sqrt{3}}{2}\int \frac{1}{u^2+1}\,du$$
$$= \frac{2}{\sqrt{3}}\tan^{-1}u + C = \frac{2}{\sqrt{3}}\tan^{-1}(\frac{2x+1}{\sqrt{3}}) + C.$$

Case 2-2: $r = 1$ and $A \neq 0$.

$$\int \frac{Ax+B}{x^2+cx+d}\,dx = \frac{A}{2}\cdot\int \frac{2x+c+(\frac{2B}{A}-c)}{x^2+cx+d}\,dx$$
$$= \frac{A}{2}\cdot\int \frac{2x+c}{x^2+cx+d}\,dx + \frac{A}{2}\cdot\int \frac{\frac{2B}{A}-c}{x^2+cx+d}\,dx$$
$$= \frac{A}{2}\cdot\ln|x^2+cx+d| + \frac{A}{2}(\frac{2B}{A}-c)\int \frac{1}{x^2+cx+d}\,dx + C'.$$

The integral $\int \frac{1}{x^2+cx+d}\,dx$ can be computed by the method in Case 2-1.

Example 7.19. *Integrate.*

$$\int \frac{x}{x^2+x+1}\,dx = \frac{1}{2}\int \frac{2x+1}{x^2+x+1}\,dx - \frac{1}{2}\int \frac{1}{x^2+x+1}\,dx$$
$$= \frac{1}{2}\ln|x^2+x+1| - \frac{1}{\sqrt{3}}\tan^{-1}(\frac{2x+1}{\sqrt{3}}) + C.$$

Case 2-3: $r > 1$ and $A = 0$.

$$\int \frac{B}{(x^2+cx+d)^r}\,dx = \int \frac{B}{[(x+\frac{c}{2})^2+(d-\frac{c^2}{4})]^r}\,dx = \frac{B}{(d-\frac{c^2}{4})^r}\cdot\int \frac{1}{[1+(\frac{2x+c}{\sqrt{4d-c^2}})^2]^r}\,dx.$$

Let $u = \frac{2x+c}{\sqrt{4d-c^2}}$. Then $du = \frac{2}{\sqrt{4d-c^2}}\,dx$, and

$$\int \frac{B}{(x^2+cx+d)^r}\,dx = \frac{B}{(d-\frac{c^2}{4})^r}\cdot\frac{\sqrt{4d-c^2}}{2}\cdot\int \frac{1}{(u^2+1)^r}\,du.$$

Let $u = \tan v$. Then $du = \sec^2 v\,dv$, and

$$\int \frac{B}{(x^2+cx+d)^r}\,dx = \int \frac{1}{(\sec^2 v)^r}\cdot\sec^2 v\,dv = \int \cos^{2(r-1)} v\,dv.$$

Example 7.20. *Integrate.*

$$\begin{aligned}\int \frac{1}{(x^2+x+1)^2}\,dx &= \int \frac{1}{[(x+\frac{1}{2})^2+\frac{3}{4}]^2}\,dx = \frac{16}{9}\int \frac{1}{[(\frac{2x+1}{\sqrt{3}})^2+1]^2}\,dx \\ &= \frac{8\sqrt{3}}{9}\int \frac{1}{(u^2+1)^2}\,du \qquad (u=\frac{2x+1}{\sqrt{3}}, du=\frac{2}{\sqrt{3}}\,dx) \\ &= \frac{8\sqrt{3}}{9}\int \frac{1}{\sec^4 v}\cdot \sec^2 v\,dv \qquad (u=\tan v, du=\sec^2 v\,dv) \\ &= \frac{8\sqrt{3}}{9}\int \cos^2 v\,dv = \frac{8\sqrt{3}}{9}\int \frac{\cos 2v+1}{2}\,dv \\ &= \frac{2\sqrt{3}}{9}\sin 2v + \frac{4\sqrt{3}}{9}v + C \\ &= \frac{4\sqrt{3}}{9}\sin v\cos v + \frac{4\sqrt{3}}{9}v + C.\end{aligned}$$

Since $\tan v = u = \frac{2x+1}{\sqrt{3}}$*, one has*

$$v=\tan^{-1}\frac{2x+1}{\sqrt{3}}, \qquad \sin v = \frac{2x+1}{\sqrt{4x^2+4x+4}}, \qquad \cos v = \frac{\sqrt{3}}{\sqrt{4x^2+4x+4}}.$$

Thus,

$$\int \frac{1}{(x^2+x+1)^2}\,dx = \frac{4\sqrt{3}}{9}\cdot\frac{\sqrt{3}(2x+1)}{4x^2+4x+4} + \frac{4\sqrt{3}}{9}\tan^{-1}(\frac{2x+1}{\sqrt{3}}) + C.$$

Case 2-4: $r>1$ and $A\neq 0$.

$$\begin{aligned}\int \frac{Ax+B}{(x^2+cx+d)^r}\,dx &= \frac{A}{2}\int \frac{2x+c}{(x^2+cx+d)^r}\,dx + \frac{A}{2}\int \frac{\frac{2B}{A}-c}{(x^2+cx+d)^r}\,dx \\ &= \frac{A}{2}\cdot\frac{1}{-r+1}(x^2+cx+d)^{-r+1} + \frac{A}{2}\int \frac{\frac{2B}{A}-c}{(x^2+cx+d)^r}\,dx,\end{aligned}$$

where the second term can be computed by Case 2-3.

Example 7.21. *Integrate.*

$$\begin{aligned}\int \frac{x}{(x^2+x+1)^2}\,dx &= \frac{1}{2}\int \frac{2x+1}{(x^2+x+1)^2}\,dx - \frac{1}{2}\int \frac{1}{(x^2+x+1)^2}\,dx \\ &= -\frac{1}{2}\cdot\frac{1}{x^2+x+1} - \frac{1}{6}\cdot\frac{2x+1}{x^2+x+1} + \frac{2\sqrt{3}}{9}\tan^{-1}(\frac{2x+1}{\sqrt{3}}) + C,\end{aligned}$$

where the second term was computed in Example 7.20.

7.4. **Improper integrals.** In [1, Section 8.8], improper integrals are divided into two types. Type I is integral of continuous function on an infinite region. Type II is integral of function with infinite discontinuity.

Definition 7.22 ([1, Section 8.8], [3, Section 11.7])**.** *Integrals with infinite upper/lower bounds are called* ***improper integrals*** *(of Type I).*

(1) *If $f(x)$ is continuous on $[a,\infty)$, then*

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx.$$

(2) *If $f(x)$ is continuous on $(-\infty,b]$, then*

$$\int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty}\int_a^b f(x)\,dx.$$

(3) *If $f(x)$ is continuous on $(-\infty,\infty)$, then*

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx,$$

where c is any real number.

In each case, if the limit exists and finite, we say the improper integral ***converges*** *and the limit is the* ***value*** *of the improper integral. If the limit does not exist, we say the improper integral* ***diverges****.*

Example 7.23. *Evaluate.*

(1) $\displaystyle\int_1^\infty \frac{\ln x}{x^2}\,dx = \lim_{b\to\infty}\int_1^b \frac{\ln x}{x^2}\,dx = \lim_{b\to\infty}\Big(\frac{-1}{x}\ln x\Big|_1^b + \int_1^b \frac{1}{x^2}\,dx\Big) = \lim_{b\to\infty}\Big(-\frac{\ln b}{b} - \frac{1}{b} + 1\Big) = 1.$

(2) $\displaystyle\int_{-\infty}^\infty \frac{dx}{1+x^2} = \int_0^\infty \frac{dx}{1+x^2} + \int_{-\infty}^0 \frac{dx}{1+x^2} = \pi$, *because*

$$\int_0^\infty \frac{dx}{1+x^2} = \lim_{b\to\infty}\int_0^b \frac{dx}{1+x^2} = \lim_{b\to\infty}\tan^{-1}x\Big|_0^b = \frac{\pi}{2} = \int_{-\infty}^0 \frac{dx}{1+x^2}.$$

Example 7.24. *The integral*

$$\int_1^\infty \frac{1}{x^p}\,dx = \begin{cases} \dfrac{1}{p-1}, & p>1,\\ \infty \ (\textit{diverges}), & p\le 1.\end{cases}$$

Definition 7.25. *Integrals of functions that become infinite at a point within the interval of integration are called* ***improper integrals*** *(of Type II).*

(1) *If $f(x)$ is continuous on $(a,b]$ and discontinuous at a, then*

$$\int_a^b f(x)\,dx = \lim_{c\to a^+}\int_c^b f(x)\,dx.$$

(2) *If $f(x)$ is continuous on $[a,b)$ and discontinuous at b, then*

$$\int_a^b f(x)\,dx = \lim_{c\to b^-}\int_a^c f(x)\,dx.$$

(3) *If $f(x)$ is discontinuous at c, where $a<c<b$, and continuous on $[a,c)\cup(c,b]$, then*

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

In each case, if the limit exists and is finite, we say the improper integral ***converges*** *and the limit is the* ***value*** *of the improper integral. If the limit does not exist, we say the integral* ***diverges****.*

Example 7.26. *The integral*

$$\int_0^1 \frac{1}{x^p}\,dx = \begin{cases} \dfrac{1}{1-p}, & p < 1, \\ \infty \text{ (diverges)}, & p \geq 1. \end{cases}$$

Example 7.27. *Integrate.*

(1) $\displaystyle\int_0^1 \frac{1}{1-x}\,dx = \lim_{c\to 1^-} \int_0^c \frac{1}{1-x}\,dx = \lim_{c\to 1^-} \Big(-\ln|1-x| \Big|_0^c \Big)$ *diverges.*

(2) $\displaystyle\int_0^3 \frac{dx}{(x-1)^{2/3}} = \int_0^1 \frac{dx}{(x-1)^{2/3}} + \int_1^3 \frac{dx}{(x-1)^{2/3}} = 3 + 3\sqrt[3]{2}.$

Theorem 7.28 ([1, Theorem 2 in Section 8.8], [3, (11.7.2)])**.** *Let f and g be continuous on $[a,\infty)$ with $0 \leq f(x) \leq g(x)$ for all $x \geq a$. Then*

(1) *If $\displaystyle\int_a^\infty g(x)\,dx$ converges, then $\displaystyle\int_a^\infty f(x)\,dx$ also converges.*

(2) *If $\displaystyle\int_a^\infty f(x)\,dx$ diverges, then $\displaystyle\int_a^\infty g(x)\,dx$ also diverges.*

Theorem 7.29 ([1, Theorem 3, Section 8.8])**.** *If the positive functions f and g are continuous on $[a,\infty)$, and if*

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L, \qquad 0 < L < \infty,$$

then

$$\int_a^\infty f(x)\,dx \qquad \text{and} \qquad \int_a^\infty g(x)\,dx$$

either both converge or both diverge.

Example 7.30. *Determine the following integrals converge or diverge.*

(1) $\displaystyle\int_1^\infty \frac{\sin^2 x}{x^2}\,dx$ *converges because* $0 \leq \dfrac{\sin^2 x}{x^2} \leq \dfrac{1}{x^2}$ *on* $[1,\infty)$ *and* $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$ *converges.*

(2) $\displaystyle\int_1^\infty \frac{1}{\sqrt{x^2-1/2}}\,dx$ *diverges because* $0 \leq \dfrac{1}{x} \leq \dfrac{1}{\sqrt{x^2-1/2}}$ *on* $[1,\infty)$ *and* $\displaystyle\int_1^\infty \frac{1}{x}\,dx$ *diverges.*

(3) $\displaystyle\int_1^\infty \frac{dx}{1+x^2}$ *converges because*

$$\lim_{x\to\infty} \frac{1/(1+x^2)}{1/x^2} = 1 \qquad \text{and} \qquad \int_1^\infty \frac{1}{x^2}\,dx \text{ converges.}$$

(4) $\displaystyle\int_1^\infty \frac{1-e^{-x}}{x}\,dx$ *diverges because*

$$\lim_{x\to\infty} \frac{(1-e^{-x})/x}{1/x} = 1 \qquad \text{and} \qquad \int_1^\infty \frac{1}{x}\,dx \text{ diverges.}$$

8. Infinite series

In this section, we consider "infinite sums"

$$a_1 + a_2 + a_3 + \cdots .$$

Basic question: do we really get a number by this infinite sum, i.e. is this infinite sum convergent? To make this precise, let us briefly summarize basic facts on sequences.

8.1. Review of Sequences.

Definition 8.1. *A sequence*

$$(a_n)_{n=1}^{\infty} = a_1, a_2, \cdots, a_n, \cdots .$$

is said to be ***convergent*** *if there exists a number L with the property: for each $\epsilon > 0$, there exists $N = N_\epsilon$ such that*

$$|a_n - L| < \epsilon \qquad \text{whenever} \qquad n \geq N.$$

In this case, we say L is the ***limit*** *of $(a_n)_{n=1}^{\infty}$, denoted by*

$$\lim_{n\to\infty} a_n = L, \qquad \text{or} \qquad a_n \to L \quad (\text{as } n \to \infty).$$

A sequence $(a_n)_{n=1}^{\infty}$ is said to be ***divergent*** *if it is not convergent.*

Example 8.2. *One can prove that $\lim_{n\to\infty} \frac{1}{n} = 0$ and $\lim_{n\to\infty} r^n = 0$ for any $|r| < 1$.*

To determine the convergence of a sequence, we have the following useful results:

(1) Every convergent sequence is bounded, and equivalently, every unbounded sequence is divergent. For example, we know that $(n)_{n=1}^{\infty}$ is divergent since it is unbounded.

(2) A bounded above increasing sequence is convergent, and equivalently, a bounded below decreasing sequence is convergent. For example, $(1/(n+1))_{n=1}^{\infty}$ is convergent since it is decreasing and bounded below.

(3) If $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are convergent sequences, and γ a real number, then the sequences $(a_n + b_n)_{n=1}^{\infty}$, $(\gamma \cdot a_n)_{n=1}^{\infty}$ and $(a_n b_n)_{n=1}^{\infty}$ and also convergent, and $\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$; $\lim_{n\to\infty}(\gamma \cdot a_n) = \gamma \cdot \lim_{n\to\infty} a_n$; $\lim_{n\to\infty}(a_n b_n) = (\lim_{n\to\infty} a_n)(\lim_{n\to\infty} b_n)$. For example, $(1/n^2)_{n=1}^{\infty}$ converges to 0.

   Furthermore, if $\lim_{n\to\infty} b_n \neq 0$, then the sequence $(a_n/b_n)_{n=1}^{\infty}$ (whose terms are well-defined for n large enough) is convergent, and $\lim_{n\to\infty}(a_n/b_n) = (\lim_{n\to\infty} a_n)\big/(\lim_{n\to\infty} b_n)$. For example, $((n^2+1)/(n^2+n) = (1+1/n^2)/(1+1/n))_{n=1}^{\infty}$ converges to 1.

(4) If $(a_n)_{n=1}^{\infty}$ converges to c, and if a function f is continuous at c, then $(f(a_n))_{n=1}^{\infty}$ is convergent and $\lim_{n\to\infty} f(a_n) = f(c)$. For example, $(\cos(1/n^2))_{n=1}^{\infty}$ converges to $\cos 0 = 1$.

(5) Suppose that for all n sufficiently large $a_n \leq b_n \leq c_n$. If $\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} c_n$, then $\lim_{n\to\infty} b_n = L$. For example, $(\sin(n^2)/n)_{n=1}^{\infty}$ converges to 0.

(6) A bounded sequence $(a_n)_{n=1}^{\infty}$ converges iff every subsequence of it converges to the same number iff $\overline{\lim_{n\to\infty}} a_n = \underline{\lim_{n\to\infty}} a_n$. In that case, $\lim_{n\to\infty} a_n = \overline{\lim_{n\to\infty}} a_n = \underline{\lim_{n\to\infty}} a_n$. For example, $((-1)^n)_{n=1}^{\infty}$ is divergent since $\overline{\lim_{n\to\infty}}(-1)^n = 1 \neq -1 = \underline{\lim_{n\to\infty}}(-1)^n$.

(7) A sequence of real numbers is convergent iff it is a Cauchy sequence. Here, a sequence $(a_n)_{n=1}^{\infty}$ is called a **Cauchy sequence** if for each $\epsilon > 0$, there exists $N = N_\epsilon$ such that

$$|a_n - a_m| < \epsilon \qquad \text{whenever} \qquad n \geq N.$$

This criterion is particularly useful when we have no clue on the value of the limit. For example, if we know that $|a_{n+1} - a_n| < (1/2)^n$ for every n, then $(a_n)_{n=1}^{\infty}$ is convergent.

8.2. **Infinite series.** Let $(a_k)_{k=0}^{\infty}$ be a sequence. Consider the sequence $\Big(\sum_{k=0}^{n} a_k\Big)_{n=0}^{\infty}$ of **partial sums**:

$$a_0, (a_0 + a_1), (a_0 + a_1 + a_2), \cdots .$$

Definition 8.3. *We say that the series* $\sum_{k=0}^{\infty} a_k$ ***converges*** *to L if the sequence* $\Big(\sum_{k=0}^{n} a_k\Big)_{n=0}^{\infty}$ *converges to L:*

$$\sum_{k=0}^{\infty} a_k = \lim_{n\to\infty} \sum_{k=0}^{n} a_k = L.$$

We call L the ***sum*** *of the series. If the sequence of partial sums diverges, we say that the series* $\sum_{k=0}^{\infty} a_k$ ***diverges****.*

Example 8.4. *Consider the following series.*

(1) Since

$$\sum_{k=0}^{n} \frac{1}{(k+1)(k+2)} = \sum_{k=0}^{n} \Big(\frac{1}{k+1} - \frac{1}{k+2}\Big) = 1 - \frac{1}{n+2},$$

we have

$$\sum_{k=0}^{\infty} \frac{1}{(k+1)(k+2)} = \lim_{n\to\infty} \Big(1 - \frac{1}{n+2}\Big) = 1.$$

(2) Since

$$\sum_{k=0}^{n} (-1)^k = \begin{cases} 1, & \text{if } n \text{ is even,} \\ 0, & \text{if } n \text{ is odd,} \end{cases}$$

the series $\sum_{k=0}^{\infty} (-1)^k$ *diverges.*

Example 8.5 (Geometric series)**.** *Since*

$$\sum_{k=0}^{n} x^k = 1 + x + x^2 + \cdots + x^n = \begin{cases} \frac{1-x^{n+1}}{1-x}, & \text{if } x \neq 1, \\ n+1, & \text{if } x = 1, \end{cases}$$

we have

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}, \qquad \text{if} \qquad |x| < 1,$$

$$\sum_{k=0}^{\infty} x^k \quad \text{diverges,} \qquad \text{if} \qquad |x| \geq 1.$$

Remark 8.6. *Let p be a positive integer. The series* $\sum_{k=0}^{\infty} a_k$ *converges iff* $\sum_{k=p}^{\infty} a_k$ *converges.*

Theorem 8.7 ([3, Theorem 12.2.4]). *Let a_k, b_k, α and β be numbers. If $\sum_{k=0}^{\infty} a_k$ converges to L and $\sum_{k=0}^{\infty} b_k$ converges to M, then $\sum_{k=0}^{\infty}(\alpha\, a_k + \beta\, b_k)$ converges to $\alpha\, L + \beta\, M$.*

Example 8.8. *Consider the series.*

(1) $\sum_{k=0}^{\infty}(1/2)^k + 2(1/3)^k = 2 + 2 \cdot 3/2 = 5.$

(2) $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$ *diverges because*

$$\sum_{k=1}^{n} \frac{1}{\sqrt{k}} \geq \frac{1}{\sqrt{n}} + \cdots + \frac{1}{\sqrt{n}} = \sqrt{n} \to \infty.$$

8.3. **Convergence of series with non-negative terms.** In this section, we will introduce several criteria for the convergence of series. The most basic one is the following:

Theorem 8.9 (n^{th}-term Test for Divergence, [3, Theorem 12.2.5]). *If $\sum_{k=0}^{\infty} a_k$ converges, then $\lim_{k\to\infty} a_k = 0$. Equivalently, if $\lim_{k\to\infty} a_k \neq 0$, then $\sum_{k=0}^{\infty} a_k$ diverges.*

Example 8.10. *Consider the series.*

(1) $\sum_{k=0}^{\infty} \frac{k}{k+1}$ *diverges.*

(2) $\sum_{k=1}^{\infty} \cos(1/k)$ *diverges.*

Throughout the remainder of this subsection, we consider series with non-negative terms.

As an immediate corollary of Theorem 1.19 (4), we have:

Theorem 8.11 ([3, Theorem 12.3.1]). *A series with nonnegative terms converges iff the sequence of partial sums is bounded.*

The next criterion is referred to as the *integral test*.

Theorem 8.12 (Integral Test, [3, Theorem 12.3.2]). *If f is continuous, positive and decreasing on $[1, \infty)$, then*

$$\sum_{k=1}^{\infty} f(k) \quad \text{converges} \qquad \text{iff} \qquad \int_1^{\infty} f(x)\,dx \quad \text{converges.}$$

Recall that $\int_1^{\infty} f(x)\,dx = \lim_{b\to\infty} \int_1^{b} f(x)\,dx$ can be regarded as the area of the region between bounded by $x = 1$, the graph of f and the x-axis.

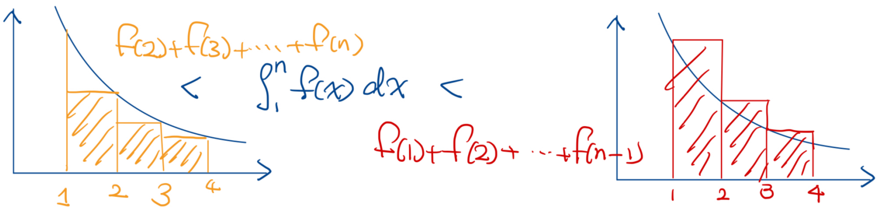


FIGURE 51. Proof of integral test.

Example 8.13 (p-series)**.** *Consider $f(x) = 1/x$ which is continuous, positive and decreasing on* $[1, \infty)$. *Since*

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{b\to\infty} \ln b \qquad \textit{diverges,}$$

it follows from Theorem 8.12 that

$$\sum_{k=1}^\infty \frac{1}{k} \qquad \textit{diverges.}$$

More generally, since

$$\int_1^\infty \frac{1}{x^p}\,dx \quad \textit{converges} \qquad \textit{iff} \qquad p > 1,$$

we have

$$\sum_{k=1}^\infty \frac{1}{k^p} \qquad \textit{iff} \qquad p > 1.$$

Theorem 8.14 (Basic Comparison Test, [3, Theorem 12.3.6])**.** *Let a_k and b_k be nonnegative numbers. Suppose*

$$a_k \le b_k \qquad \textit{for all } k \textit{ sufficiently large.}$$

(1) If $\sum_{k=1}^\infty b_k$ converges, then $\sum_{k=1}^\infty a_k$ converges.
(2) If $\sum_{k=1}^\infty a_k$ diverges, then $\sum_{k=1}^\infty b_k$ diverges.

Example 8.15. *Determine the convergence of sequences.*

(1) $\sum_{k=1}^\infty \frac{1}{2k^3+1}$ converges because

$$\frac{1}{2k^3+1} < \frac{1}{k^3} \quad \forall\, k \ge 1 \qquad \textit{and} \qquad \sum_{k=1}^\infty \frac{1}{k^3} \quad \textit{converges.}$$

(2) $\sum_{k=1}^\infty \frac{k^3}{k^5+5k^4+7}$ converges because

$$\frac{k^3}{k^5+5k^4+7} < \frac{k^3}{k^5} = \frac{1}{k^2} \quad \forall\, k \ge 1 \qquad \textit{and} \qquad \sum_{k=1}^\infty \frac{1}{k^2} \quad \textit{converges.}$$

(3) $\sum_{k=1}^\infty \frac{1}{3k+1}$ diverges because

$$\frac{1}{3k+1} \ge \frac{1}{3k+k} = \frac{1}{4k} \quad \forall\, k \ge 1 \qquad \textit{and} \qquad \sum_{k=1}^\infty \frac{1}{4k} = \frac{1}{4}\sum_{k=1}^\infty \frac{1}{k} \quad \textit{diverges.}$$

Theorem 8.16 (Limit Comparison Test, [3, Theorem 12.3.7]). *Let a_k and b_k be positive numbers. If $\lim_{k\to\infty} a_k/b_k \neq 0$, then*

$$\sum_{k=1}^{\infty} a_k \quad \textit{converges} \qquad \textit{iff} \qquad \sum_{k=1}^{\infty} b_k \quad \textit{converges.}$$

Example 8.17. *Determine whether the series converges or diverges.*

(1) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{5^k - 3}$. *It converges because* $\displaystyle\lim_{k\to\infty} \frac{1}{5^k-3} \Big/ \frac{1}{5^k} = 1 \neq 0$ *and* $\displaystyle\sum_{k=1}^{\infty} \frac{1}{5^k}$ *converges.*

(2) $\displaystyle\sum_{k=1}^{\infty} \frac{2k+1}{\sqrt{k^3+1}}$. *It diverges because* $\displaystyle\lim_{k\to\infty} \frac{2k+1}{\sqrt{k^3+1}} \Big/ \frac{1}{\sqrt{k}} = 2 \neq 0$ *and* $\displaystyle\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$ *diverges.*

(3) $\displaystyle\sum_{k=1}^{\infty} \frac{2k+5}{\sqrt{k^6+3k^3}}$. *It converges because* $\displaystyle\lim_{k\to\infty} \frac{2k+5}{\sqrt{k^6+3k^3}} \Big/ \frac{1}{k^2} = 2 \neq 0$ *and* $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2}$ *converges.*

Theorem 8.18 (Root Test, [3, Theorem 12.4.1]). *Let a_k be nonnegative numbers. Suppose that*

$$\rho = \overline{\lim_{k\to\infty}} (a_k)^{1/k}. \qquad (\textit{Note that } \rho = \lim_{k\to\infty} (a_k)^{1/k} \textit{ if it converges.})$$

(1) *If $\rho < 1$, then $\displaystyle\sum_{k=1}^{\infty} a_k$ converges.*

(2) *If $\rho > 1$, then $\displaystyle\sum_{k=1}^{\infty} a_k$ diverges.*

(3) *If $\rho = 1$, then $\displaystyle\sum_{k=1}^{\infty} a_k$ may converge or diverge (no conclusion). (Consider $\sum \frac{1}{k}$ and $\sum \frac{1}{k^2}$.)*

Proof. Recall that if $\overline{\lim_{k\to\infty}} b_k = \beta$, then for any $\epsilon > 0$, there exists $N = N_\epsilon$ such that

(i) $b_k < \beta + \epsilon$ for any $k \geq N_\epsilon$;

(ii) there exists a subsequence $(b_{k_j})_{j=1}^{\infty}$ of $(b_k)_{k=1}^{\infty}$ such that $b_{k_j} > \beta - \epsilon$ for any $j \geq 1$. (Draw a picture.)

If $\rho < 1$, choose $\epsilon = (1-\rho)/2 > 0$ so that $\rho + \epsilon < 1$. Since $\overline{\lim_{k\to\infty}} (a_k)^{1/k} = \rho$, there exists N_ϵ such that $(a_k)^{1/k} < \rho + \epsilon$ for $k \geq N_\epsilon$. Thus,

$$a_k < (\rho+\epsilon)^k \qquad \text{for all} \quad k \geq N_\epsilon.$$

Since $\displaystyle\sum_{k=1}^{\infty} (\rho+\epsilon)^k$ converges, it follows from the basic comparison test that $\displaystyle\sum_{k=1}^{\infty} a_k$ converges.

If $\rho > 1$, choose $\epsilon' = \dfrac{\rho - 1}{2} > 0$. There exists a subsequence $(a_{k_j})_{j=1}^{\infty}$ of $(a_k)_{k=1}^{\infty}$ such that $(a_{k_j})^{1/k_j} \geq \rho - \epsilon' = \dfrac{\rho+1}{2} > 1$ for all $j \geq 1$. Since $(\rho - \epsilon') > 1$ and $k_j \geq j$, we have

$$a_{k_j} \geq (\rho - \epsilon')^{k_j} \geq (\rho - \epsilon')^j \qquad \text{for all } j \geq 1.$$

This shows that the subsequence $(a_{k_j})_{j=1}^{\infty}$ is unbounded, and so is $(a_k)_{k=1}^{\infty}$. In particular, $\lim_{k\to\infty} a_k \neq 0$. By the n-th term test, we conclude that $\displaystyle\sum_{k=1}^{\infty} a_k$ diverges, as desired. □

Example 8.19. *Determine whether the series converges or diverges.*

(1) *The series* $\sum_{k=2}^{\infty} \frac{1}{(\ln k)^k}$ *converges because* $\lim_{k\to\infty}(a_k)^{1/k} = \lim_{k\to\infty} \frac{1}{\ln k} = 0 < 1.$

(2) *The series* $\sum_{k=1}^{\infty} \frac{2^k}{k^3}$ *diverges because* $\lim_{k\to\infty}(a_k)^{1/k} = \lim_{k\to\infty} \frac{2}{(k^{1/k})^3} = 2 > 1.$

Theorem 8.20 (Ratio Test, [3, Theorem 12.4.2])**.** *Let a_k be positive numbers. Suppose that*

$$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lambda.$$

(1) *If $\lambda < 1$, then $\sum_{k=1}^{\infty} a_k$ converges.*

(2) *If $\lambda > 1$, then $\sum_{k=1}^{\infty} a_k$ diverges.*

(3) *If $\lambda = 1$, then $\sum_{k=1}^{\infty} a_k$ may converge or diverge (no conclusion). (Consider $\sum \frac{1}{k}$ and $\sum \frac{1}{k^2}$.)*

Proof. By the root test, it suffices to show that if $\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lambda$, then $\lim_{k\to\infty}(a_k)^{1/k} = \lambda$.

Let $\epsilon > 0$ be arbitrary. Since $\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lambda$, there exists $N = N_\epsilon \in \mathbb{N}$ such that for all $k \geq N$,

$$\max\{0, \lambda - \epsilon\} < \frac{a_{k+1}}{a_k} < \lambda + \epsilon.$$

For $n > N$, we can write a_n as a telescoping product:

$$a_n = a_N \cdot \prod_{k=N}^{n-1} \frac{a_{k+1}}{a_k}.$$

Using the upper bound $(\lambda + \epsilon)$ for each of the $n - N$ terms in the product, we have:

$$a_n < a_N(\lambda + \epsilon)^{n-N} \implies (a_n)^{1/n} < (a_N)^{1/n}(\lambda + \epsilon)^{1-\frac{N}{n}}.$$

Taking the limit superior as $n \to \infty$:

$$\overline{\lim_{n\to\infty}}(a_n)^{1/n} \leq 1 \cdot (\lambda + \epsilon)^1 = \lambda + \epsilon.$$

For the lower bound, we consider two cases:

(1) If $\lambda = 0$, then $\underline{\lim}_{n\to\infty}(a_n)^{1/n} \geq 0$ naturally since $a_k > 0$.

(2) If $\lambda > 0$, we choose ϵ small enough such that $\lambda - \epsilon > 0$. Then:

$$a_n > a_N(\lambda - \epsilon)^{n-N} \implies (a_n)^{1/n} > (a_N)^{1/n}(\lambda - \epsilon)^{1-\frac{N}{n}}.$$

Taking the limit inferior as $n \to \infty$ gives $\underline{\lim}_{n\to\infty}(a_n)^{1/n} \geq \lambda - \epsilon$.

In either case, for any $\epsilon > 0$, we have:

$$\lambda - \epsilon \leq \underline{\lim}_{n\to\infty}(a_n)^{1/n} \leq \overline{\lim_{n\to\infty}}(a_n)^{1/n} \leq \lambda + \epsilon.$$

Since this holds for all $\epsilon > 0$, it follows that $\lim_{n\to\infty}(a_n)^{1/n} = \lambda$. □

Example 8.21. *Determine whether the series converges or diverges.*

(1) *The series* $\sum_{k=1}^{\infty} \frac{1}{k!}$ *converges because* $\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \frac{1}{k+1} = 0.$

(2) *The series* $\sum_{k=1}^{\infty} \frac{k}{10^k}$ *converges because* $\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \frac{1}{10}\frac{k+1}{k} = \frac{1}{10} < 1.$

(3) *The series* $\sum_{k=1}^{\infty} \frac{k^k}{k!}$ *diverges because* $\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \left(\frac{k+1}{k}\right)^k = \lim_{k\to\infty} \left(1+\frac{1}{k}\right)^k = e > 1.$

Theorem 8.22 (Generalized Ratio Test)**.** *Let a_k be positive numbers.*

(1) *If* $\overline{\lim_{k\to\infty}} \frac{a_{k+1}}{a_k} < 1$*, then* $\sum_{k=1}^{\infty} a_k$ *converges.*

(2) *If* $\underline{\lim_{k\to\infty}} \frac{a_{k+1}}{a_k} > 1$*, then* $\sum_{k=1}^{\infty} a_k$ *diverges.*

(3) *If* $\underline{\lim_{k\to\infty}} \frac{a_{k+1}}{a_k} \leq 1 \leq \overline{\lim_{k\to\infty}} \frac{a_{k+1}}{a_k}$*, then the test is inconclusive. (Consider* $\sum \frac{1}{k}$ *and* $\sum \frac{1}{k^2}$*.)*

Proof. (1) Suppose $\overline{\lim_{k\to\infty}} \frac{a_{k+1}}{a_k} = \lambda < 1$. For $\epsilon = \frac{1-\lambda}{2}$, there exists $N = N_\epsilon \in \mathbb{N}$ such that for all $k \geq N$, $\frac{a_{k+1}}{a_k} < \lambda + \epsilon$. Denote $r = \lambda + \epsilon = \frac{\lambda+1}{2} < 1$. Then we have

$$a_{N+1} \leq r a_N, \quad a_{N+2} \leq r a_{N+1} \leq r^2 a_N, \quad \ldots, \quad a_{N+m} \leq r^m a_N$$

That is, $a_k \leq a_N r^{k-N}$ for any $k > N$. Since the geometric series $\sum_{k=N+1}^{\infty} a_N r^{k-N}$ converges for $r < 1$, we conclude that $\sum_{k=1}^{\infty} a_k$ converges by the basic comparison test.

(2) Suppose $\underline{\lim_{k\to\infty}} \frac{a_{k+1}}{a_k} = \mu > 1$. For $\epsilon' = \frac{\mu-1}{2}$, there exists $N' = N_{\epsilon'} \in \mathbb{N}$ such that for all $k \geq N'$, $\frac{a_{k+1}}{a_k} > \mu - \epsilon = \frac{\mu+1}{2} > 1$. In particular, we have $a_k > a_{k-1} > \cdots > a_{N'} > 0$ for all $k > N'$. This implies that $\lim_{k\to\infty} a_k \neq 0$, and thus the series diverges. □

Remark 8.23. *Let $(a_k)_{k=1}^{\infty}$ be a sequence of positive numbers. One has the inequality*

$$\underline{\lim_{k\to\infty}} \frac{a_{k+1}}{a_k} \leq \underline{\lim_{k\to\infty}} (a_k)^{1/k} \leq \overline{\lim_{k\to\infty}} (a_k)^{1/k} \leq \overline{\lim_{k\to\infty}} \frac{a_{k+1}}{a_k}$$

which shows that the root test is stronger than the ratio test.

In fact, the root test is strictly stronger than the ratio test:

Example 8.24 (Root test vs. ratio test)**.** *Consider the sequence a_k defined by:*

$$a_k = \begin{cases} (1/2)^k & \text{if } k \text{ is even} \\ (1/3)^k & \text{if } k \text{ is odd.} \end{cases}$$

The series is $\sum_{k=0}^{\infty} a_k = 1 + \frac{1}{3} + \frac{1}{4} + \frac{1}{27} + \cdots.$

(1) ***Ratio Test:*** *For* $k = 2m$,

$$\frac{a_{2m+1}}{a_{2m}} = \frac{(1/3)^{2m+1}}{(1/2)^{2m}} = \frac{1}{3}\left(\frac{2}{3}\right)^{2m} \to 0$$

as $m \to \infty$. *For* $k = 2m - 1$ *odd,*

$$\frac{a_{2m}}{a_{2m-1}} = \frac{(1/2)^{2m}}{(1/3)^{2m-1}} = \frac{1}{2}\left(\frac{3}{2}\right)^{2m-1} \to \infty$$

as $m \to \infty$. *Since* $\underline{\lim}_{k\to\infty} \frac{a_{k+1}}{a_k} = 0 \leq 1 \leq \overline{\lim}_{k\to\infty} \frac{a_{k+1}}{a_k} = \infty$, *the test is* inconclusive.

(2) ***Root Test:*** *For* k *even,* $(a_k)^{1/k} = \left((1/2)^k\right)^{1/k} = 1/2$. *For* k *odd,* $(a_k)^{1/k} = \left((1/3)^k\right)^{1/k} = 1/3$. *Thus,* $\overline{\lim}_{k\to\infty}(a_k)^{1/k} = \frac{1}{2} < 1$. *The root test confirms the series* converges.

8.4. **Absolute and conditional convergence.** In this subsection, we consider the convergence of arbitrary series $\sum_{k=1}^{\infty} a_k$. Note that $\sum_{k=1}^{\infty} |a_k|$ is a series with nonnegative terms.

Definition 8.25 (Absolute Convergence)**.** *A series* $\sum_{k=1}^{\infty} a_k$ *is* ***absolutely convergent*** *if the series* $\sum_{k=1}^{\infty} |a_k|$ *converges. A series is* ***conditionally convergent*** *if it is convergent and not absolutely convergent.*

Theorem 8.26 ([3, Theorem 12.5.1])**.** *Absolutely convergent series are convergent.*

Proof. Let $\sum_{k=1}^{\infty} a_k$ be an absolutely convergent series. Since $\sum_{k=1}^{\infty} 2|a_k|$ converges and , for each k,

$$0 \leq a_k + |a_k| \leq 2|a_k|,$$

it follows from the basic comparison test that $\sum_{k=1}^{\infty} a_k + |a_k|$ converges. Thus,

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty}(a_k + |a_k|) - \sum_{k=1}^{\infty} |a_k|$$

is also convergent. □

Example 8.27. *Consider the series.*

(1) $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2} = 1 - \frac{1}{4} + \frac{1}{9} - \cdots$ *is absolutely convergent.*

(2) $\frac{1}{2} - \frac{1}{2^2} - \frac{1}{2^3} + \frac{1}{2^4} - \frac{1}{2^5} - \frac{1}{2^6} + \cdots$ *is absolutely convergent.*

(3) $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \cdots$ *is not absolutely convergent.*

By considering $\sum |a_k|$, we do not know whether the series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$ converges or not. We need more criteria to determine the convergence. We start with a technical lemma.

Lemma 8.28. *Let* $(a_k)_{k=1}^{\infty}$ *and* $(b_k)_{k=1}^{\infty}$ *be two sequences satisfying the conditions:*

(1) there exists $M > 0$ such that $\left|\sum_{k=1}^{n} a_k\right| \leq M$ for all $n \in \mathbb{N}$;

(2) the sequence $(b_k)_{k=1}^{\infty}$ is monotone.

Then for any $n \in \mathbb{N}$,

$$|a_1 b_1 + \cdots + a_n b_n| \leq M(|b_1| + 2|b_n|).$$

Proof. Let $s_0 = 0$ and $s_n = a_1 + \cdots + a_n$. Then

$$\begin{aligned} a_1 b_1 + \cdots + a_n b_n &= (s_1 - s_0) b_1 + (s_2 - s_1) b_2 + \cdots + (s_n - s_{n-1}) b_n \\ &= s_1(b_1 - b_2) + s_2(b_2 - b_3) + \cdots + s_{n-1}(b_{n-1} - b_n) + s_n b_n. \end{aligned}$$

Since $(b_k)_{k=1}^{\infty}$ is monotone, we have

$$\begin{aligned} |s_1(b_1 - b_2)| + \cdots + |s_{n-1}(b_{n-1} - b_n)| &= \Big| |s_1|(b_1 - b_2) + \cdots + |s_{n-1}|(b_{n-1} - b_n) \Big| \\ &\leq \Big| M(b_1 - b_2) + \cdots + M(b_{n-1} - b_n) \Big| = M\,|b_1 - b_n|. \end{aligned}$$

Thus,

$$\begin{aligned} |a_1 b_1 + \cdots + a_n b_n| &\leq |s_1(b_1 - b_2)| + \cdots + |s_{n-1}(b_n - b_{n-1})| + |s_n|\,|b_n| \\ &\leq M\,|b_1 - b_n| + M\,|b_n| \\ &\leq M(|b_1| + 2|b_n|). \end{aligned}$$

This completes the proof. □

Theorem 8.29 (Dirichlet's Test)**.** *Let $(a_k)_{k=1}^{\infty}$ and $(b_k)_{k=1}^{\infty}$ be sequences of real numbers. Suppose that*

(1) there exists $M > 0$ such that $\left|\sum_{k=1}^{n} a_k\right| \leq M$ for all $n \in \mathbb{N}$;

(2) the sequence $(b_k)_{k=1}^{\infty}$ is monotone; and

(3) $\lim_{k\to\infty} b_k = 0$.

Then the series $\sum_{k=1}^{\infty} a_k b_k$ converges.

Proof. Let $s_n = \sum_{k=1}^{n} a_k$. By (1), we have

$$|a_{m+1} + \cdots + a_n| = |s_n - s_m| \leq 2M$$

for any $n, m \in \mathbb{N}$ with $n > m$. By Lemma 8.28 and (2), we have

$$|a_{m+1} b_{m+1} + \cdots + a_n b_n| \leq 2M(|b_{m+1}| + 2|b_n|).$$

By (3), for any $\epsilon > 0$, there exists $N = N_\epsilon$ such that $|b_m| < \dfrac{\epsilon}{6M}$ whenever $m \geq N$. Therefore, for $m \geq N$,

$$|a_{m+1} b_{m+1} + \cdots + a_n b_n| \leq 2M(|b_{m+1}| + 2|b_n|) < 2M\Big(\frac{\epsilon}{6M} + \frac{2\epsilon}{6M}\Big) = \epsilon.$$

This shows that $\Big(\sum_{k=1}^{n} a_k b_k\Big)_{n=1}^{\infty}$ is a Cauchy sequence, and thus the series $\sum_{k=1}^{\infty} a_k b_k$ converges. □

Corollary 8.30 (Alternating Series Test, a.k.a. Leibniz's Test, [3, Theorem 12.5.3]). *Let $(b_k)_{k=1}^{\infty}$ be a decreasing sequence of non-negative numbers. The series $\sum_{k=1}^{\infty}(-1)^{k+1}b_k$ converges iff $\lim_{k\to\infty} b_k = 0$.*

Proof. Let $a_k = (-1)^{k+1}$. Then $\left|\sum_{k=1}^{n} a_k\right| \leq 1$ for all $n \in \mathbb{N}$ ($M = 1$). By Theorem 8.29, the series $\sum_{k=1}^{\infty}(-1)^{k+1}b_k$ converges if $\lim_{k\to\infty} b_k = 0$. The converse implication follows from the n-th term test. □

Remark 8.31. *A* rearrangement *of a series $\sum_{k=1}^{\infty} a_k$ is a series that has exactly same terms but in a different order. For example, the series*

$$a_1 + a_3 + a_2 + a_5 + a_7 + a_4 + \cdots$$

is a rearrangement of $\sum_{k=1}^{\infty} a_k$.

It is a theorem of Riemann that all rearrangements of an absolutely convergent series converge absolutely to the same limit. In a sharp contrast, a series that is only conditionally convergent can be rearranged to converge to any real number, to diverge to $+\infty$, to diverge to $-\infty$, or even to oscillate between any two numbers we choose.

The following theorem is useful for the study of power series.

Theorem 8.32 (Abel's Test). *Suppose that*

(1) the series $\sum_{k=1}^{\infty} a_k$ converges, and

(2) the sequence $(b_k)_{k=1}^{\infty}$ is monotone and bounded.

Then the series $\sum_{k=1}^{\infty} a_k b_k$ converges.

Proof. Since $(b_k)_{k=1}^{\infty}$ is bounded, there exists $M > 0$ such that $|b_k| < M$ for all $k \in \mathbb{N}$. Since $\sum_{k=1}^{\infty} a_k$ converges, the sequence $\left(\sum_{k=1}^{n} a_k\right)_{n=1}^{\infty}$ is Cauchy. Thus, for any $\epsilon > 0$ there exists $N = N_\epsilon$ such that

$$|a_{m+1} + \cdots + a_n| < \frac{\epsilon}{3M}$$

whenever $n > m \geq N$. By Lemma 8.28, for any $n > m \geq N$,

$$|a_{m+1}b_{m+1} + \cdots + a_n b_n| \leq \frac{\epsilon}{3M}(|b_{m+1}| + 2|b_n|) < \frac{\epsilon}{3M} \cdot 3M = \epsilon.$$

This shows that the sequence $\left(\sum_{k=1}^{n} a_k b_k\right)_{n=1}^{\infty}$ is Cauchy, and thus the series $\sum_{k=1}^{\infty} a_k b_k$ converges. □

Corollary 8.33. *If the series $\sum_{k=1}^{\infty} a_k$ converges, then the power series $\sum_{k=1}^{\infty} a_k x^k$ converges for all $x \in [0, 1]$.*

9. Series of functions

A significant class of series emerges from the study of functions. In this section, we examine two primary examples: *Taylor series* and *Fourier series*.

9.1. **Taylor expansion.** Having established the rules for the differentiation and integration of polynomials, we now seek to approximate more complicated functions using polynomial forms. For an approximation of the form

$$f(x) \approx a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n,$$

we require that the function and the polynomial share the same derivatives at $x = 0$ up to the n-th order. This yields the following relations:

$$f(0) = a_0, \quad f'(0) = a_1, \quad f''(0) = 2! \cdot a_2, \quad \cdots, \quad f^{(n)}(0) = n! \cdot a_n.$$

We consider the polynomial

$$P_n(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n$$

as the best n-th degree polynomial approximation of $f(x)$ centered at $x = 0$. As n increases, the accuracy of this approximation typically improves. To optimize this approach, we consider the formal power series:

$$f(x) \approx \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots .$$

This transition from finite polynomials to infinite series raises two fundamental questions:

(1) For which values of x does the series $\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k$ converge?
(2) If the series converges at a point x, is the sum necessarily equal to $f(x)$?

We need the following theorem to answer these questions.

Theorem 9.1 (Taylor's Theorem, [3, Theorem 12.6.1])**.** *If f has $n+1$ continuous derivatives on an open interval I that contains 0, then for each $x \in I$*

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + R_n(x), \tag{4}$$

where

$$R_n(x) = \frac{1}{n!}\int_0^x f^{(n+1)}(t) \cdot (x-t)^n dt$$

is called the ***remainder****.*

Proof. Fix $x \in I$. Since

$$\int_0^x f'(t)\,dt = f(x) - f(0),$$

we have

$$f(x) = f(0) + \int_0^x f'(t)\,dt. \qquad (Case\ n = 0)$$

Let $u(t) = f'(t)$ and $v(t) = t - x$. Integrating by parts, we have

$$\begin{aligned} f(x) &= f(0) + (t-x)f'(t)\Big|_{t=0}^{x} - \int_0^x (t-x)f''(t)\,dt \\ &= f(0) + f'(0)x + \int_0^x f''(t)(x-t)\,dt. \qquad (\textit{Case } n = 1) \end{aligned}$$

Assume Equation (4) is true for k, $k \le n-1$. Let $u(t) = f^{(k+1)}(t)$ and $v(t) = -(x-t)^{k+1}/(k+1)!$. Integrating by parts, we have

$$\begin{aligned} R_k(x) &= -\frac{(x-t)^{k+1}}{(k+1)!}\Big|_{t=0}^{x} + \int_0^x \frac{(x-t)^{k+1}}{(k+1)!} f^{(k+2)}(t)\,dt \\ &= \frac{f^{(k+1)}(0)}{(k+1)!}x^{k+1} + R_{k+1}(x). \qquad (\textit{Case } n = k+1) \end{aligned}$$

This proves the theorem by induction. □

The remainder $R_n(x)$ can be expressed in another form by the following theorem:

Theorem 9.2 (Second Mean Value Theorem for Integrals, [3, Theorem 5.9.3])**.** *If u and v are continuous on $[a,b]$ and $v \ge 0$ on $[a,b]$, then there exists $c \in [a,b]$ such that*

$$\int_a^b u(x)v(x)\,dx = u(c) \cdot \int_a^b v(x)\,dx.$$

*Such $u(c)$ is called the v-**weight average** of u on $[a,b]$.*

Proof. If $v \equiv 0$, the theorem is clear. Thus, we may assume that v is not a zero function, and this implies that $\int_a^b v(x)\,dx > 0$.

Since u is continuous on $[a,b]$, by the extreme value theorem, u takes on a minimum value m and a maximum value M. Since $v \ge 0$,

$$m \cdot v(x) \le u(x) \cdot v(x) \le M \cdot v(x), \qquad \forall x \in [a,b].$$

This implies that

$$m \int_a^b v(x)\,dx \le \int_a^b u(x)v(x)\,dx \le M \int_a^b v(x)\,dx,$$

and thus

$$m \quad \le \quad \frac{\int_a^b u(x)v(x)\,dx}{\int_a^b v(x)\,dx} \quad \le \quad M.$$

It follows from the intermediate value theorem that there exists $c \in [a,b]$ such that

$$u(c) = \frac{\int_a^b u(x)v(x)\,dx}{\int_a^b v(x)\,dx}$$

which implies the assertion. □

Applying Theorem 9.2 to $R_n(x)$ with

$$u(t) = \begin{cases} f^{(n+1)}(t), & \text{if } x \geq 0, \\ (-1)^n f^{(n+1)}(t), & \text{if } x < 0, \end{cases} \qquad \text{and} \qquad v(t) = \begin{cases} \dfrac{(x-t)^n}{n!}, & \text{if } x \geq 0, \\ \dfrac{(t-x)^n}{n!}, & \text{if } x < 0, \end{cases}$$

we get the *Lagrange formula for the remainder*:

Corollary 9.3. *The remainder $R_n(x)$ can be written as*

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1}$$

with c some number between 0 and x. In other words,

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1} \tag{5}$$

with c some number (depending on x) between 0 and x.

Note that Equation (5) can be considered as an extension of the mean value theorem.

Remark 9.4. *By Theorem 9.1, the series $\sum_{k=0}^{\infty} \frac{f^{(k)}}{k!} x^k$ converges to $f(x)$ if $\lim_{n\to\infty} R_n(x) = 0$. To study $\lim_{n\to\infty} R_n(x)$, it is useful to notice that, due to Corollary 9.3, we have*

$$|R_n(x)| \leq \left(\max_{t \in J_x} |f^{(n+1)}(t)| \right) \frac{|x|^{n+1}}{(n+1)!},$$

where J_x is the closed interval that joins 0 and x.

Example 9.5. *For any real number x,*

$$e^x = \sum_{k=0}^{\infty} \frac{1}{k!} x^k = 1 + x + \frac{1}{2!}x^2 + \cdots,$$

since

$$|R_n(x)| \leq \left(\max_{t \in J_x} e^t \right) \frac{|x|^{n+1}}{(n+1)!} \leq e^{|x|} \frac{|x|^{n+1}}{(n+1)!} \to 0 \qquad \text{as } n \to \infty.$$

(Recall that $\lim_{n\to\infty} \frac{a^n}{n!} = 0$. See Proposition 6.40 or [3, Section 11.4]*.)*

Example 9.6. *For any real number x,*

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \cdots,$$

since

$$|R_n(x)| \leq \left(\max_{t \in J_x} |\sin^{(n+1)} t| \right) \frac{|x|^{n+1}}{(n+1)!} \leq \frac{|x|^{n+1}}{(n+1)!} \to 0 \qquad \text{as } n \to \infty.$$

Similarly, for any real number x, we have

$$\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \cdots.$$

Showing that $\lim_{n\to\infty} R_n(x) = 0$ is not easy in general. We will introduce another method to determine the convergence of Taylor series in the next section.

Consider Taylor series in $x - a$. The following theorem can be shown by similar arguments.

Theorem 9.7 (Taylor's Theorem, [3, Theorem 12.7.1])**.** *If f has $n+1$ continuous derivatives on an open interval I that contains a, then for each $x \in I$*

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x), \tag{6}$$

where

$$R_n(x) = \frac{1}{n!}\int_a^x f^{(n+1)}(t)(x-t)^n dt.$$

Furthermore, there exists a number c (depending on x) between a and x such that

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}.$$

Example 9.8. *The polynomial*

$$f(x) = 4x^3 - 3x^2 + 5x - 1$$

can be expanded in powers of $x-2$ by computing $f^{(n)}(2)$: $f(2) = 29$, $f'(2) = 41$, $f''(2) = 42$, $f'''(2) = 24$, and $f^{(n)}(2) = 0$ for all $n \geq 4$. Consequently,

$$\begin{aligned} f(x) &= 29 + 41(x-2) + \frac{42}{2!}(x-2)^2 + \frac{24}{3!}(x-2)^3 \\ &= 29 + 41(x-2) + 21(x-2)^2 + 4(x-2)^3. \end{aligned}$$

9.2. **Power series.** In this section, we approach the problem in the other direction: We consider the series

$$\sum_{k=0}^{\infty} a_k x^k \qquad \text{and} \qquad \sum_{k=0}^{\infty} a_k (x-a)^k$$

without a given function $f(x)$. For simplicity, we will state the definitions and theorems for $\sum_{k=0}^{\infty} a_k x^k$. All the parallel definitions and theorems hold for $\sum_{k=0}^{\infty} a_k (x-a)^k$.

Definition 9.9. *A power series $\sum_{k=0}^{\infty} a_k x^k$ is said to* ***converge***

(1) ***at*** *c if $\sum_{k=0}^{\infty} a_k c^k$ converges;*

(2) ***on the set*** *S if $\sum_{k=0}^{\infty} a_k x^k$ converges at each $x \in S$.*

Theorem 9.10 ([3, Theorem 12.8.2])**.** *If $\sum_{k=0}^{\infty} a_k x^k$ converges at $c \neq 0$, then it converges absolutely at all x with $|x| < |c|$. If $\sum_{k=0}^{\infty} a_k x^k$ diverges at d, then it diverges at all x with $|x| > |d|$.*

Proof. It suffices to show the first assertion. Suppose $\sum_{k=0}^{\infty} a_k c^k$ converges. Then $\lim_{k\to\infty} a_k c^k = 0$, and, for k sufficiently large,

$$|a_k c^k| \leq 1.$$

This implies that

$$|a_k x^k| = |a_k c^k| \left|\frac{x}{c}\right|^k \leq \left|\frac{x}{c}\right|^k.$$

Since $|x| < |c|$, the series $\sum_k |x/c|^k$ converges. If follows from the comparison theorem that $\sum_k |a_k x^k|$ converges. □

It follows from this theorem that there are exactly three possibilities for a power series:

Case 1. The series $\sum_k a_k x^k$ converges only at $x = 0$. For example, $a_k = k^k$.

Case 2. The series $\sum_k a_k x^k$ converges absolutely at all real numbers x. For example, $\sum_k x^k/k!$.

Case 3. There exists a positive number r such that the series $\sum_k a_k x^k$ converges absolutely for $|x| < r$ and diverges for $|x| > r$. For example, $\sum_k x^k$ converges absolutely for $|x| < 1$ and diverges for $|x| > 1$.

Definition 9.11. *The* ***radius of convergence*** *of the series $\sum_k a_k x^k$ is*

(1) 0 if it is Case 1;
(2) ∞ if it is Case 2;
(3) r if it is Case 3.

The ***interval of convergence*** *of a power series is the maximal interval on which it converges.*

Example 9.12. *The radius of convergence of $\sum k^k x^k$ is 0. The radius of convergence of $\sum x^k/k!$ is ∞. The radius of convergence of $\sum x^k$ is 1.*

The interval of convergence of $\sum k^k x^k$ is $\{0\}$. The interval of convergence of $\sum x^k/k!$ is $(-\infty, \infty)$. The interval of convergence of $\sum x^k$ is $(-1, 1)$. *Explain.*

By the Root Test, we establish the following fundamental result:

Theorem 9.13. *The radius of convergence r of the power series $\sum_{k=0}^{\infty} a_k x^k$ is given by $r = \dfrac{1}{\overline{\lim}_{k\to\infty} \sqrt[k]{|a_k|}}$.*

Here, the value r is understood to be ∞ if $\overline{\lim}_{k\to\infty} \sqrt[k]{|a_k|} = 0$, and 0 if $\overline{\lim}_{k\to\infty} \sqrt[k]{|a_k|} = \infty$.

Remark 9.14. *One can also employ the Ratio Test to determine the radius of convergence. Specifically, by Remark 8.23 and Theorem 9.13, if the limit*

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$$

exists, then the radius of convergence is $r = \dfrac{1}{L}$. *Explain*

Example 9.15. *Find the interval of convergence of the series. Step 1: Find the radius of convergence. Step 2: Check the convergence of the series at the endpoints.*

(1) The interval of convergence of $\sum_{k=1}^{\infty} x^k/k$ is $[-1, 1)$.

(2) *The interval of convergence of* $\sum_{k=1}^{\infty} x^k/k^2$ *is* $[-1,1]$.

(3) *The interval of convergence of* $\sum_{k=1}^{\infty} \frac{k}{6^k}x^k$ *is* $(-6,6)$.

(4) *The interval of convergence of* $\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2 3^k}(x+2)^k$ *is* $[-5,1]$.

Example 9.16. *We compare the convergence properties of two related power series:*

(1) *Consider the power series defined by*

$$f(x) = \sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k}} x^k.$$

Using the Ratio Test, we find the radius of convergence is $r = 1$*, meaning the series converges absolutely for* $x \in (-1,1)$*. Regarding the boundary behavior:*

- *At* $x = -1$*, the series is* $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$*, which is a p-series with* $p = 1/2$ *and thus diverges.*
- *At* $x = 1$*, the series is* $\sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k}}$*, which converges by the Alternating Series Test.*

(2) *Now consider the modified series*

$$g(x) = \sum_{k=1}^{\infty} \left(\frac{(-1)^k}{\sqrt{k}} \cdot \frac{k+1}{k} \right) x^k.$$

First, we note that the radius of convergence for $g(x)$ *remains* $r = 1$ *because* $\lim_{k\to\infty} \frac{k+1}{k} = 1$*. We then examine the behavior at the endpoints* $x = 1$ *and* $x = -1$*. Regarding the boundary behavior:*

- ***At*** $x = 1$***:** The series for* $g(1)$ *takes the form* $\sum_{k=1}^{\infty} a_k b_k$*, where:*

$$a_k = \frac{(-1)^k}{\sqrt{k}} \quad \text{and} \quad b_k = \frac{k+1}{k} = 1 + \frac{1}{k}.$$

We check the conditions for Abel's Test (Theorem 8.32):

(a) *The series* $\sum a_k$ *converges by the Alternating Series Test.*

(b) *The sequence* b_k *is bounded* $(1 < b_k \le 2)$ *and monotone decreasing* $(b_{k+1} < b_k)$*.*

By Abel's Test, the product series converges. Thus, $g(x)$ *converges at* $x = 1$*.*

- ***At*** $x = -1$***:** Substituting* $x = -1$ *into the expression for* $g(x)$ *yields:*

$$g(-1) = \sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k}} \cdot \frac{k+1}{k} \cdot (-1)^k = \sum_{k=1}^{\infty} \frac{k+1}{k\sqrt{k}} = \sum_{k=1}^{\infty} \left(\frac{1}{\sqrt{k}} + \frac{1}{k^{3/2}} \right).$$

The resulting series is the sum of a divergent p-series $(p = 1/2)$ *and a convergent p-series* $(p = 3/2)$*. Consequently, the series diverges at* $x = -1$*.*

Therefore, the interval of convergence for $g(x)$ *is* $(-1,1]$*.*

Next, we examine the calculus of power series. Within their radius of convergence, these series can be differentiated and integrated term-by-term (see [3, Theorems 12.9.1–12.9.3]).

Theorem 9.17. *Suppose r is the radius of convergence of the power series $\sum_{k=0}^{\infty} a_k x^k$.*

(1) The radius of convergence of the power series $\sum_{k=1}^{\infty} k a_k x^k$ is also r.

(2) Let $f(x) = \sum_{k=0}^{\infty} a_k x^k$. Then $f(x) \in C^{\infty}(-r, r)$, and

$$f'(x) = \sum_{k=1}^{\infty} k a_k x^k, \qquad \forall x \in (-r, r).$$

In particular, the Taylor expansion of $f(x)$ at 0 is the given series $\sum_{k=0}^{\infty} a_k x^k$, i.e. $a_k = \dfrac{f^{(k)}(0)}{k!}$.

(3) If $[a, b] \subset (-r, r)$, then

$$\int_a^b f(x)\,dx = \sum_{k=0}^{\infty} a_k \left(\int_a^b x^k\,dx \right).$$

The proof of the first assertion follows from the observation:

$$\varlimsup_{k\to\infty} \sqrt[k]{|a_k|} = L \quad \Longrightarrow \quad \varlimsup_{k\to\infty} \sqrt[k]{k|a_k|} = \left(\lim_{k\to\infty} \sqrt[k]{k} \right) \left(\varlimsup_{k\to\infty} \sqrt[k]{|a_k|} \right) = 1 \cdot L = L.$$

For the remaining two assertions, the proof requires the theory of uniform convergence. We postpone the formal proof to the course of Advanced Calculus.

Example 9.18. *Since the geometric series*

$$\sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \cdots$$

converges to $\dfrac{1}{1-x}$ on $(-1, 1)$, the series

$$\sum_{k=1}^{\infty} k x^{k-1}, \qquad \sum_{k=2}^{\infty} k(k-1) x^{k-2} \qquad \text{and} \qquad \sum_{k=3}^{\infty} k(k-1)(k-2) x^{k-3}$$

converge to $\left(\dfrac{1}{1-x}\right)'$, $\left(\dfrac{1}{1-x}\right)''$ and $\left(\dfrac{1}{1-x}\right)'''$ on $(-1, 1)$, respectively.

Example 9.19. *Recall that*

$$e^x = \sum_{k=0}^{\infty} \frac{1}{k!} x^k = 1 + x + \frac{1}{2!} x^2 + \cdots,$$

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} = x - \frac{1}{3!} x^3 + \frac{1}{5!} x^5 + \cdots,$$

$$\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} = 1 - \frac{1}{2!} x^2 + \frac{1}{4!} x^4 + \cdots$$

on $(-\infty, \infty)$. One can verify

$$(e^x)' = e^x, \qquad (\sin x)' = \cos x, \qquad (\cos x)' = -\sin x$$

by differentiating the power series directly. One also can solve

$$y'' + y = 0$$

by power series.

Example 9.20. *By termwise integration, one can show that*

(1) $\displaystyle\sum_{k=1}^{\infty} \frac{x^k}{k} = \ln\Big(\frac{1}{1-x}\Big)$ *for all* $x \in (-1, 1)$*;*

(2) $\displaystyle\ln(1 + x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} x^{k+1}$ *for all* $x \in (-1, 1)$. *Taylor expansion of natural logarithm.*

About the expansion of $\ln(1 + x)$, we actually have

$$\ln 2 = \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} 1^{k+1} = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

by the following theorem.

Theorem 9.21 (Abel's Theorem, [3, Theorem 12.9.5])**.** *Suppose that* $\sum_{k=0}^{\infty} a_k x^k$ *converges on* $(-c, c)$ *and* $f(x) = \sum_{k=0}^{\infty} a_k x^k$ *on this interval.*

(1) *If* f *is left continuous at* c *and* $\sum_{k=0}^{\infty} a_k c^k$ *converges, then* $f(c) = \sum_{k=0}^{\infty} a_k c^k$.

(2) *If* f *is right continuous at* $-c$ *and* $\sum_{k=0}^{\infty} a_k (-c)^k$ *converges, then* $f(-c) = \sum_{k=0}^{\infty} a_k (-c)^k$.

Proof. Without loss of generality, let $c = 1$. Suppose the series $\sum_{k=0}^{\infty} a_k$ converges to S. We wish to show that $\lim_{x\to 1^-} f(x) = S$.

Let $s_n = \sum_{k=0}^{n} a_k$ denote the n-th partial sum of the coefficients, with $s_{-1} = 0$. Then $a_k = s_k - s_{k-1}$. For $|x| < 1$, the power series $f(x)$ can be rewritten using summation by parts:

$$f(x) = \sum_{k=0}^{\infty} (s_k - s_{k-1}) x^k = \sum_{k=0}^{\infty} s_k x^k - \sum_{k=0}^{\infty} s_{k-1} x^k = \sum_{k=0}^{\infty} s_k x^k - x \sum_{k=0}^{\infty} s_k x^k$$

Factoring out $(1 - x)$, we obtain the identity:

$$f(x) = (1 - x) \sum_{k=0}^{\infty} s_k x^k.$$

Since $(1 - x) \sum_{k=0}^{\infty} x^k = 1$ for $|x| < 1$, we have $S = (1 - x) \sum_{k=0}^{\infty} S x^k$. Thus,

$$|f(x) - S| = \left|(1 - x) \sum_{k=0}^{\infty} (s_k - S) x^k\right| \le (1 - x) \sum_{k=0}^{\infty} |s_k - S| x^k.$$

Given $\epsilon > 0$, since $s_k \to S$, there exists N such that $|s_k - S| < \epsilon/2$ for all $k > N$. We split the sum into two parts:

$$|f(x) - S| \leq (1-x)\sum_{k=0}^{N} |s_k - S|x^k + (1-x)\sum_{k=N+1}^{\infty} \frac{\epsilon}{2}x^k.$$

The second term is bounded by $(1-x)\cdot\frac{\epsilon/2}{1-x} = \epsilon/2$. For the first term, since it is a finite sum, we can choose $\delta > 0$ such that $(1-x)\sum_{k=0}^{N}|s_k - S| < \epsilon/2$ whenever $0 < 1-x < \delta$. This shows that $|f(x) - S| < \epsilon/2 + \epsilon/2 = \epsilon$ when $0 < 1-x < \delta$, i.e. $S = \lim_{x\to 1^-} f(x) = f(c)$. The proof for (1) is thus complete. The proof for the case $x = -c$ follows by a similar argument or by considering $g(x) = f(-x)$. □

Corollary 9.22. *For any* $x \in (-1, 1]$,

$$\ln(1+x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1}x^{k+1}.$$

Corollary 9.23. *For any* $x \in [-1, 1]$,

$$\arctan x = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}x^{2k+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots.$$

Proof. Note that

$$(\arctan x)' = \frac{1}{1+x^2} = \sum_{k=0}^{\infty}(-x^2)^k, \qquad \forall x \in (-1,1).$$

Thus,

$$\arctan x = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}x^{2k+1} + C, \qquad \forall x \in (-1,1).$$

By taking $x = 0$, one can see that $C = 0$. Furthermore, by the alternating series test, the series $\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}x^{2k+1}$ converges at $x = \pm 1$. Since $\arctan x$ is continuous at $x \pm 1$, the corollary follows. □

Example 9.24. *Expand the following functions in power of x.*

(1) $\cosh x = \dfrac{e^x + e^{-x}}{2} = \dfrac{1}{2}\Big(\sum_{k=0}^{\infty}\dfrac{x^k}{k!} + \sum_{k=0}^{\infty}\dfrac{(-x)^k}{k!}\Big) = \sum_{k=0}^{\infty}\dfrac{x^{2k}}{(2k)!}$.

(2) $x^2\cos x^3 = x^2\Big(\sum_{k=0}^{\infty}\dfrac{(-1)^k}{(2k)!}(x^3)^{2k}\Big) = \sum_{k=0}^{\infty}\dfrac{(-1)^k}{(2k)!}x^{6k+2}$.

Summary of Taylor series and power series:

$$\text{smooth functions } f(x) \;\underset{\text{Compute sum}}{\overset{\text{Taylor series}}{\rightleftarrows}}\; \text{power series } \sum_{k=0}^{\infty} a_k x^k$$

- Taylor series: it starts with a function.
  - Taylor series of a smooth function f (at $x = 0$) is $\sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{k!}x^k$.

– The series $\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$ does NOT necessarily converge to $f(x)$. For example, if

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0, \end{cases}$$

then $f^{(n)}(0) = 0$ for all n. Thus, its Taylor series is the zero series which converges everywhere to zero, not $f(x)$.

– To get actual $f(x)$, we can use the finite expansion:

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k + R_n(x),$$

where

$$R_n(x) = \frac{1}{n!} \int_0^x f^{(n+1)}(t) \cdot (x-t)^n dt = \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1}$$

for some c between x and 0.

– The Taylor series $\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$ converges to $f(x)$ iff $R_n(x)$ converges to 0 as $n \to \infty$.

– But $\lim_{n\to\infty} R_n(x)$ is difficult in general. Thus, we usually try to get the desired expansions by the power series techniques and the well-known expansions:

$$e^x = \sum_{k=0}^{\infty} \frac{1}{k!} x^k = 1 + x + \frac{1}{2!} x^2 + \cdots, \qquad x \in (-\infty, \infty);$$

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} = x - \frac{1}{3!} x^3 + \frac{1}{5!} x^5 + \cdots, \qquad x \in (-\infty, \infty);$$

$$\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} = 1 - \frac{1}{2!} x^2 + \frac{1}{4!} x^4 + \cdots, \qquad x \in (-\infty, \infty);$$

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \cdots, \qquad x \in (-1, 1);$$

$$\ln(1+x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} x^{k+1} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots, \qquad x \in (-1, 1].$$

• Power series: it start with a series $\sum_{k=0}^{\infty} a_k x^k = a_0 + a_1 x + a_2 x^2 + \cdots$. There is no function at beginning.

– Many nice properties: radius of convergence, differentiation, integration, etc.

– If $\sum_{k=0}^{\infty} a_k x^k$ converges on $(-r, r)$, $r > 0$, then we can consider the function

$$f(x) = \sum_{k=0}^{\infty} a_k x^k, \qquad x \in (-r, r).$$

The Taylor series of $f(x)$ is exactly $\sum_{k=0}^{\infty} a_k x^k$.

9.3. **Fourier series.** The theory of Fourier series is a fundamental tool in mathematical analysis, allowing us to represent complicated periodic functions as infinite sums of simple sine and cosine waves. Originally developed by Joseph Fourier (1768–1830) to solve the heat equation, this decomposition transforms difficult problems involving periodic signals into manageable algebraic ones involving trigonometric functions.

9.3.1. *Periodic functions and their Fourier coefficients.*

Definition 9.25. *A function $f : \mathbb{R} \to \mathbb{R}$ is said to be* **periodic** *with period $T > 0$ if*

$$f(x + T) = f(x)$$

for all $x \in \mathbb{R}$.

For example, the trigonometric functions, $\cos nx$ and $\sin mx$ are periodic functions with period $T = 2\pi$.

To simplify our integration limits later, we first establish that the integral of a periodic function is invariant under translation of the interval.

Lemma 9.26. *If $f(x)$ is continuous on $\mathbb{R}$ and periodic with period $T > 0$, then for any real number a,*

$$\int_a^{a+T} f(x)\,dx = \int_0^T f(x)\,dx.$$

Proof. Let $F(a) = \displaystyle\int_a^{a+T} f(x)\,dx$. By the Fundamental Theorem of Calculus, the derivative is:

$$F'(a) = f(a + T) - f(a).$$

Since f is periodic, $f(a + T) = f(a)$, which implies $F'(a) = 0$. Thus, $F(a)$ is constant for all a. Choosing $a = 0$ gives the result. □

Without loss of generality, we consider periodic functions $f(x)$ with period $T = 2\pi$. We aim to approximate $f(x)$ using a linear combination of trigonometric functions:

$$f(x) \approx \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx).$$

We seek to determine the coefficients by requiring that the integral properties hold for both sides of the equation. Specifically, assuming the series allows for term-by-term integration, we require that for any $m \in \mathbb{N}$:

$$\int_{-\pi}^{\pi} f(x) \cos mx\,dx = \int_{-\pi}^{\pi} \frac{a_0}{2} \cos mx\,dx + \sum_{k=1}^{\infty} \left(a_k \int_{-\pi}^{\pi} \cos kx \cos mx\,dx + b_k \int_{-\pi}^{\pi} \sin kx \cos mx\,dx \right),$$

$$\int_{-\pi}^{\pi} f(x) \sin mx\,dx = \int_{-\pi}^{\pi} \frac{a_0}{2} \sin mx\,dx + \sum_{k=1}^{\infty} \left(a_k \int_{-\pi}^{\pi} \cos kx \sin mx\,dx + b_k \int_{-\pi}^{\pi} \sin kx \sin mx\,dx \right).$$

Due to the periodicity of the functions, the choice of the integration interval — whether $[-\pi, \pi]$ or $[0, 2\pi]$ — does not affect the result.

The method for finding Fourier coefficients α_k and β_l relies on the orthogonality of the basis functions $\{1, \cos mx, \sin nx\}_{m,n=1}^{\infty}$ on the interval $[-\pi, \pi]$.

Lemma 9.27. *For non-negative integers m and n, the following hold:*

(1) $$\int_{-\pi}^{\pi} \cos(mx)\cos(nx)\,dx = \begin{cases} 0 & \text{if } m \neq n \\ \pi & \text{if } m = n \neq 0 \\ 2\pi & \text{if } m = n = 0 \end{cases}$$

(2) $$\int_{-\pi}^{\pi} \sin(mx)\sin(nx)\,dx = \begin{cases} 0 & \text{if } m \neq n \\ \pi & \text{if } m = n \neq 0 \\ 0 & \text{if } m = n = 0 \end{cases}$$

(3) $$\int_{-\pi}^{\pi} \sin(mx)\cos(nx)\,dx = 0 \quad \text{for all } m, n.$$

By Lemma 9.27, our requirements imply that

$$\int_{-\pi}^{\pi} f(x)\,dx = \int_{-\pi}^{\pi} \frac{a_0}{2}\cos(0x)\,dx = \pi a_0,$$
$$\int_{-\pi}^{\pi} f(x)\cos mx\,dx = a_m \int_{-\pi}^{\pi} \cos mx \cos mx\,dx = \pi a_m,$$
$$\int_{-\pi}^{\pi} f(x)\sin mx\,dx = b_m \int_{-\pi}^{\pi} \sin mx \sin mx\,dx = \pi b_m.$$

This observation leads to the following formal definition.

Definition 9.28. *Let $f(x)$ be a periodic, piecewise continuous function with period 2π. The values*

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx\,dx, \qquad k \geq 0;$$
$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx\,dx, \qquad k \geq 1,$$

are defined as the ***Fourier coefficients*** *of the function $f(x)$. The trigonometric series*

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx)$$

is called the ***Fourier series*** *of $f(x)$. We denote this relationship by*

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx).$$

Example 9.29. *Let $f(x)$ be the periodic function with period 2π such that*

$$f(x) = x, \qquad -\pi < x \leq \pi.$$

Find the Fourier series of $f(x)$. $\quad f(x) \sim \sum_{k=1}^{\infty} \frac{2(-1)^{k+1}}{k} \sin kx.$

Example 9.30. *Let $g(x)$ be the periodic function with period 2π such that*

$$g(x) = x^2, \qquad 0 < x \leq 2\pi.$$

Find the Fourier series of $g(x)$. $g(x) \sim \frac{4\pi^2}{3} + 4\sum_{k=1}^{\infty}\frac{\cos kx}{k^2} - 4\pi\sum_{k=1}^{\infty}\frac{\sin kx}{k}.$

9.3.2. *Sums of Fourier series.* In a manner analogous to the study of Taylor series, two fundamental questions naturally arise regarding Fourier series:

(1) For which values of x does the Fourier series $\frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx)$ converge?
(2) If the series converges at a specific point x, is the resulting sum necessarily equal to the function value $f(x)$?

To obtain clear and useful answers, we consider the following class of functions.

Definition 9.31. *A function f is said to be* ***piecewise continuously differentiable*** *on $\mathbb{R}$, denoted by $f \in PC^1$, if for every $x_0 \in \mathbb{R}$, one of the following conditions holds:*

(1) f and its derivative f' are both continuous at x_0.
(2) The one-sided limits $f(x_0^+) = \lim_{x\to x_0^+} f(x)$ and $f(x_0^-) = \lim_{x\to x_0^-} f(x)$ exist, and the following one-sided derivatives exist:

$$f'(x_0^+) := \lim_{u\to 0^+}\frac{f(x_0+u) - f(x_0^+)}{u},$$
$$f'(x_0^-) := \lim_{u\to 0^-}\frac{f(x_0+u) - f(x_0^-)}{u}.$$

Note: This definition ensures that at any point of discontinuity (of either the function or its derivative), the function approaches a finite, well-defined slope from both the left and the right.

Theorem 9.32 (Pointwise Convergence Theorem)**.** *Let $f(x) \in PC^1$ be a periodic function with period 2π. If*

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx),$$

then for any $x_0 \in \mathbb{R}$, the series converges to the arithmetic mean of the limits from the left and right:

$$\frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kx_0 + b_k \sin kx_0) = \frac{f(x_0^+) + f(x_0^-)}{2}.$$

To prove Theorem 9.32, we need the following lemma.

Lemma 9.33 (Riemann–Lebesgue)**.** *If f is a piecewise continuous function on $[a, b]$, then*

$$\lim_{\lambda\to\infty}\int_a^b f(x)\sin\lambda x\,dx = 0, \qquad \textit{and} \qquad \lim_{\lambda\to\infty}\int_a^b f(x)\cos\lambda x\,dx = 0.$$

Proof. Since f is piecewise continuous, it is integrable. Thus, for any $\epsilon > 0$, there exists a partition $P_\epsilon = \{x_0, x_1, \ldots, x_{n_\epsilon}\}$ of $[a, b]$, where

$$a = x_0 < x_1 < \cdots < x_{n_\epsilon} = b,$$

such that $U_f(P_\epsilon) - L_f(P_\epsilon) < \epsilon$. Here $U_f(P_\epsilon)$ and $L_f(P_\epsilon)$ are the upper Riemann sum and lower Riemann sum, respectively. More explicitly,

$$U_f(P_\epsilon) = \sum_{j=1}^{n_\epsilon} M_j(x_j - x_{j-1}) \qquad \text{and} \qquad L_f(P_\epsilon) = \sum_{j=1}^{n_\epsilon} m_j(x_j - x_{j-1}),$$

where $M_j = \sup_{x\in[x_{j-1},x_j]} f(x)$ and $m_j = \inf_{x\in[x_{j-1},x_j]} f(x)$.

We decompose $\displaystyle\int_a^b f(x)\sin\lambda x\,dx$ as

$$\int_a^b f(x)\sin\lambda x\,dx = \sum_{j=1}^{n_\epsilon}\int_{x_{j-1}}^{x_j} (f(x) - m_j)\sin\lambda x\,dx + \sum_{j=1}^{n_\epsilon}\int_{x_{j-1}}^{x_j} m_j \sin\lambda x\,dx.$$

Note that

$$\begin{aligned}\left|\sum_{j=1}^{n_\epsilon}\int_{x_{j-1}}^{x_j} (f(x) - m_j)\sin\lambda x\,dx\right| &\le \sum_{j=1}^{n_\epsilon}\int_{x_{j-1}}^{x_j} \left|f(x) - m_j\right| \cdot |\sin\lambda x|\,dx \\ &\le \sum_{j=1}^{n_\epsilon}\int_{x_{j-1}}^{x_j} (M_j - m_j)\,dx = U_f(P_\epsilon) - L_f(P_\epsilon) < \epsilon\end{aligned}$$

and

$$\left|\sum_{j=1}^{n_\epsilon}\int_{x_{j-1}}^{x_j} m_j\sin\lambda x\,dx\right| \le \sum_{j=1}^{n_\epsilon}|m_j|\cdot\left|\int_{x_{j-1}}^{x_j}\sin\lambda x\,dx\right| \le \sum_{j=1}^{n_\epsilon}|m_j|\cdot\frac{2}{\lambda}.$$

Consequently,

$$\left|\int_a^b f(x)\sin\lambda x\,dx\right| \le \epsilon + \sum_{j=1}^{n_\epsilon}|m_j|\cdot\frac{2}{\lambda}.$$

Given any $n_k \to +\infty$,

$$0 \le \varliminf_{k\to\infty}\left|\int_a^b f(x)\sin n_k x\,dx\right| \le \varlimsup_{k\to\infty}\left|\int_a^b f(x)\sin n_k x\,dx\right| \le \varlimsup_{k\to\infty}\left(\epsilon + \sum_{j=1}^{n_\epsilon}|m_j|\cdot\frac{2}{k}\right) = \epsilon.$$

Since $\epsilon > 0$ is arbitrary, we have

$$0 = \varliminf_{k\to\infty}\left|\int_a^b f(x)\sin n_k x\,dx\right| = \varlimsup_{k\to\infty}\left|\int_a^b f(x)\sin n_k x\,dx\right| = \lim_{k\to\infty}\left|\int_a^b f(x)\sin n_k x\,dx\right|.$$

This proves the first equality. The proof for the second is similar. □

Proof of Theorem 9.32. Let $S_n(x)$ denote the n-th partial sum of the Fourier series of $f(x)$:

$$S_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n}(a_k\cos kx + b_k\sin kx).$$

Since

$$a_k\cos kx + b_k\sin kx = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)(\cos kt\cdot\cos kx + \sin kt\cdot\sin kx)\,dt = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kt - kx)\,dt,$$

we have

$$S_n(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\left[\frac{1}{2} + \sum_{k=1}^{n}\cos(k(t-x))\right]dt.$$

By performing the change of variable $u = t - x$ and using Lemma 9.26, we have

$$S_n(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+u)D_n(u)\,du,$$

where $D_n(u) = \frac{1}{2} + \sum_{k=1}^{n}\cos(ku)$ is called the **Dirichlet Kernel**. Note that

$$\begin{aligned} 2\sin(u/2)\cdot D_n(u) &= \sin(u/2) + \sum_{k=1}^{n} 2\sin(u/2)\cos(ku) \\ &= \sin(u/2) + \sum_{k=1}^{n}\big(\sin((k+\frac{1}{2})u) - \sin((k-\frac{1}{2})u)\big) = \sin((n+\frac{1}{2})u). \end{aligned}$$

This shows that $D_n(u) = \dfrac{\sin((n+\frac{1}{2})u)}{2\sin(u/2)}$. Also note that

- $D_n(u)$ is an even function; and
- $\frac{1}{\pi}\int_{-\pi}^{\pi} D_n(u)\,du = \frac{1}{\pi}\int_{-\pi}^{\pi}\Big(\frac{1}{2} + \sum_{k=1}^{n}\cos(ku)\Big)\,du = 1.$

We can therefore write the target value as:

$$\frac{f(x_0^+) + f(x_0^-)}{2} = \frac{1}{\pi}\int_0^{\pi} f(x_0^+)D_n(u)\,du + \frac{1}{\pi}\int_0^{\pi} f(x_0^-)D_n(u)\,du.$$

Subtracting this from $S_n(x_0)$, we obtain the error term:

$$\begin{aligned} S_n(x_0) - \frac{f(x_0^+) + f(x_0^-)}{2} &= \frac{1}{\pi}\int_0^{\pi} f(x_0+u)D_n(u)\,du + \frac{1}{\pi}\int_0^{\pi} f(x_0-u)D_n(u)\,du - \frac{f(x_0^+) + f(x_0^-)}{2} \\ &= \frac{1}{2\pi}\int_0^{\pi}\Big(\frac{f(x_0+u) + f(x_0-u) - f(x_0^+) - f(x_0^-)}{u}\,\frac{u}{\sin(u/2)}\Big)\sin((n+\frac{1}{2})u)\,du. \end{aligned}$$

Since $f \in PC^1$, the function

$$g(u) := \frac{f(x_0+u) + f(x_0-u) - f(x_0^+) - f(x_0^-)}{u}\,\frac{u}{\sin(u/2)}$$

is piecewise continuous on $[0, \pi]$. By Lemma 9.33, we therefore have

$$\lim_{n\to\infty}\Big(S_n(x_0) - \frac{f(x_0^+) + f(x_0^-)}{2}\Big) = \lim_{n\to\infty}\frac{1}{\pi}\int_0^{\pi} g(u)\sin\big((n+\tfrac{1}{2})u\big)\,du = 0.$$

This concludes the proof. □

Conclusion to the fundamental questions:

(1) The Fourier series of a PC^1 periodic function converges for any $x \in \mathbb{R}$.
(2) Suppose that a function f is a PC^1 periodic function. Its Fourier series converges to $\dfrac{f(x_0^+) + f(x_0^-)}{2}$. In particular, the sum equals $f(x_0)$ if and only if f is continuous at x_0.

	Taylor series	Fourier series
Basis functions	Powers of x (x^n)	Trigonometric ($\sin nx, \cos nx$)
Requirement	f must be C^∞	f can be discontinuous
Convergence	Within a radius R	Everywhere (if $f \in PC^1$)
Representation	Local (near a point x_0)	Global (over the entire period)
Condition for sum to equal $f(x)$	$\lim_{n\to\infty} \frac{1}{n!} \int_{x_0}^{x} f^{(n+1)}(t)(x-t)^n dt = 0$	f is continuous (and PC^1) at x

Example 9.34. *Let $f(x)$ be the periodic function with period 2π such that*

$$f(x) = x^2, \qquad 0 < x \le 2\pi.$$

Recall that $f(x) \sim \frac{4\pi^2}{3} + 4\sum_{k=1}^{\infty} \frac{\cos kx}{k^2} - 4\pi \sum_{k=1}^{\infty} \frac{\sin kx}{k}$*. At $x = 0$, by Theorem 9.32, we have*

$$\frac{f(0^+) + f(0^-)}{2} = \frac{4\pi^2}{3} + 4\sum_{k=1}^{\infty} \frac{\cos k0}{k^2} - 4\pi \sum_{k=1}^{\infty} \frac{\sin k0}{k} = \frac{4\pi^2}{3} + 4\sum_{k=1}^{\infty} \frac{1}{k^2}.$$

This shows that

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{1}{4}\Big(\frac{f(0^+) + f(0^-)}{2} - \frac{4\pi^2}{3}\Big) = \frac{1}{4}\Big(\frac{0 + 4\pi^2}{2} - \frac{4\pi^2}{3}\Big) = \frac{\pi^2}{6}.$$

9.3.3. *Linear algebra perspectives of Fourier series.* Let V be the set of continuous periodic functions with period 2π. It can be easily verified that V is a vector space over $\mathbb{R}$ with the operations: for $r \in \mathbb{R}$ and $f, g \in V$,

$$(r \cdot f)(x) = rf(x) \qquad \text{and} \qquad (f + g)(x) = f(x) + g(x).$$

Define

$$\langle -, - \rangle : V \times V \to \mathbb{R}, \qquad \langle f, g \rangle := \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)g(x)\, dx. \tag{7}$$

and $\| f \| := \sqrt{\langle f, f \rangle}$, called the **norm** of f.

Lemma 9.35. *The pair $(V, \langle -, - \rangle)$ is an inner product space; that is, for any $f, g, h \in V$ and any scalar $r \in \mathbb{R}$, the following properties hold:*

(1) Linearity: $\langle rf + g, h \rangle = r\langle f, h \rangle + \langle g, h \rangle$
(2) Symmetry: $\langle f, g \rangle = \langle g, f \rangle$
(3) Positive-definiteness: $\langle f, f \rangle \ge 0,$ *and* $\langle f, f \rangle = 0 \iff f(x) = 0$ *for all* $x \in \mathbb{R}$.

By the positive-definiteness, the norm $\| f \|$ is well-defines.

Remark 9.36. *The map defined in (7) can be extended to the space of piecewise continuous functions. However, it does not constitute a formal inner product on this larger space. Specifically, the positive-definiteness property fails because there exist non-zero functions f such that $\langle f, f \rangle = 0$. For example, $f(0) = 1$ and $f(x) = 0$ for $x \neq 1$.*

Definition 9.37. *A subset S of V is said to be **orthogonal** if $0 \notin S$ and any two distinct elements $f, g \in S$ satisfy $\langle f, g \rangle = 0$. A subset S is said to be **orthonormal** if it is orthogonal and every element $f \in S$ satisfies $\langle f, f \rangle = 1$ (i.e., each vector has unit norm).*

Note that any orthonormal set is orthogonal; furthermore, if S is an orthogonal set, one can easily construct an orthonormal set $S' = \left\{ \frac{f}{\sqrt{\langle f,f\rangle}} \mid f \in S \right\}$ such that $\operatorname{span} S = \operatorname{span} S'$.

Lemma 9.38. *An orthogonal set is linearly independent.*

Proof. Let $\{v_1, v_2, \ldots, v_n\}$ be a finite subset of an orthogonal set S. Suppose there exist scalars $c_1, c_2, \ldots, c_n \in \mathbb{R}$ such that:

$$c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = 0.$$

To show linear independence, we must prove that $c_i = 0$ for all $i \in \{1, \ldots, n\}$. For any fixed index j, we take the inner product of both sides with v_j:

$$c_j \left\langle v_j, v_j \right\rangle = \sum_{i=1}^{n} c_i \left\langle v_i, v_j \right\rangle = \left\langle \sum_{i=1}^{n} c_i v_i, v_j \right\rangle = \left\langle 0, v_j \right\rangle.$$

By the definition of an orthogonal set, $v_j \neq 0$, which implies $\left\langle v_j, v_j \right\rangle > 0$ by the positive-definiteness of the inner product. Therefore, we must have $c_j = 0$. Since this holds for every j, all coefficients are zero, and the set is linearly independent. □

By Lemma 9.27, the set

$$S = \{1\} \cup \{\cos mx \mid m \in \mathbb{N}\} \cup \{\sin nx \mid n \in \mathbb{N}\}$$

is an orthogonal set in V under the inner product $\langle f, g\rangle = \dfrac{1}{\pi} \displaystyle\int_{-\pi}^{\pi} f(x)g(x)\,dx$. Furthermore,

$$S' = \left\{ \frac{1}{\sqrt{2}} \right\} \cup \{\cos mx \mid m \in \mathbb{N}\} \cup \{\sin nx \mid n \in \mathbb{N}\}$$

is an orthonormal set. In particular, by Lemma 9.38, the set S forms a basis for its span, denoted by $W := \operatorname{span} S$. Note that W consists of all finite trigonometric series:

$$W = \left\{ \frac{a_0}{2} + \sum_{k=1}^{N} (a_k \cos kx + b_k \sin kx) \mid N \in \mathbb{N},\ a_k, b_k \in \mathbb{R} \right\}.$$

This subspace W is strictly smaller than V because elements of W are always smooth (C^∞), whereas V contains continuous functions that may not be differentiable. Nevertheless, by Theorem 9.32, any function in V can be represented as the limit of a sequence in W as $N \to \infty$.

Let W_N be the $(2N + 1)$-dimensional subspace of V defined by:

$$W_N = \operatorname{span}\{1, \cos x, \sin x, \ldots, \cos Nx, \sin Nx\} = \left\{ \frac{a_0}{2} + \sum_{k=1}^{N} (a_k \cos kx + b_k \sin kx) \mid a_k, b_k \in \mathbb{R} \right\}.$$

Our goal is to find the best approximation $S_N \in W_N$ of a function $f \in V$. By "best," we mean the element S_N that minimizes the distance to f; that is,

$$\left\| f - S_N \right\| \leq \left\| f - A_N \right\| \qquad \text{for all } A_N \in W_N.$$

The following theorem guarantees the existence and uniqueness of the best approximation within the subspace W_N:

Theorem 9.39 (Best Approximation Theorem)**.** *Let f be a piecewise continuous periodic function with period 2π, and let its Fourier series be given by:*

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx).$$

For any choice of coefficients $\alpha_0, \alpha_1, \ldots, \alpha_N$ and $\beta_1, \ldots, \beta_N \in \mathbb{R}$, let

$$A_N(x) = \frac{\alpha_0}{2} + \sum_{k=1}^{N} (\alpha_k \cos kx + \beta_k \sin kx) \in W_N.$$

Then the N-th partial sum $S_N(x) = \frac{a_0}{2} + \sum_{k=1}^{N} (a_k \cos kx + b_k \sin kx)$ satisfies:

$$\left\| f - S_N \right\| \leq \left\| f - A_N \right\|.$$

Furthermore, equality holds if and only if $\alpha_k = a_k$ for $k = 0, 1, \ldots, N$ and $\beta_k = b_k$ for $k = 1, 2, \ldots, N$.

Proof. Let $A_N \in W_N$ be an arbitrary finite trigonometric series with coefficients α_k, β_k. We evaluate the squared error norm:

$$\left\| f - A_N \right\|^2 = \langle f - A_N, f - A_N \rangle = \left\| f \right\|^2 - 2 \langle f, A_N \rangle + \left\| A_N \right\|^2.$$

Using the definition of A_N and the linearity of the inner product:

$$\langle f, A_N \rangle = \left\langle f, \frac{\alpha_0}{2} + \sum_{k=1}^{N} (\alpha_k \cos kx + \beta_k \sin kx) \right\rangle = \frac{\alpha_0}{2} \langle f, 1 \rangle + \sum_{k=1}^{N} \alpha_k \langle f, \cos kx \rangle + \sum_{k=1}^{N} \beta_k \langle f, \sin kx \rangle .$$

Recall that the Fourier coefficients satisfy $a_0 = \langle f, 1 \rangle$, $a_k = \langle f, \cos kx \rangle$, and $b_k = \langle f, \sin kx \rangle$. Thus:

$$\langle f, A_N \rangle = \frac{\alpha_0 a_0}{2} + \sum_{k=1}^{N} (\alpha_k a_k + \beta_k b_k).$$

Next, we calculate $\left\| A_N \right\|^2$ using Lemma 9.27:

$$\left\| A_N \right\|^2 = \frac{1}{\pi} \int_{-\pi}^{\pi} \left(\frac{\alpha_0}{2} + \sum_{k=1}^{N} (\alpha_k \cos kx + \beta_k \sin kx) \right)^2 dx = \frac{\alpha_0^2}{2} + \sum_{k=1}^{N} (\alpha_k^2 + \beta_k^2).$$

Substituting these into the expression for $\left\| f - A_N \right\|^2$:

$$\left\| f - A_N \right\|^2 = \left\| f \right\|^2 - \left(\alpha_0 a_0 + 2 \sum_{k=1}^{N} (\alpha_k a_k + \beta_k b_k) \right) + \left(\frac{\alpha_0^2}{2} + \sum_{k=1}^{N} (\alpha_k^2 + \beta_k^2) \right). \tag{8}$$

In particular, if $A_N = S_N$, we have

$$\left\| f - S_N \right\|^2 = \left\| f \right\|^2 - \left(\frac{a_0^2}{2} + \sum_{k=1}^{N} (a_k^2 + b_k^2) \right).$$

We now complete the square for each coefficient in Equation (8):

$$\begin{aligned}\|f - A_N\|^2 &= \|f\|^2 - \left(\frac{a_0^2}{2} + \sum_{k=1}^{N}(a_k^2 + b_k^2)\right) + \left(\frac{(\alpha_0 - a_0)^2}{2} + \sum_{k=1}^{N}[(\alpha_k - a_k)^2 + (\beta_k - b_k)^2]\right)\\ &\geq \|f\|^2 - \left(\frac{a_0^2}{2} + \sum_{k=1}^{N}(a_k^2 + b_k^2)\right) = \|f - S_N\|^2.\end{aligned}$$

The minimum is achieved if and only if the squared difference terms are zero, which implies $\alpha_k = a_k$ and $\beta_k = b_k$ for all k. □

In fact, the relationship between a function and its Fourier coefficients is captured by a nice identity that serves as an infinite-dimensional version of the Pythagorean theorem:

Theorem 9.40 (Parseval's Identity)**.** *Let f be a piecewise continuous periodic function with period 2π. If*

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx),$$

then

$$\|f\|^2 = \frac{1}{\pi}\int_{-\pi}^{\pi}(f(x))^2\,dx = \frac{a_0^2}{2} + \sum_{k=1}^{\infty}(a_k^2 + b_k^2).$$

The proof of Parseval's identity is omitted here, as it requires the theory of uniform convergence or L^2 completeness to rigorously justify the limit.

Corollary 9.41. *Let f and g be piecewise continuous periodic functions with period 2π. If*

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx) \quad \textit{and} \quad g(x) \sim \frac{\alpha_0}{2} + \sum_{k=1}^{\infty}(\alpha_k \cos kx + \beta_k \sin kx),$$

then

$$\langle f, g\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)g(x)\,dx = \frac{a_0\alpha_0}{2} + \sum_{k=1}^{\infty}(a_k\alpha_k + b_k\beta_k).$$

Proof. By an easy verification, we have

$$\langle f, g\rangle = \frac{1}{4}\left(\|f+g\|^2 - \|f-g\|^2\right).$$

Applying Parseval's Identity to both $\|f+g\|^2$ and $\|f-g\|^2$, we have

$$\begin{aligned}\langle f, g\rangle &= \frac{1}{4}\left[\left(\frac{(a_0+\alpha_0)^2}{2} + \sum(a_k+\alpha_k)^2 + (b_k+\beta_k)^2\right) - \left(\frac{(a_0-\alpha_0)^2}{2} + \sum(a_k-\alpha_k)^2 + (b_k-\beta_k)^2\right)\right]\\ &= \frac{1}{4}\left[\frac{4a_0\alpha_0}{2} + \sum(4a_k\alpha_k + 4b_k\beta_k)\right]\\ &= \frac{a_0\alpha_0}{2} + \sum_{k=1}^{\infty}(a_k\alpha_k + b_k\beta_k).\end{aligned}$$

This completes the proof. □

Example 9.42. *Let $f(x)$ be the periodic function with period 2π such that*

$$f(x) = x, \qquad -\pi < x \leq \pi.$$

Recall that $f(x) \sim \sum_{k=1}^{\infty} \frac{2(-1)^{k+1}}{k} \sin kx$*. By the Parseval's identity, we have*

$$\frac{2\pi^2}{3} = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \, dx = \sum_{k=1}^{\infty} \frac{4}{k^2}.$$

This provides an alternative way to show that $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$.

10. Special functions

We consider a few special functions in this section.

10.1. **Legendre polynomials.** Recall that when we studied Fourier series, we considered the inner product and norm on the space of continuous periodic functions. Here we consider a polynomial version of that story. Instead of periodic functions on $[-\pi, \pi]$, we focus on the space of continuous functions $V = C([-1, 1])$ and the subspace of polynomials $\mathbb{R}[x]$.

We define the inner product on V as:

$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)\,dx$$

The standard basis for the space $\mathbb{R}[x]$ of polynomials, $\{1, x, x^2, x^3, \dots\}$, is *not* orthogonal under this inner product. (One can see this by a direct computation: $\left\langle 1, x^2\right\rangle = \int_{-1}^{1} x^2\,dx = \frac{2}{3} \neq 0$.)

10.1.1. *Step-by-step Gram-Schmidt construction.* To obtain an orthogonal basis for the space of polynomials (analogous to the role of $\{1, \cos mx, \sin nx\}$ in Fourier series), we apply the Gram-Schmidt process to the standard basis $\{1, x, x^2, \dots\}$ in $\mathbb{R}[x]$. By convention, we impose the normalization condition $P_n(1) = 1$.

(1) **For $n = 0$:** Let $v_0 = 1$. Since $\left\| v_0 \right\|$ is constant and $v_0(1) = 1$, we have $P_0(x) = 1$.

(2) **For $n = 1$:** We subtract the projection of $v_1 = x$ onto P_0:

$$u_1 = x - \frac{\langle x, P_0\rangle}{\left\| P_0 \right\|^2} P_0 = x - \frac{\int_{-1}^{1} x \cdot 1\,dx}{\int_{-1}^{1} 1^2\,dx}(1) = x - 0 = x.$$

Since $u_1(1) = 1$, we have $P_1(x) = x$.

(3) **For $n = 2$:** We subtract the projections of $v_2 = x^2$ onto P_0 and P_1:

$$u_2 = x^2 - \frac{\left\langle x^2, P_0\right\rangle}{\left\| P_0 \right\|^2} P_0 - \frac{\left\langle x^2, P_1\right\rangle}{\left\| P_1 \right\|^2} P_1.$$

Evaluating the inner products: $\left\langle x^2, P_0\right\rangle = 2/3$, $\left\langle x^2, P_1\right\rangle = 0$, and $\left\| P_0 \right\|^2 = 2$.

$$u_2 = x^2 - \frac{2/3}{2}(1) - 0 = x^2 - \frac{1}{3}.$$

To satisfy $P_2(1) = 1$, we scale u_2 by $3/2$: $P_2(x) = \frac{1}{2}(3x^2 - 1)$.

(4) **For $n = 3$:** Subtracting projections of $v_3 = x^3$ onto the previous polynomials:

$$u_3 = x^3 - \frac{\left\langle x^3, P_1\right\rangle}{\left\| P_1 \right\|^2} P_1 = x^3 - \frac{2/5}{2/3} x = x^3 - \frac{3}{5} x.$$

Normalization yields: $P_3(x) = \frac{1}{2}(5x^3 - 3x)$.

By continuing this inductive process, one can obtain all Legendre polynomials $P_n(x)$.

10.1.2. *Rodrigues' formula.* A more direct way to generate these polynomials without recursion is through the **Rodrigues' formula**:

$$R_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$$

For $n = 2$, this yields: $R_2(x) = \frac{1}{8}\frac{d^2}{dx^2}(x^4 - 2x^2 + 1) = \frac{1}{2}(3x^2 - 1) = P_2(x)$.

Proposition 10.1. *For any $n \in \mathbb{N} \cup \{0\}$, $R_n(x) = P_n(x)$.*

Proof. By repeated integration by parts, we show R_n is orthogonal to any x^m for $m < n$:

$$\langle R_n, x^m\rangle = \frac{1}{2^n n!}\int_{-1}^{1}\frac{d^n}{dx^n}(x^2-1)^n x^m\,dx = \frac{(-1)^n}{2^n n!}\int_{-1}^{1}(x^2-1)^n\frac{d^n}{dx^n}(x^m)\,dx = 0. \tag{9}$$

Furthermore, using the Leibniz rule:

$$\frac{d^n}{dx^n}[(x-1)^n(x+1)^n]\Big|_{x=1} = \binom{n}{n}\left[\frac{d^n}{dx^n}(x-1)^n\right](x+1)^n\Big|_{x=1} = n!2^n.$$

Thus $R_n(1) = \frac{1}{2^n n!}(n!2^n) = 1$. Since the set of polynomials of degree n that are orthogonal to $\{1, x, \ldots, x^{n-1}\}$ forms a one-dimensional subspace of $\mathbb{R}_n[x]$, and both R_n and P_n reside in this subspace with the same normalization $P_n(1) = 1 = R_n(1)$, we conclude that $R_n(x) = P_n(x)$. □

10.1.3. *Orthogonality.* Equation (9) actually shows the orthogonality of Legendre polynomials:

Theorem 10.2 (Orthogonality of Legendre Polynomials)**.** *The Legendre polynomials satisfy:*

$$\langle P_m, P_n\rangle = \int_{-1}^{1}P_m(x)P_n(x)\,dx = \begin{cases}0 & m \neq n\\ \dfrac{2}{2n+1} & m = n\end{cases}$$

Proof. It remains to evaluate $\left\| P_n \right\|^2 = \langle P_n, P_n\rangle$. Using the same integration by parts argument as above:

$$\langle P_n, P_n\rangle = \frac{(-1)^n}{2^n n!}\int_{-1}^{1}(x^2-1)^n\frac{d^n}{dx^n}P_n(x)\,dx.$$

From the definition of $P_n(x)$, the leading term is $\dfrac{(2n)!}{2^n(n!)^2}x^n$. Therefore, its n-th derivative is the constant:

$$\frac{d^n}{dx^n}P_n(x) = \frac{(2n)!}{2^n n!}.$$

Substituting this back into the integral:

$$\left\| P_n \right\|^2 = \frac{(-1)^n(2n)!}{(2^n n!)^2}\int_{-1}^{1}(x^2-1)^n\,dx = \frac{(2n)!}{(2^n n!)^2}\int_{-1}^{1}(1-x^2)^n\,dx.$$

To evaluate $I_n = \displaystyle\int_{-1}^{1}(1-x^2)^n\,dx$, we use the substitution $x = \cos\theta$:

$$I_n = \int_0^{\pi}\sin^{2n+1}\theta\,d\theta = \frac{2^{2n+1}(n!)^2}{(2n+1)!}.$$

Combining these results:

$$\left\| P_n \right\|^2 = \frac{(2n)!}{2^{2n}(n!)^2}\cdot\frac{2^{2n+1}(n!)^2}{(2n+1)!} = \frac{2\cdot(2n)!}{(2n+1)!} = \frac{2}{2n+1}.$$

This completes the proof. □

Using an argument similar to the proof of Theorem 9.39, we obtain the following result:

Theorem 10.3 (Best Polynomial Approximation)**.** *Given $f \in C([-1,1])$, the N-degree polynomial S_N that minimizes the distance $\left\| f - S_N \right\|$ is the partial Legendre sum:*

$$S_N(x) = \sum_{n=0}^{N} c_n P_n(x), \quad \textit{where} \quad c_n = \frac{\langle f, P_n \rangle}{\left\| P_n \right\|^2} = \frac{2n+1}{2} \int_{-1}^{1} f(x) P_n(x)\, dx.$$

10.2. **Bernoulli polynomials.** As a motivating question, let us seek a formula for the sum of powers $1^k + 2^k + \cdots + n^k$. Suppose we can find a polynomial $B_k(t)$ of degree k such that

$$\int_x^{x+1} B_k(t)\, dt = \frac{x^k}{k!}.$$

Then the sum of the first n powers can be expressed as:

$$1^k + 2^k + \cdots + n^k = k! \int_0^{n+1} B_k(t)\, dt.$$

Definition 10.4 (Bernoulli polynomial)**.** *The n-th **Bernoulli polynomial** $B_n(x)$ is the polynomial of degree n such that*

$$\int_x^{x+1} B_n(t)\, dt = \frac{x^n}{n!}, \qquad \forall x \in \mathbb{R}. \tag{10}$$

To confirm that this relation indeed defines a polynomial, we must show that there exists exactly one polynomial of degree n satisfying Equation (10). We begin with the proof of uniqueness:

Lemma 10.5. *There is at most one polynomial of degree n satisfying Equation* (10).

Proof. Suppose there are two such polynomials, $P(x)$ and $Q(x)$, of degree n. Let $D(x) = P(x) - Q(x)$. Then $D(x)$ is a polynomial of degree at most n satisfying:

$$\int_x^{x+1} D(t)\, dt = 0, \qquad \forall x \in \mathbb{R}.$$

Differentiating both sides with respect to x using the Fundamental Theorem of Calculus, we obtain:

$$D(x+1) - D(x) = 0, \qquad \forall x \in \mathbb{R}.$$

This implies that $D(x)$ is a periodic polynomial with period 1. However, since any polynomial has finite zeros, the only periodic polynomial is a constant polynomial C. Since $\int_x^{x+1} C\, dt = C = 0$, it follows that $D(x) = 0$ for all x, so $P(x) = Q(x)$. □

The following lemma shows the existence of Bernoulli polynomials:

Lemma 10.6. *For every $n \in \mathbb{N} \cup \{0\}$, there exists a polynomial $B_n(x)$ of degree n satisfying Equation* (10).

Proof. We proceed by induction on n.

Base Case ($n = 0$): We require a constant $B_0(x) = c$ such that $\int_x^{x+1} c\,dt = \frac{x^0}{0!} = 1$. Evaluating the integral gives $[ct]_x^{x+1} = c(x+1) - cx = c$. Thus, $c = 1$, and $B_0(x) = 1$ satisfies the condition.

Inductive Step: Suppose $B_{n-1}(x)$ exists and is a polynomial of degree $n-1$. We seek $B_n(x)$ such that:

$$\int_x^{x+1} B_n(t)\,dt = \frac{x^n}{n!}. \tag{11}$$

Differentiating our goal in Equation (11), we require:

$$B_n(x+1) - B_n(x) = \frac{x^{n-1}}{(n-1)!}.$$

By the inductive hypothesis, we know that $\int_x^{x+1} B_{n-1}(t)\,dt = \frac{x^{n-1}}{(n-1)!}$. Therefore, we seek $B_n(x)$ such that:

$$B_n(x+1) - B_n(x) = \int_x^{x+1} B_{n-1}(t)\,dt.$$

This relation is satisfied if we define $B_n(x)$ as an antiderivative of $B_{n-1}(x)$, i.e., $B_n'(x) = B_{n-1}(x)$. Let:

$$B_n(x) = \int_0^x B_{n-1}(t)\,dt + C_n,$$

where C_n is a constant of integration. To satisfy the original integral condition at $x = 0$, we must have:

$$\int_0^1 B_n(t)\,dt = \frac{0^n}{n!} = 0 \quad (\text{for } n \geq 1).$$

Substituting our expression for $B_n(x)$:

$$\int_0^1 \left(\int_0^t B_{n-1}(s)\,ds + C_n \right) dt = 0 \implies C_n = -\int_0^1 \int_0^t B_{n-1}(s)\,ds\,dt.$$

Since B_{n-1} is a polynomial of degree $n-1$, its integral is a polynomial of degree n. Thus, $B_n(x)$ is a well-defined polynomial of degree n, completing the induction. □

The Bernoulli polynomials can be computed more elegantly using the following generating function:

$$G(s,t) := \sum_{n=0}^{\infty} B_n(s) \frac{t^n}{n!}. \tag{12}$$

Theorem 10.7. *For $|t| < 2\pi$, the generating function is given by:*

$$G(s,t) = \frac{te^{st}}{e^t - 1}.$$

The proof of this theorem requires deeper results from analysis. We skip the formal proof here.

Let $B_n := B_n(0)$, which are called the **Bernoulli numbers**. The above theorem shows that

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}.$$

To compute these numbers, we multiply by the Taylor series of $e^t - 1$:

$$t = \left(\sum_{j=1}^{\infty} \frac{t^j}{j!} \right) \left(\sum_{n=0}^{\infty} B_n \frac{t^n}{n!} \right).$$

Equating the coefficients of t^m, we obtain the recurrence relation $\sum_{k=0}^{n-1}\binom{n}{k}B_k = 0$ for $n > 1$, with $B_0 = 1$. The first few values are:

$$B_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_3 = 0, \quad B_4 = -\frac{1}{30}.$$

The relationship between Bernoulli polynomials $B_n(s)$ and Bernoulli numbers B_k is derived by expressing the generating function as a product of two power series:

$$G(s,t) = \frac{t}{e^t - 1}e^{st} = \left(\sum_{k=0}^{\infty} B_k \frac{t^k}{k!}\right)\left(\sum_{j=0}^{\infty} \frac{s^j t^j}{j!}\right) = \sum_{n=0}^{\infty} B_n(s)\frac{t^n}{n!}. \tag{13}$$

By applying the Cauchy product rule, we compare the coefficients of t^n on both sides:

$$\frac{B_n(s)}{n!} = \sum_{k=0}^{n} \frac{B_k}{k!}\frac{s^{n-k}}{(n-k)!}. \tag{14}$$

Multiplying by $n!$ yields the explicit binomial formula:

$$B_n(s) = \sum_{k=0}^{n}\binom{n}{k}B_k s^{n-k}. \tag{15}$$

This formula allows for the direct computation of $B_n(s)$ given the first n Bernoulli numbers. Using $B_0 = 1, B_1 = -1/2, B_2 = 1/6$, and $B_3 = 0$:

- $B_2(s) = \binom{2}{0}B_0 s^2 + \binom{2}{1}B_1 s + \binom{2}{2}B_2 = (1)s^2 + 2(-\frac{1}{2})s + \frac{1}{6} = s^2 - s + \dfrac{1}{6}$.
- $B_3(s) = \binom{3}{0}B_0 s^3 + \binom{3}{1}B_1 s^2 + \binom{3}{2}B_2 s + \binom{3}{3}B_3 = (1)s^3 + 3(-\frac{1}{2})s^2 + 3(\frac{1}{6})s + 0 = s^3 - \dfrac{3}{2}s^2 + \dfrac{1}{2}s$.

Back to our motivating question, we have the following result:

Proposition 10.8. *For $k \in \mathbb{N}$,*

$$1^k + 2^k + \cdots + n^k = \frac{B_{k+1}(n+1) - B_{k+1}}{k+1}.$$

Proof. Consider the difference between generating functions:

$$G(s+1,t) - G(s,t) = \frac{te^{(s+1)t}}{e^t - 1} - \frac{te^{st}}{e^t - 1} = te^{st}.$$

Expanding into power series:

$$\sum_{n=0}^{\infty}(B_n(s+1) - B_n(s))\frac{t^n}{n!} = \sum_{n=1}^{\infty} ns^{n-1}\frac{t^n}{n!}.$$

Comparing coefficients of $t^{k+1}/(k+1)!$, we obtain the identity:

$$B_{k+1}(s+1) - B_{k+1}(s) = (k+1)s^k.$$

Summing from $s = 1$ to n:

$$B_{k+1}(n+1) - B_{k+1} = \sum_{s=1}^{n}(B_{k+1}(s+1) - B_{k+1}(s)) = (k+1)\sum_{s=1}^{n} s^k,$$

which completes the proof. □

For the sum of squares $1^2 + 2^2 + \cdots + n^2$, the formula gives:

$$\sum_{i=1}^{n} i^2 = \frac{B_3(n+1) - B_3}{3}. \tag{16}$$

Recalling that $B_3 = 0$ and $B_3(s) = s^3 - \frac{3}{2}s^2 + \frac{1}{2}s$, we substitute $s = n+1$:

$$\begin{aligned}
\sum_{i=1}^{n} i^2 &= \frac{(n+1)^3 - \frac{3}{2}(n+1)^2 + \frac{1}{2}(n+1) - 0}{3} = \frac{n+1}{3}\left((n+1)^2 - \frac{3}{2}(n+1) + \frac{1}{2}\right) \\
&= \frac{n+1}{3}\left(n^2 + 2n + 1 - \frac{3}{2}n - \frac{3}{2} + \frac{1}{2}\right) = \frac{n+1}{3}\left(n^2 + \frac{1}{2}n\right) = \frac{n(n+1)(n+1/2)}{3} \\
&= \frac{n(n+1)(2n+1)}{6}.
\end{aligned}$$

10.3. **Gamma function.** The Gamma function $\Gamma(x)$ is a differentiable function $\Gamma : (0, \infty) \to \mathbb{R}$ that satisfies the property $\Gamma(n+1) = n!$ for all $n \in \mathbb{N}$.

Definition 10.9. *For $x > 0$, the* **Gamma function** $\Gamma(x)$ *is defined by the improper integral:*

$$\Gamma(x) := \int_0^\infty e^{-t} t^{x-1}\, dt \tag{17}$$

Lemma 10.10. *The improper integral (17) converges.*

Proof. To analyze the convergence of the integral, we split it into two parts:

$$\Gamma(x) = \underbrace{\int_0^1 e^{-t} t^{x-1}\, dt}_{I_1} + \underbrace{\int_1^\infty e^{-t} t^{x-1}\, dt}_{I_2}.$$

Convergence of I_1: For $t \in (0, 1]$, we observe that $0 < e^{-t} \le 1$. Thus,

$$0 < e^{-t} t^{x-1} \le t^{x-1}.$$

The integral $\int_0^1 t^{x-1}\, dt$ converges if and only if $x - 1 > -1$, which simplifies to $x > 0$. This shows that I_1 converges for all $x > 0$.

Convergence of I_2: Since $\lim_{t\to\infty} e^{-t/2} t^{x-1} = 0$, for any fixed x, there exists a constant $M = M_x$ such that $t^{x-1} \le M e^{t/2}$ for sufficiently large t. Thus, for sufficiently large t,

$$e^{-t} t^{x-1} \le M e^{-t} e^{t/2} = M e^{-t/2}.$$

Since the integral $\int_1^\infty M e^{-t/2}\, dt$ is convergent, I_2 converges for all $x \in \mathbb{R}$. □

Theorem 10.11. *For $x > 0$, the Gamma function satisfies the functional equation:*

$$\Gamma(x+1) = x\Gamma(x).$$

Proof. Applying the integration by parts, we have

$$\Gamma(x+1) = \int_0^\infty e^{-t} t^x\, dt = \lim_{b\to\infty}\left(-e^{-t} t^x \Big|_0^b + x \int_0^b e^{-t} t^{x-1}\, dt\right) = x\Gamma(x),$$

which completes the proof. □

Corollary 10.12. *For $n \in \mathbb{N} \cup \{0\}$,*

$$\Gamma(n+1) = n!.$$

Proof. A direction computation shows that $\Gamma(1) = 1$. Thus, the corollary follows from an induction argument. □

The Beta function is another special function that is closely related to the Gamma function.

Definition 10.13. *For $x, y > 0$, the* **Beta function** *$B(x, y)$ is defined by the improper integral:*

$$B(x,y) := \int_0^1 t^{x-1}(1-t)^{y-1}\,dt. \tag{18}$$

Lemma 10.14. *The integral (18) converges.*

Proof. To prove convergence, we split the integral at $t = \frac{1}{2}$:

$$B(x,y) = \int_0^{1/2} t^{x-1}(1-t)^{y-1}\,dt + \int_{1/2}^1 t^{x-1}(1-t)^{y-1}\,dt. \tag{19}$$

In the first integral, as $t \to 0$, we have $(1-t)^{y-1} \to 1$. The integrand behaves like t^{x-1}. The integral $\int_0^{1/2} t^{x-1}\,dt$ converges if $x > 0$.

In the second integral, as $t \to 1$, we have $t^{x-1} \to 1$. Using the substitution $u = 1 - t$, the integrand behaves like u^{y-1} as $u \to 0$. This integral converges if $y > 0$.

Thus, $B(x, y)$ converges for all $x, y \in \mathbb{R}$ such that $x > 0$ and $y > 0$. □

Remark 10.15. *The trigonometric form of the Beta function,*

$$B(x,y) = 2\int_0^{\pi/2} (\sin\theta)^{2x-1}(\cos\theta)^{2y-1}\,d\theta, \tag{20}$$

is obtained by applying the substitution $t = \sin^2\theta$ to the standard definition $B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt$.

Under this substitution:

- *The differential becomes $dt = 2\sin\theta\cos\theta\,d\theta$.*
- *The limits of integration change: when $t = 0$, $\theta = 0$; when $t = 1$, $\theta = \frac{\pi}{2}$.*
- *The term t^{x-1} becomes $(\sin^2\theta)^{x-1} = (\sin\theta)^{2x-2}$.*
- *The term $(1-t)^{y-1}$ becomes $(\cos^2\theta)^{y-1} = (\cos\theta)^{2y-2}$.*

Combining these yields:

$$\begin{aligned} B(x,y) &= \int_0^{\pi/2} (\sin\theta)^{2x-2}(\cos\theta)^{2y-2} \cdot (2\sin\theta\cos\theta)\,d\theta \\ &= 2\int_0^{\pi/2} (\sin\theta)^{2x-1}(\cos\theta)^{2y-1}\,d\theta. \end{aligned}$$

This form is especially useful for evaluating definite integrals of products of sines and cosines over the interval $[0, \pi/2]$.

Theorem 10.16. *The Beta function is related to the Gamma function by the identity:*

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

Proof. Skip. Consider the product of two Gamma functions expressed as integrals:

$$\Gamma(x)\Gamma(y) = \left(\int_0^\infty e^{-u}u^{x-1}\,du\right)\left(\int_0^\infty e^{-v}v^{y-1}\,dv\right). \tag{21}$$

We use the substitution $u = a^2$ and $v = b^2$, which implies $du = 2a\,da$ and $dv = 2b\,db$:

$$\Gamma(x)\Gamma(y) = 4\int_0^\infty \int_0^\infty e^{-(a^2+b^2)}a^{2x-1}b^{2y-1}\,da\,db. \tag{22}$$

Transforming to polar coordinates where $a = r\cos\theta$ and $b = r\sin\theta$, the Jacobian is r:

$$\begin{aligned}\Gamma(x)\Gamma(y) &= 4\int_0^{\pi/2}\int_0^\infty e^{-r^2}(r\cos\theta)^{2x-1}(r\sin\theta)^{2y-1}r\,dr\,d\theta\\ &= \left(2\int_0^\infty e^{-r^2}r^{2(x+y)-1}\,dr\right)\left(2\int_0^{\pi/2}(\cos\theta)^{2x-1}(\sin\theta)^{2y-1}\,d\theta\right).\end{aligned}$$

By setting $t = r^2$ in the first integral, we identify it as $\Gamma(x+y)$. The second integral is the trigonometric representation of $B(x,y)$:

$$\Gamma(x)\Gamma(y) = \Gamma(x+y)\cdot B(x,y). \tag{23}$$

Solving for $B(x,y)$ yields the result:

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \tag{24}$$

□

Example 10.17. *The value* $\Gamma\left(\frac{1}{2}\right) = \int_0^\infty e^{-t}t^{-1/2}\,dt$ *can be computed using the Beta function properties. Since*

$$B\left(\frac{1}{2},\frac{1}{2}\right) = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}\right)} = \frac{\left[\Gamma\left(\frac{1}{2}\right)\right]^2}{\Gamma(1)} = \left[\Gamma\left(\frac{1}{2}\right)\right]^2,$$

we can evaluate the value of $B\left(\frac{1}{2},\frac{1}{2}\right)$ *independently using the trigonometric form:*

$$B\left(\frac{1}{2},\frac{1}{2}\right) = 2\int_0^{\pi/2}(\sin\theta)^{2(1/2)-1}(\cos\theta)^{2(1/2)-1}\,d\theta = \pi.$$

Thus, we have $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

Example 10.18. *The integral* $I = \int_{-\infty}^\infty e^{-x^2}\,dx$ *can be computed by the Gamma function. Since the integrand is an even function, we have* $I = 2\int_0^\infty e^{-x^2}\,dx$. *Applying the substitution* $t = x^2$, *which gives* $dx = \frac{1}{2}t^{-1/2}\,dt$, *the integral becomes:*

$$I = 2\int_0^\infty e^{-t}\left(\frac{1}{2}t^{-1/2}\right)dt = \int_0^\infty e^{-t}t^{1/2-1}\,dt = \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

Example 10.19. *The integral* $I = \int_0^{2\pi} \sin^{100} x\, dx$ *can be computed by the Beta function. By the symmetry of* $\sin^{100} x$*, we have:*

$$I = 4\int_0^{\pi/2} \sin^{100} x\, dx.$$

Applying the trigonometric form $B(x, y) = 2\int_0^{\pi/2} \sin^{2x-1}\theta\cos^{2y-1}\theta\, d\theta$ *with* $x = \frac{101}{2}$ *and* $y = \frac{1}{2}$*:*

$$I = 2B\left(\frac{101}{2}, \frac{1}{2}\right) = 2\frac{\Gamma\left(50 + \frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(51)}.$$

Since

$$\Gamma\left(50 + \frac{1}{2}\right) = \frac{1}{2}\cdot\frac{3}{2}\cdot\cdots\cdot\frac{99}{2}\cdot\Gamma\left(\frac{1}{2}\right) = \frac{100!}{2^{50}\cdot 2\cdot 4\cdots 100}\cdot\sqrt{\pi} = \frac{100!}{2^{100}50!}\sqrt{\pi},$$

we have

$$I = 2\frac{\left(\frac{100!}{2^{100}50!}\sqrt{\pi}\right)\sqrt{\pi}}{50!} = \frac{100!}{2^{99}(50!)^2}\pi.$$

Example 10.20. *The integral* $I = \int_2^5 (5-t)^{50}(t-2)^{50}\, dt$ *can be computed by the Beta function. We use the substitution* $u = \frac{t-2}{3}$*, which implies* $dt = 3\, du$ *and transforms the interval* $[2, 5]$ *to* $[0, 1]$*. Substituting* $t - 2 = 3u$ *and* $5 - t = 3(1-u)$*, we obtain:*

$$I = \int_0^1 [3(1-u)]^{50}(3u)^{50}\cdot 3\, du = 3^{101}\int_0^1 u^{50}(1-u)^{50}\, du$$

The integral is the definition of $B(51, 51)$*. Thus:*

$$I = 3^{101}B(51, 51) = 3^{101}\frac{\Gamma(51)\Gamma(51)}{\Gamma(102)} = 3^{101}\frac{(50!)^2}{101!}$$

More generally, for the integral $J = \int_a^b (b-t)^{x-1}(t-a)^{y-1}\, dt$*, we use the substitution* $u = \frac{t-a}{b-a}$*:*

- $t - a = (b-a)u$
- $b - t = (b-a)(1-u)$
- $dt = (b-a)\, du$

Substituting these into J:

$$J = \int_0^1 [(b-a)(1-u)]^{x-1}[(b-a)u]^{y-1}(b-a)\, du = (b-a)^{x+y-1}\int_0^1 u^{y-1}(1-u)^{x-1}\, du.$$

Using the definition of the Beta function:

$$\int_a^b (b-t)^{x-1}(t-a)^{y-1}\, dt = (b-a)^{x+y-1}B(x, y)$$

11. Vector calculus

We have considered calculus of real-valued functions with one variable. From now on, we will consider more general functions $f : \mathbb{R}^n \to \mathbb{R}^m$. In particular, we will consider the following types of functions:

- $f : \mathbb{R} \to \mathbb{R}^3$ — curves in $\mathbb{R}^3$ or vector-valued functions,
- $f : \mathbb{R}^3 \to \mathbb{R}$ — multivariable real-valued functions.

We start with a study of $\mathbb{R}^3$.

11.1. Vectors in a three-dimensional space.

A point in $\mathbb{R}^3$ is usually represented by a triple (a_1, a_2, a_3) of real numbers. Such a triple also can be considered as the vector from $(0, 0, 0)$ to (a_1, a_2, a_3). These vectors are equipped with two important operations: *addition* and *scalar product*.

Definition 11.1 ([3, Section 13.2])**.** *A **vector** $\vec{a}$ in $\mathbb{R}^3$ is an ordered triple of real numbers:*

$$\vec{a} = (a_1, a_2, a_3) \in \mathbb{R}^3.$$

Let $\vec{a} = (a_1, a_2, a_3)$, $\vec{b} = (b_1, b_2, b_3)$ and $\alpha \in \mathbb{R}$.

(1) ***Equal**:*

$$\vec{a} = \vec{b} \qquad \textit{iff} \qquad a_1 = b_1, a_2 = b_2, a_3 = b_3.$$

(2) ***Addition**:*

$$\vec{a} + \vec{b} = (a_1 + b_1, a_2 + b_2, a_3 + b_3).$$

(3) ***Scalar product**:*

$$\alpha \cdot \vec{a} = (\alpha a_1, \alpha a_2, \alpha a_3).$$

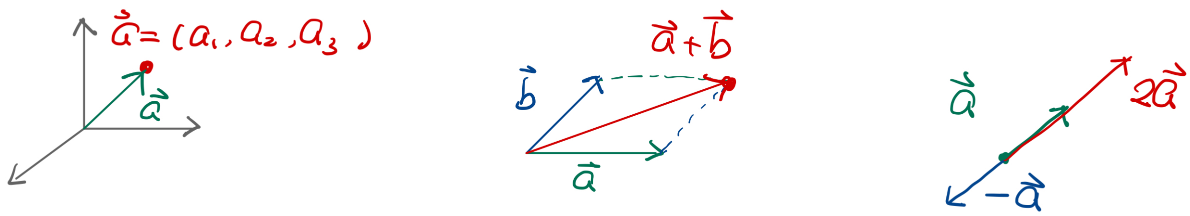


Figure 52. Operations of vectors.

We often use the following notations:

$$-\vec{a} = (-1) \cdot \vec{a},$$
$$\vec{a} - \vec{b} = \vec{a} + (-1)\vec{b},$$
$$\vec{0} = (0, 0, 0).$$

It is straightforward to show the following

Proposition 11.2. *Let $\vec{a} = (a_1, a_2, a_3)$, $\vec{b} = (b_1, b_2, b_3)$, $\vec{c} = (c_1, c_2, c_3)$ and $\alpha, \beta \in \mathbb{R}$.*

(1) $\vec{a} + \vec{b} = \vec{b} + \vec{a}$.
(2) $(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$.

(3) $\vec{a} + \vec{0} = \vec{a}$.
(4) $\vec{a} + (-\vec{a}) = \vec{0}$.
(5) $1 \cdot \vec{a} = \vec{a}$.
(6) $(\alpha\beta)\vec{a} = \alpha(\beta\vec{a})$.
(7) $\alpha(\vec{a} + \vec{b}) = \alpha\vec{a} + \alpha\vec{b}$.
(8) $(\alpha + \beta)\vec{a} = \alpha\vec{a} + \beta\vec{a}$.

Example 11.3. *Let* $\vec{a} = (1, -1, 2)$, $\vec{b} = (2, 3, -1)$, $\vec{c} = (0, 1, 0)$.

(1) $\vec{a} - \vec{b} = (1 - 2, -1 - 3, 2 + 1) = (-1, -4, 3)$.
(2) $2\vec{a} + 3\vec{b} - \vec{c} = (2 + 6 - 0, -2 + 9 - 1, 4 - 3 - 0) = (8, 6, 1)$.

Definition 11.4. *The* ***norm*** *(or* ***magnitude, length****) of the vector* $\vec{a} = (a_1, a_2, a_3)$ *is the number*

$$\|\vec{a}\| = \sqrt{a_1^2 + a_2^2 + a_3^2}.$$

Proposition 11.5. *Let* $\vec{a}, \vec{b} \in \mathbb{R}^3$ *and* $\alpha \in \mathbb{R}$.

(1) $\|\vec{a}\| \geq 0$, *and* $\|\vec{a}\| = 0$ *iff* $\vec{a} = \vec{0}$.
(2) $\|\alpha \cdot \vec{a}\| = |\alpha| \cdot \|\vec{a}\|$.
(3) $\|\vec{a} + \vec{b}\| \leq \|\vec{a}\| + \|\vec{b}\|$.

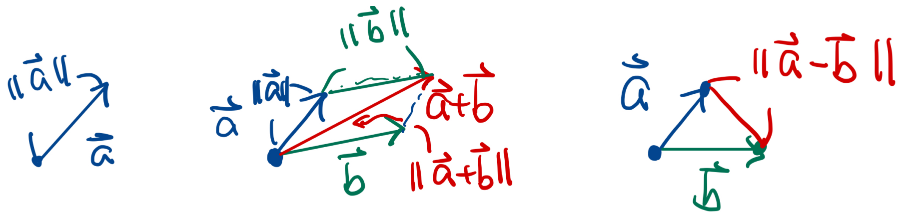


FIGURE 53. Norm of vector.

Example 11.6. *Let* $\vec{a} = (1, -1, 2)$, $\vec{b} = (2, 3, -1)$.

(1) $\|\vec{a}\| = \sqrt{6}$.
(2) $\|\vec{b}\| = \sqrt{14}$.
(3) $\|\vec{a} + \vec{b}\| = \sqrt{3^2 + 2^2 + 1^2} = \sqrt{14} \leq \|\vec{a}\| + \|\vec{b}\|$.
(4) $\|\vec{a} - \vec{b}\| = \sqrt{(1 - 2)^2 + (-1 - 3)^2 + (2 - (-1))^2} = \sqrt{26}$.

Definition 11.7 ([3, Definition 13.3.1])**.** *For vectors* $\vec{a} = (a_1, a_2, a_3), \vec{b} = (b_1, b_2, b_3) \in \mathbb{R}^3$, *we define the* ***dot product*** $\vec{a} \cdot \vec{b}$ *by setting*

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3.$$

The geometric meaning of the dot product is the following:

Theorem 11.8. *Let* θ *be the angle from* $\vec{a}$ *to* $\vec{b}$. *Then*

$$\vec{a} \cdot \vec{b} = \|\vec{a}\|\|\vec{b}\| \cos\theta.$$

Proof. By the law of cosines,

$$\begin{aligned} 2\|\vec{a}\|\|\vec{b}\|\cos\theta &= \|\vec{a}\|^2 + \|\vec{b}\|^2 - \|\vec{a}-\vec{b}\|^2 \\ &= (a_1^2+a_2^2+a_3^2) + (b_1^2+b_2^2+b_3^2) - ((a_1-b_1)^2 + (a_2-b_2)^2 + (a_3-b_3)^2) \\ &= 2(\vec{a}\cdot\vec{b}). \end{aligned}$$

Thus, the theorem follows. □

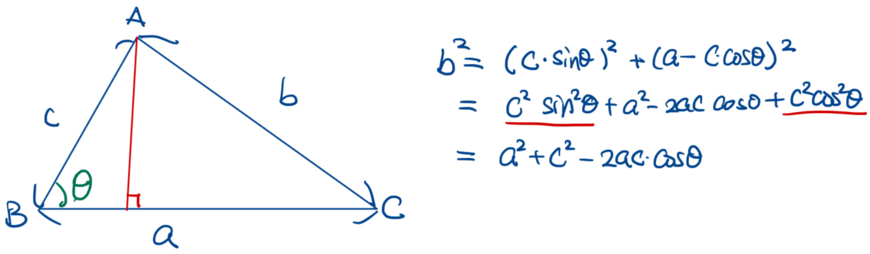


FIGURE 54. Law of cosines.

In particular,

$$\vec{a} \perp \vec{b} \qquad \text{iff} \qquad \vec{a}\cdot\vec{b} = 0.$$

Example 11.9. *The vectors* $(2, 1, 1)$ *and* $(1, 1-3)$ *are perpendicular because*

$$(2, 1, 1)\cdot(1, 1-3) = 0.$$

The angle between $(1, 1, 0)$ *and* $(0, 1, 1)$ *is*

$$\arccos\Big(\frac{(1,1,0)\cdot(0,1,1)}{\|(1,1,0)\|\|(0,1,1)\|}\Big) = \arccos\Big(\frac{1}{2}\Big) = \frac{\pi}{3}.$$

The dot product satisfies the following properties:

Proposition 11.10 ([3, Section 13.3])**.** *Let* $\vec{a}, \vec{b}, \vec{c} \in \mathbb{R}^3$ *and* $\alpha, \beta \in \mathbb{R}$.

(1) $\vec{a}\cdot\vec{a} = \|\vec{a}\|^2$.
(2) $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$.
(3) $\vec{a}\cdot(\alpha\vec{b}+\beta\vec{c}) = \alpha(\vec{a}\cdot\vec{b}) + \beta(\vec{a}\cdot\vec{c})$.

Example 11.11. *Given that*

$$\|\vec{a}\| = 1, \qquad \|\vec{b}\| = 3, \qquad \|\vec{c}\| = 4, \qquad \vec{a}\cdot\vec{b} = 0, \qquad \vec{a}\cdot\vec{c} = 1, \qquad \vec{b}\cdot\vec{c} = -2,$$

find

(1) $3\vec{a}\cdot(\vec{b}+4\vec{c})$. *ans:* 12
(2) $(\vec{a}-\vec{b})\cdot(2\vec{a}+\vec{b})$. *ans:* -7
(3) $((\vec{b}\cdot\vec{c})\vec{a} - (\vec{a}\cdot\vec{c})\vec{b})\cdot\vec{c}$. *ans:* 0

Remark 11.12. *All the definitions and properties about vectors can be established for vectors in* $\mathbb{R}^2$*, and even in* $\mathbb{R}^n$ *for any* n.

Remark 11.13. *In* [3]*, the authors use the "i, j, k" notations for vectors in* $\mathbb{R}^3$*:*

$$\vec{\imath} = (1,0,0), \qquad \vec{\jmath} = (0,1,0), \qquad \vec{k} = (0,0,1),$$

and

$$\vec{a} = (a_1, a_2, a_3) = a_1\,\vec{\imath} + a_2\,\vec{\jmath} + a_3\,\vec{k}.$$

11.2. **Limit and vector derivative.** Now we consider functions of the type $\vec{f} : \mathbb{R} \to \mathbb{R}^3$. Such functions can be expressed as

$$\vec{f}(t) = (f_1(t), f_2(t), f_3(t)) = f_1(t)\,\vec{\imath} + f_2(t)\,\vec{\jmath} + f_3(t)\,\vec{k}$$

and can be considered as curves in $\mathbb{R}^3$.

Example 11.14. *Following are examples of functions of the type* $\vec{f} : \mathbb{R} \to \mathbb{R}^3$*:*

(1) $\vec{f}(t) = \cos t\,\vec{\imath} + \sin t\,\vec{\jmath}$.
(2) $\vec{g}(t) = \cos(2\pi t)\,\vec{\imath} + \sin(2\pi t)\,\vec{\jmath} + t\,\vec{k}$.

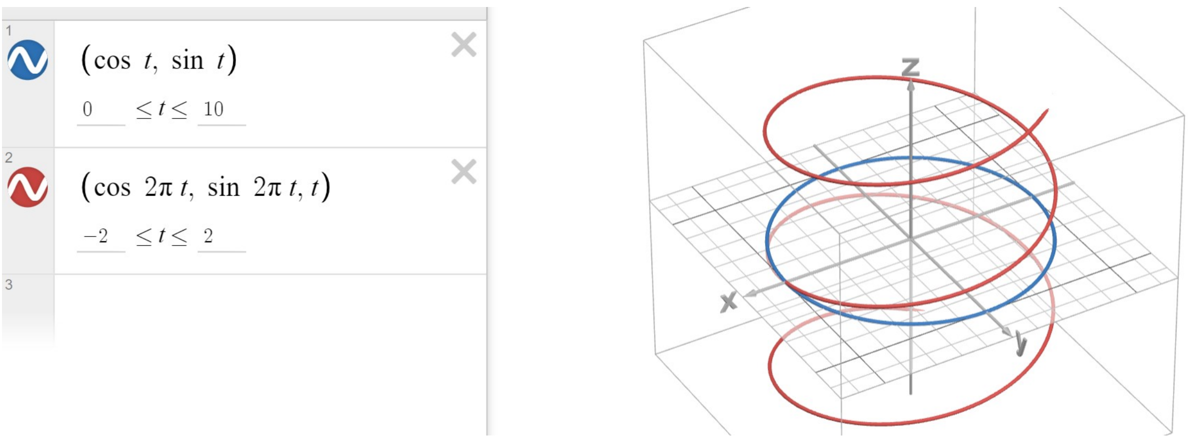


FIGURE 55. Curves in $\mathbb{R}^3$.

Definition 11.15 ([3, Definition 14.1.1])**.** *Let $\vec{f}(t)$ be a function valued in $\mathbb{R}^3$, and $t_0 \in \mathbb{R}$. We say that the* **limit** $\lim_{t\to t_0} \vec{f}(t) = \vec{L}$ *exists if there exists $\vec{L} \in \mathbb{R}^3$ such that*

$$\lim_{t\to t_0} \left\| \vec{f}(t) - \vec{L} \right\| = 0.$$

Theorem 11.16 ([3, (14.1.4)])**.** *Let $\vec{L} = (L_1, L_2, L_3) \in \mathbb{R}^3$ and $\vec{f}(t) = (f_1(t), f_2(t), f_3(t))$ be a vector-valued function. Then*

$$\lim_{t\to t_0} \vec{f}(t) = \vec{L}$$

if and only if

$$\lim_{t\to t_0} f_1(t) = L_1, \qquad \lim_{t\to t_0} f_2(t) = L_2, \qquad \lim_{t\to t_0} f_3(t) = L_3.$$

Proof. By the definition,

$$\lim_{t\to t_0} \vec{f}(t) = \vec{L} \qquad \text{iff} \qquad \lim_{t\to t_0} \sqrt{(f_1(t)-L_1)^2 + (f_2(t)-L_2)^2 + (f_3(t)-L_3)^2} = 0.$$

Since

$$|f_i(t) - L_i| \leq \sqrt{(f_1(t)-L_1)^2 + (f_2(t)-L_2)^2 + (f_3(t)-L_3)^2},$$

it follows from the pinching theorem that $\lim_{t\to t_0}|f_1(t)-L_1| = \lim_{t\to t_0}|f_2(t)-L_2| = \lim_{t\to t_0}|f_3(t)-L_3| = 0$. Thus, we have $\lim_{t\to t_0} f_1(t) = L_1, \lim_{t\to t_0} f_2(t) = L_2, \lim_{t\to t_0} f_3(t) = L_3$. The converse follows from the basic properties of limit. □

Example 11.17. *Let*

$$\vec{f}(t) = \cos(t+\pi)\,\vec{\imath} + \sin(t+\pi)\,\vec{\jmath} + e^{-t^2}\,\vec{k}.$$

Then

$$\begin{aligned}\lim_{t\to 0}\vec{f}(t) &= \Big(\lim_{t\to 0}\cos(t+\pi)\Big)\vec{\imath} + \Big(\lim_{t\to 0}\sin(t+\pi)\Big)\vec{\jmath} + \Big(\lim_{t\to 0} e^{-t^2}\Big)\vec{k}\\ &= -\vec{\imath} + \vec{k}.\end{aligned}$$

The following theorem can be verified by Theorem 11.16.

Theorem 11.18 ([3, Theorem 14.1.3])**.** *Let $\vec{f}$ and $\vec{g}$ be vector-valued functions and u a real-valued function. Suppose that*

$$\lim_{t\to t_0}\vec{f}(t) = \vec{L}, \qquad \lim_{t\to t_0}\vec{g}(t) = \vec{M}, \qquad \lim_{t\to t_0} u(t) = A.$$

Then

$$\lim_{t\to t_0}(\alpha\vec{f}(t) + \beta\vec{g}(t)) = \alpha\vec{L} + \beta\vec{M},$$

$$\lim_{t\to t_0}(u(t)\vec{f}(t)) = A\vec{L}, \qquad \lim_{t\to t_0}\big\|\vec{f}(t)\big\| = \big\|\vec{L}\big\|, \qquad \lim_{t\to t_0}\vec{f}(t)\cdot\vec{g}(t) = \vec{L}\cdot\vec{M}.$$

Definition 11.19. *A vector-valued function $\vec{f}$ is **continuous** at c if $\lim_{t\to c}\vec{f}(t) = \vec{f}(c)$.*

Remark 11.20. *By Theorem 11.16, a vector-valued function $\vec{f}(t) = (f_1(t), f_2(t), f_3(t))$ is continuous at $t = c$ iff f_1, f_2 and f_3 are continuous at $t = c$.*

Definition 11.21 ([3, Definition 14.1.5])**.** *A vector-valued function $\vec{f}$ is said to be **differentiable** at t if*

$$\lim_{h\to 0}\frac{\vec{f}(t+h) - \vec{f}(t)}{h} \qquad \text{exists.}$$

*If this limit exists, it is called the **derivative** of $\vec{f}$ at t and is denoted by $\vec{f}'(t)$ or $\dfrac{d\vec{f}}{dt}$.*

Theorem 11.22 ([3, Page 697])**.** *Let $\vec{f}(t) = (f_1(t), f_2(t), f_3(t))$ be a vector-valued function. Then $\vec{f}$ is differentiable at t iff all f_1, f_2 and f_3 are differentiable at t. In this case,*

$$\vec{f}'(t) = (f_1'(t), f_2'(t), f_3'(t)).$$

Proof. It can be verified directly by the definition:

$$\begin{aligned}\vec{f}'(t) &= \lim_{h\to 0}\frac{\vec{f}(t+h) - \vec{f}(t)}{h}\\ &= \Big(\lim_{h\to 0}\frac{f_1(t+h) - f_1(t)}{h},\ \lim_{h\to 0}\frac{f_2(t+h) - f_2(t)}{h},\ \lim_{h\to 0}\frac{f_3(t+h) - f_3(t)}{h}\Big)\\ &= (f_1'(t), f_2'(t), f_3'(t)),\end{aligned}$$

where the second equality follows from Theorem 11.16. □

Example 11.23. *If $\vec{f}(t) = (1, 2, 3)$ is constant, then $\vec{f}'(t) = \vec{0}$ for all t. If $\vec{g}(t) = t\,\vec{\imath} + t^2\,\vec{\jmath} - e^t\,\vec{k}$, then $\vec{g}'(t) = \vec{\imath} + 2t\,\vec{\jmath} - e^t\,\vec{k}$.*

By Remark 11.20 and Theorem 11.22, we have

Corollary 11.24. *Let $\vec{f}(t) = (f_1(t), f_2(t), f_3(t))$ be a vector-valued function. If $\vec{f}$ is differentiable at $t = c$, then $\vec{f}$ is continuous at $t = c$.*

By Theorem 11.22, one can easily verify the following theorem:

Theorem 11.25 ([3, Section 14.2])**.** *Let $\vec{f}, \vec{g}$ be differentiable vector-valued functions, u a differentiable real-valued function, and $\alpha, \beta \in \mathbb{R}$.*

(1) $(\alpha\vec{f} + \beta\vec{g})'(t) = \alpha\vec{f}'(t) + \beta\vec{g}'(t)$.
(2) $(u\vec{f})'(t) = u'(t)\vec{f}(t) + u(t)\vec{f}'(t)$.
(3) $(\vec{f} \cdot \vec{g})'(t) = \vec{f}'(t) \cdot \vec{g}(t) + \vec{f}(t) \cdot \vec{g}'(t)$.
(4) $(\vec{f} \circ u)'(t) = $ "$\vec{f}'(u(t))u'(t)$" $= u'(t)\vec{f}'(u(t))$. *(chain rule)*

11.3. **Geometry of curves.** A differentiable vector-valued function

$$\vec{f}(t) = f_1(t)\,\vec{\imath} + f_2(t)\,\vec{\jmath} + f_3(t)\,\vec{k}, \qquad t \in (a, b) \subset \mathbb{R},$$

traces out a directed curved path in $\mathbb{R}^3$. It is called a **differentiable parametrized curve** in $\mathbb{R}^3$. We consider $\vec{f}(t)$ as a parametrization of the curve $\mathrm{im}(\vec{f}) \subset \mathbb{R}^3$. For instance,

$$\vec{f}(t) = \cos t\,\vec{\imath} + \sin t\,\vec{\jmath}, \qquad t \in \mathbb{R},$$
$$\vec{g}(t) = \cos(2\pi t)\,\vec{\imath} + \sin(2\pi t)\,\vec{\jmath}, \qquad t \in \mathbb{R}$$

are two parametrizations of the circle $\{(x, y, 0) \in \mathbb{R}^3 \mid x^2 + x^2 = 1\}$.

Definition 11.26 ([3, Definition 14.3.1])**.** *Let $\vec{f}(t) = f_1(t)\,\vec{\imath} + f_2(t)\,\vec{\jmath} + f_3(t)$ be a differentiable parametrized curve in $\mathbb{R}^3$. The vector*

$$\vec{f}'(t) = f_1'(t)\,\vec{\imath} + f_2'(t)\,\vec{\jmath} + f_3'(t)\,\vec{k}$$

*is called a **tangent vector** of the curve $\mathrm{im}(\vec{f})$ at the point $\vec{f}(t)$.*

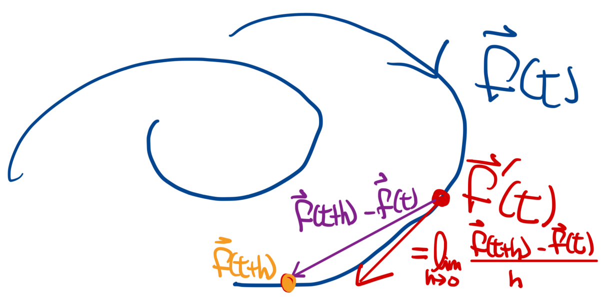


FIGURE 56. Tangent vector.

Remark 11.27. *If f is a differentiable real-valued function, then its graph $\Gamma = \{(x, f(x)) \in \mathbb{R}^2 \mid x \in \mathbb{R}\}$ is a curve in $\mathbb{R}^2$ which is parametrized by*

$$\vec{\gamma}(t) = t\,\vec{\imath} + f(t)\,\vec{\jmath}.$$

The vector

$$\vec{\gamma}'(t) = \vec{\imath} + f'(t)\,\vec{\jmath}$$

is a tangent vector of Γ *at the point* $(t, f(t))$*. The slope of this vector is* $f'(t)$*.*

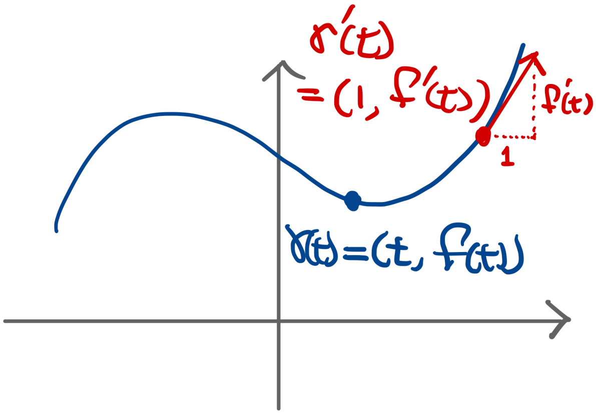


FIGURE 57. Tangent vector of the graph of a function.

The **arc length** of a curve is the least upper bound of the set of all lengths of polygonal paths inscribed in the curve. Recall from Remark A.5 that the arc length L of the curve $y = f(x)$ can be computed by the formula

$$L = \int_a^b \sqrt{1 + (f'(x))^2}\,dx.$$

More generally, the arc length of a differentiable parametrized curve can be computed by the following theorem:

Theorem 11.28 ([3, Theorem 14.4.2])**.** *Let* $\vec{f}(t)$*,* $t \in [a, b]$*, be a continuously differentiable parametrized curve in* $\mathbb{R}^3$*. The arc length L of* $\vec{f}([a, b])$ *is given by the formula*

$$L = \int_a^b \left\|\vec{f}'(t)\right\| dt.$$

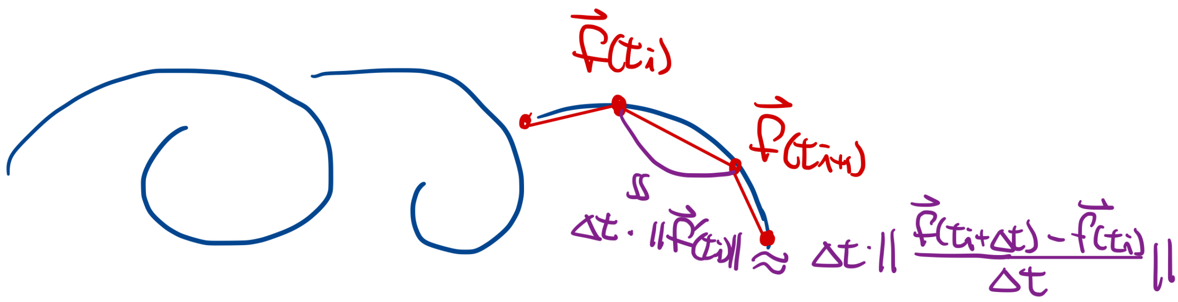


FIGURE 58. Arc length of a curve.

Example 11.29. *The circle with radius r can be parametrized by*

$$\vec{f}(t) = r\cos t\,\vec{\imath} + r\sin t\,\vec{\jmath}, \qquad t \in [0, 2\pi].$$

Its arc length is

$$\int_0^{2\pi} \left\| -r\sin t\,\vec{\imath} + r\cos\ \vec{\jmath}\right\| dt = \int_0^{2\pi} r\,dt = 2\pi r$$

which is same as the well-known formula.

Example 11.30. *Find the length of the curve*

$$\vec{f}(t) = \cos t\,\vec{\imath} + \sin t\,\vec{\jmath} + t\,\vec{k}, \qquad t \in [0, \pi/2],$$

and compare it to the straight-line distance between the endpoints of the curve.

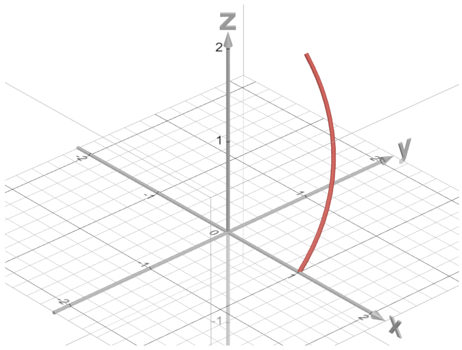


FIGURE 59. $\cos t\,\vec{\imath} + \sin t\,\vec{\jmath} + t\,\vec{k}$, $t \in [0, \pi/2]$.

12. Functions of several variables and their derivatives

From now on, we will consider *(real-valued) functions of several variables*, such as

$$f(x, y) = x^2 + y^2,$$
$$g(x, y, z) = x^2 + y^2 + z^2.$$

Both functions are quite common in mathematics. The first function $f(x, y)$ appears in the descriptions of a circle $\{f(x, y) = r^2\} \subset \mathbb{R}^2$, a disk $\{f(x, y) \leq r^2\} \subset \mathbb{R}^2$ and a cone $\{z = f(x, y)\} \subset \mathbb{R}^3$. The second function $g(x, y, z)$ appears in the descriptions of a sphere $\{g(x, y, z) = r^2\} \subset \mathbb{R}^3$ and a ball $\{g(x, y, z) \leq r^2\} \subset \mathbb{R}^3$. We will introduce derivatives of such functions in this chapter.

12.1. Partial derivatives.

If a function has several variables, one can consider the derivative of it with respect to a certain variable. For simplicity, we start with functions of two variables.

Definition 12.1 ([3, Definition 15.4.1]). *Let f be a function of two variables x, y. The* ***partial derivatives*** *of f with respect to x and with respect to y are the functions f_x and f_y (also denoted by $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$, respectively) defined by setting*

$$f_x(x, y) = \frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h}$$
$$f_y(x, y) = \frac{\partial f}{\partial y} = \lim_{h \to 0} \frac{f(x, y + h) - f(x, y)}{h}$$

provided these limits exist.

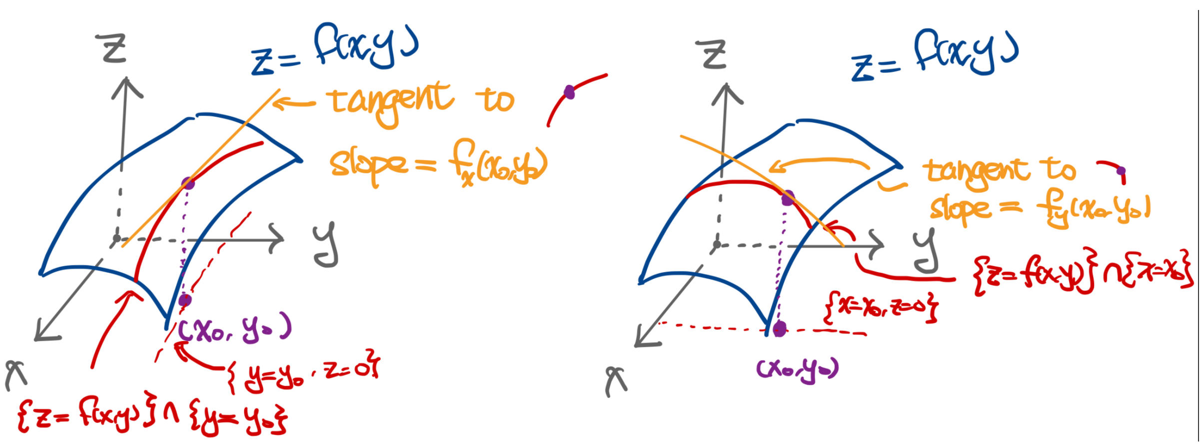


Figure 60. Partial derivatives.

Example 12.2. *Calculate f_x and f_y.*

(1) $f(x, y) = x^2 + y^2$. $\quad f_x = 2x, \quad f_y = 2y.$

(2) $f(x, y) = x \arctan(xy)$. $\quad f_x = \arctan(xy) + \frac{xy}{1 + x^2y^2}, \quad f_y = \frac{x^2}{1 + x^2y^2}.$

(3) $f(x, y) = e^{xy} + \ln(x^2 + y)$. $\quad f_x = ye^{xy} + \frac{2x}{x^2 + y}, \quad f_y = xe^{xy} + \frac{1}{x^2 + y}.$

Definition 12.3 ([3, Definition 15.4.2]). *Let f be a function of three variables x, y, z. The* ***partial derivatives*** *of f with respect to x, with respect to y, and with respect to z are the functions f_x, f_y and f_z (also denoted by $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial z}$, respectively) defined by setting*

$$f_x(x,y,z) = \frac{\partial f}{\partial x} = \lim_{h\to 0} \frac{f(x+h,y,z) - f(x,y,z)}{h}$$
$$f_y(x,y,z) = \frac{\partial f}{\partial y} = \lim_{h\to 0} \frac{f(x,y+h,z) - f(x,y,z)}{h}$$
$$f_z(x,y,z) = \frac{\partial f}{\partial z} = \lim_{h\to 0} \frac{f(x,y,z+h) - f(x,y,z)}{h}$$

provided these limits exist.

Example 12.4. *Calculate f_x, f_y and f_z. Evaluate $f_x(0,1,2)$, $f_y(0,1,2)$ and $f_z(0,1,2)$.*

(1) $f(x,y,z) = xy^2z^3$. $\quad f_x = y^2z^3, \quad f_y = 2xyz^3, \quad f_z = 3xy^2z^2.$

(2) $f(x,y,z) = x^2e^{y/z}$, $z \neq 0$. $\quad f_x = 2xe^{y/z}, \quad f_y = \frac{x^2}{z}e^{y/z}, \quad f_z = -\frac{x^2y}{z^2}e^{y/z}.$

(3) $f(x,y,z) = xy^2 - yz^2$. $\quad f_x = y^2, \quad f_y = 2xy - z^2, \quad f_z = -2yz.$

Remark 12.5. *For a function f of n variables, $x_1, \cdots, x_n$, one can similarly define*

$$f_{x_k} = \frac{\partial f}{\partial x_k} = \lim_{h\to 0} \frac{f(x_1,\cdots,x_k+h,\cdots,x_n) - f(x_1,\cdots,x_k,\cdots x_n)}{h}.$$

12.2. **Higher order partial derivatives and continuity.** Similar to functions of one variable, we have a higher order version of partial derivatives:

Definition 12.6 (Higher Order Partial Derivatives, [3, Page 782]). *Let f be a function f of n variables, $x_1, \cdots, x_n$. A* ***second order partial derivative*** *of f is a partial derivative of a partial derivative of f. More generally, a* ***k-th order partial derivative*** *of f is a partial derivative of a $(k-1)$-th order partial derivative.*

Example 12.7. *Let $f(x,y,z) = xy^2z^3$. Calculate all the first, second and third partial derivatives of f.*

1*st order:* $f_x = y^2z^3$, $f_y = 2xyz^3$, $f_z = 3xy^2z^2$.

2*nd order:* $f_{xx} = (f_x)_x = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = 0, \quad f_{xy} = (f_x)_y = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = 2yz^3,$

$f_{xz} = (f_x)_z = \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial z \partial x} = 3y^2z^2,$

$f_{yx} = 2yz^3, \quad f_{yy} = 2xz^3, \quad f_{yz} = 6xyz^2,$

$f_{zx} = 3y^2z^2, \quad f_{zy} = 6xyz^2, \quad f_{zz} = 6xy^2z.$

Observation: $f_{xy} = f_{yx}$, $f_{xz} = f_{zx}$, $f_{yz} = f_{zy}$.

3*rd order:* $f_{xyy} = f_{yxy} = f_{yyx} = 2z^3, \quad f_{xyz} = f_{xzy} = f_{yxz} = f_{yzx} = f_{zxy} = f_{zyx} = 6yz^2,$ $f_{xzz} = f_{zxz} = f_{zzx} = 6y^2z, \quad f_{yyz} = f_{yzy} = f_{zyy} = 6xz^2, \quad f_{yzz} = f_{zyz} = f_{zzy} = 12xyz,$ $f_{zzz} = 6xy^2,$ *and the other* 3*rd order partial derivatives vanish.*

From the computation, we observe that $f_{xy} = f_{yx}$. It is true under suitable assumptions. To describe the assumptions, we shall introduce "limit" and "continuity" for functions of several variables.

For simplicity, we will define these notions for functions of two variables. The definitions and examples can be extended to functions of n variables in a straightforward way.

Definition 12.8 ([3, Section 15.6]). *Let $f(\vec{x})$ be a function of two variables $\vec{x} = (x_1, x_2)$, and $\vec{a} = (a_1, a_2) \in \mathbb{R}^2$. We say that the* **limit** $\lim_{\vec{x}\to\vec{a}} f(\vec{x})$ *exists if there exists a number $L \in \mathbb{R}$ such that the following statement holds: for each $\epsilon > 0$, there exists a $\delta > 0$ such that*

$$\text{if} \qquad 0 < \left\| \vec{x} - \vec{a} \right\| < \delta, \qquad \text{then} \qquad |f(\vec{x}) - L| < \epsilon.$$

In this case, we write

$$\lim_{\vec{x}\to\vec{a}} f(\vec{x}) = L.$$

A function $f(\vec{x})$ is ***continuous*** *at $\vec{x} = \vec{a}$ if $\lim_{\vec{x}\to\vec{a}} f(\vec{x}) = f(\vec{a})$. We say that $f(\vec{x})$ is* ***continuous*** *on $\mathbb{R}^2$ if it is continuous at each point in $\mathbb{R}^2$.*

Limit and continuity for functions of two variables are actually much more complicated than the ones for functions of one variable. The main intuitional reason is that the arrow under limit "$\vec{x} \to \vec{a}$" actually means "$\vec{x}$ tends to $\vec{a}$ in all the possible directions." In the case of one variable, there are only two directions, but in the case of two variables, one needs to consider infinitely many directions.

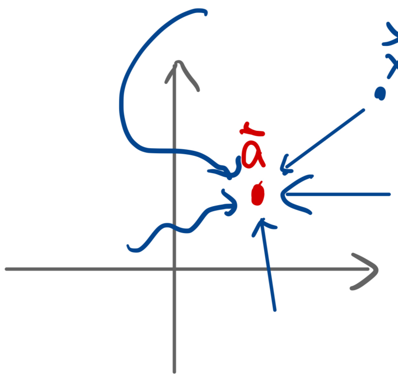


FIGURE 61. $\vec{x} \to \vec{a}$.

We will list only a few examples of continuous functions here. One can learn more about this topic in a course of Advanced Calculus.

Example 12.9. *The following functions are continuous on $\mathbb{R}^2$.*

(1) Constant functions.
(2) Polynomials of two variables: x, y, $x + y$, xy, $x^4y + y^2 - x$, $y^3 - y + 1$, etc.
(3) $e^{P(x,y)}$, $\sin P(x, y)$, $\cos P(x, y)$, combinations of them (addition, scalar product, multiplication and compositions). Here, $P(x, y)$ is a polynomial of two variables.

Now let us describe the precise theorem about our observation of higher order partial derivatives.

Theorem 12.10 ([3, Page 783]). *Let $f = f(x, y)$ be a real-valued function of two variables. Suppose that*

$$f, \quad f_x, \quad f_y, \quad f_{xy}, \quad f_{yx}$$

are continuous on $\mathbb{R}^2$. Then

$$f_{xy} = f_{yx} \qquad \text{on } \mathbb{R}^2.$$

One can find a proof of this theorem in a textbook of advanced calculus.

The analogous theorem for functions of three or more variables is also true.

12.3. **Gradient.** For convenience, unless otherwise stated, we will assume that functions are **smooth**, i.e. all the partial derivatives of any order exist and are continuous.

Instead of partial derivatives, we can also consider "total derivative" by collecting all the first order partial derivatives:

Definition 12.11 ([3, Definition 16.1.2 & Theorem 16.1.3])**.** *Let f be a real-valued function of three variables. The* ***gradient*** *of f at $\vec{x} \in \mathbb{R}^3$ is the vector*

$$\nabla f(\vec{x}) = \frac{\partial f}{\partial x}(\vec{x})\,\vec{\imath} + \frac{\partial f}{\partial y}(\vec{x})\,\vec{\jmath} + \frac{\partial f}{\partial z}(\vec{x})\,\vec{k} \qquad \in \mathbb{R}^3.$$

If f is of two variables, then the gradient is

$$\nabla f(\vec{x}) = \frac{\partial f}{\partial x}(\vec{x})\,\vec{\imath} + \frac{\partial f}{\partial y}(\vec{x})\,\vec{\jmath}.$$

Example 12.12. *For $f(x, y) = xe^y - ye^x$, we have*

$$\frac{\partial f}{\partial x}(x, y) = e^y - ye^x, \qquad \frac{\partial f}{\partial y}(x, y) = xe^y - e^x$$

and therefore

$$\nabla f(x, y) = (e^y - ye^x)\,\vec{\imath} + (xe^y - e^x)\,\vec{\jmath}.$$

Example 12.13. *Set*

$$f(x, y, z) = x\sin(\pi y) + y\cos(\pi z).$$

Evaluate ∇f at $(0, 1, 2)$. $\nabla f(0, 1, 2) = \vec{\jmath}.$

The following theorem can be verified directly by the definition.

Theorem 12.14 ([3, (16.2.1)])**.** *Let f, g be smooth functions, and $\alpha \in \mathbb{R}$. Then*

(1) $\nabla(f(\vec{x}) + g(\vec{x})) = \nabla f(\vec{x}) + \nabla g(\vec{x})$,
(2) $\nabla(\alpha f(\vec{x})) = \alpha \nabla f(\vec{x})$,
(3) $\nabla(f(\vec{x})g(\vec{x})) = f(\vec{x})\nabla g(\vec{x}) + g(\vec{x})\nabla f(\vec{x})$.

If we know the gradient, we can easily compute the partial derivatives and even all the *directional derivatives*. Here, directional derivative means the following:

Definition 12.15 ([3, Definition 16.2.2])**.** *For each* ***unit vector*** *$\vec{u}$ (i.e. $\left\|\vec{u}\right\| = 1$), the limit*

$$\nabla_{\vec{u}} f(\vec{x}) = \lim_{h \to 0} \frac{f(\vec{x} + h\vec{u}) - f(\vec{x})}{h},$$

if it exists, is called the ***directional derivative*** *of f at $\vec{x}$ in the direction $\vec{u}$.*

In [3], the directional derivative $\nabla_{\vec{u}} f$ is denoted by $f'_{\vec{u}}$.

The assumption $\left\|\vec{u}\right\| = 1$ in the definition is not important. It is there because a "direction" is conventionally represented by a unit vector.

Remark 12.16. *The directional derivative* $\nabla_{\vec{u}} f(\vec{x})$ *gives the* rate of change of f in the direction $\vec{u}$, *and*

$$\nabla_{\vec{u}} f(\vec{x}) = \frac{d}{dt}\Big|_{t=0} f(\vec{x} + t\vec{u}).$$

Remark 12.17. *Each partial derivative* f_x*,* f_y*,* f_z *is itself a directional derivative:*

$$\frac{\partial f}{\partial x}(\vec{x}) = \nabla_{\vec{i}} f(\vec{x}), \qquad \frac{\partial f}{\partial y}(\vec{x}) = \nabla_{\vec{j}} f(\vec{x}), \qquad \frac{\partial f}{\partial z}(\vec{x}) = \nabla_{\vec{k}} f(\vec{x}).$$

Theorem 12.18 ([3, Theorem 16.2.4])**.** *For each unit vector* $\vec{u}$*,*

$$\nabla_{\vec{u}} f(\vec{x}) = \nabla f(\vec{x}) \cdot \vec{u}\,.$$

Proof. We prove it for the case of two variables. The case of n variables is similar.

Let $\vec{u} = u_1\vec{i} + u_2\vec{j}$. By Mean Value Theorem, there exists c_h between x_1 and $x_1 + hu_1$ and c_h' between x_2 and $x_2 + hu_2$ such that

$$\begin{aligned} f(\vec{x} + h\vec{u}) - f(\vec{x}) &= f(x_1 + hu_1, x_2 + hu_2) - f(x_1, x_2 + hu_2) + f(x_1, x_2 + hu_2) - f(x_1, x_2) \\ &= hu_1 f_x(c_h, x_2 + hu_2) + hu_2 f_y(x_1, c_h'). \end{aligned}$$

By the smoothness assumption,

$$\begin{aligned} \nabla_{\vec{u}} f(\vec{x}) &= \lim_{h\to 0} \frac{f(\vec{x} + h\vec{u}) - f(\vec{x})}{h} \\ &= \lim_{h\to 0} \Big(u_1 f_x(c_h, x_2 + hu_2) + u_2 f_y(x_1, c_h') \Big) \\ &= u_1 f_x(x_1, x_2) + u_2 f_y(x_1, x_2) \qquad \because f_x \text{ and } f_y \text{ are continuous} \\ &= \nabla f(\vec{x}) \cdot \vec{u}\,, \end{aligned}$$

as desired. □

Remark 12.19. *It follows from Theorem 12.18 that*

$$\frac{d}{dt}\Big|_{t} f(\vec{x} + t\vec{u}) = \frac{d}{ds}\Big|_{s=0} f(\vec{x} + (t + s)\vec{u}) = \nabla f(\vec{x} + t\vec{u}) \cdot \vec{u}\,.$$

Example 12.20. *Find the directional derivative of the function* $f(x, y) = x^2 + y^2$ *at the point* $(1, 2)$ *in the direction* $\vec{u} = \frac{3}{5}\vec{i} + \frac{4}{5}\vec{j}$.

12.4. **Mean value theorem and chain rule.** The following theorem is known as the mean value theorem for functions with several variables.

Theorem 12.21 ([3, Theorem 16.3.1])**.** *Let f be a smooth function. There exists on this line segment a point $\vec{c}$ between $\vec{a}$ and $\vec{b}$ such that*

$$f(\vec{b}) - f(\vec{a}) = \nabla f(\vec{c}) \cdot (\vec{b} - \vec{a}).$$

Proof. If $\vec{a} = \vec{b}$, then theorem is clear. Assume $\vec{a} \neq \vec{b}$. Recall that the line segment between $\vec{a}$ and $\vec{b}$ can be parametrized by

$$\vec{a} + t(\vec{b} - \vec{a}), \qquad t \in [0, 1].$$

Let $\vec{u} = \frac{\vec{b} - \vec{a}}{\left\|\vec{b} - \vec{a}\right\|}$, and

$$g(t) = f(\vec{a} + t(\vec{b} - \vec{a})) = f(\vec{a} + t\left\|\vec{b} - \vec{a}\right\|\vec{u}), \qquad t \in [0, 1].$$

By Theorem 12.18 and Remark 12.19, the function $f(\vec{a}+t\vec{u})$ is differentiable and

$$\frac{d}{dt}f(\vec{a}+t\vec{u}) = \nabla f(\vec{a}+t\vec{u})\cdot\vec{u}\,.$$

Thus, the function $g(t)$ is also differentiable, and

$$g'(t) = (\nabla f(\vec{a}+t\left\|\vec{b}-\vec{a}\right\|\vec{u})\cdot\vec{u})\left\|\vec{b}-\vec{a}\right\| = \nabla f(\vec{a}+t(\vec{b}-\vec{a}))\cdot(\vec{b}-\vec{a}).$$

Applying the one-variable mean value theorem to $g(t)$, we can conclude that there exists $t_0 \in (0,1)$ such that

$$f(\vec{b})-f(\vec{a}) = g(1)-g(0) = g'(t_0)(1-0) = \nabla f(\vec{a}+t_0(\vec{b}-\vec{a}))\cdot(\vec{b}-\vec{a}).$$

Setting $\vec{c} = \vec{a}+t_0(\vec{b}-\vec{a})$, we conclude the theorem. □

Corollary 12.22. *If $\nabla f(\vec{x}) = \vec{0}$ for all $\vec{x}\in\mathbb{R}^3$, then f is constant.*

Proof. Given any $\vec{x},\vec{y}\in\mathbb{R}^3$, it follows from the mean value theorem that there exist $\vec{c}$ such that

$$f(\vec{x})-f(\vec{y}) = \nabla f(\vec{c})\cdot(\vec{x}-\vec{y}) = 0.$$

Thus, the function f is constant. □

Applying this corollary to $f-g$, we have the following

Corollary 12.23. *If $\nabla f(\vec{x}) = \nabla g(\vec{x})$ for all $\vec{x}\in\mathbb{R}^3$, then f and g differ by a constant on $\mathbb{R}^2$.*

Now let $\vec{\gamma}(t)$ be a vector-valued function with one variable. Consider the composition

$$f\circ\vec{\gamma}:\mathbb{R}\to\mathbb{R}.$$

Theorem 12.24 (Chain Rule, [3, Theorem 16.3.4])**.** *Let $\vec{\gamma}(t) = (\gamma_1(t),\gamma_2(t),\gamma_3(t))$ be a differentiable curve in $\mathbb{R}^3$ and f be a smooth function on $\mathbb{R}^3$. The composition $f\circ\vec{\gamma}$ is differentiable, and*

$$\frac{d}{dt}(f(\vec{\gamma}(t))) = \nabla f(\vec{\gamma}(t))\cdot\vec{\gamma}'(t) = f_x(\gamma_1(t))\gamma_1'(t) + f_y(\gamma_2(t))\gamma_2'(t) + f_z(\gamma_3(t))\gamma_3'(t).$$

Proof. By the mean value theorem, there exists $\vec{c}(h)$ between $\vec{\gamma}(t)$ and $\vec{\gamma}(t+h)$ such that

$$\frac{f(\vec{\gamma}(t+h))-f(\vec{\gamma}(t))}{h} = \nabla f(\vec{c}(h))\cdot\Big(\frac{\vec{\gamma}(t+h)-\vec{\gamma}(t)}{h}\Big).$$

Since $\vec{\gamma}$ is continuous,

$$\left\|\vec{c}(h)-\vec{\gamma}(t)\right\| \le \left\|\vec{\gamma}(t+h)-\vec{\gamma}(t)\right\| \to 0, \qquad \text{as } h\to 0.$$

By the continuity of $\nabla f = (f_x, f_y, f_z)$, we have

$$\lim_{h\to 0}\nabla f(\vec{c}(h)) = \nabla f(\vec{\gamma}(t)).$$

Therefore,

$$\frac{d}{dt}(f(\vec{\gamma}(t))) = \lim_{h\to 0}\frac{f(\vec{\gamma}(t+h))-f(\vec{\gamma}(t))}{h} = \lim_{h\to 0}\nabla f(\vec{c}(h))\cdot\frac{\vec{\gamma}(t+h)-\vec{\gamma}(t)}{h} = \nabla f(\vec{\gamma}(t))\cdot\vec{\gamma}'(t),$$

as desired. □

Example 12.25. *Find the rate of change of*

$$f(x, y) = \frac{1}{3}(x^3 + y^3)$$

with respect to t along the curve $\vec{\gamma}(t) = a\cos t\,\vec{\imath} + b\sin t\,\vec{\jmath}$.
ans: $(f(\vec{\gamma}(t)))' = \sin t\cos t(b^3\sin t - a^3\cos t)$

Example 12.26. *Calculate.*

(1) *Find du/dt given that $u = x^2 - y^2$, $x = t^2 - 1$ and $y = 3\sin\pi t$.* *ans:* $4t^3 - 4t - 18\pi\sin\pi t\cos\pi t$
(2) *Find $\partial u/\partial s$ and $\partial u/\partial t$ given that $u = x^2 - 2xy + 2y^3$, $x = s^2\ln t$ and $y = 2st^3$.*
ans: $\partial u/\partial s = (2s^2\ln t - 4st^3)(2s\ln t) + (-2s^2\ln t + 24s^2t^6)2t^3$,
$\partial u/\partial t = (2s^2\ln t - 4st^3)(s^2/t) + (-2s^2\ln t + 24s^2t^6)6st^2$.

Example 12.27 (Polar coordinates)**.** *Assume the $u = u(x, y)$ is differentiable. Suppose that $x = r\cos\theta$ and $y = r\sin\theta$.*

(1) *Show that*

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta,$$
$$\frac{\partial u}{\partial \theta} = -\frac{\partial u}{\partial x}r\sin\theta + \frac{\partial u}{\partial y}r\cos\theta.$$

(2) *Express $\left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2$ in terms of $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ provided $r \neq 0$.*

Example 12.28. *Suppose that y is a differentiable function of x that satisfies the equation*

$$u(x, y) = 2x^2y - y^3 + 1 - x - 2y = 0.$$

Then

$$\begin{aligned} 0 &= \frac{d}{dx}(u(x, y)) \\ &= u_x + u_y\frac{dy}{dx} \\ &= (4xy - 1) + (2x^2 - 3y^2 - 2)\frac{dy}{dx}. \end{aligned}$$

Thus,

$$\frac{dy}{dx} = -\frac{4xy - 1}{2x^2 - 3y^2 - 2}.$$

Example 12.29. *Assume that z is a differentiable function of (x, y) which satisfies the given equation. Find $\partial z/\partial x$ and $\partial z/\partial y$.*

(1) $z^4 + x^2z^3 + y^2 + xy = 2$.
(2) $\cos xyz + \ln(x^2 + y^2 + z^2) = 0$.

12.5. **Gradient and normal vector.** Recall that a **normal vector** of a line L in $\mathbb{R}^2$ is a vector that is perpendicular to L, and the **normal line** of L is the line perpendicular to L. In other words, a vector $\vec{n}$ is normal to a L if

$$\vec{n}\cdot(\vec{v} - \vec{w}) = 0, \qquad \forall \vec{v}, \vec{w} \in L.$$

If a line L in $\mathbb{R}^2$ is given by the equation $ax + by = c$, then

$$(a, b) \cdot (\vec{v} - \vec{w}) = a(v_1 - w_1) + b(v_2 - w_2) = c - c = 0,$$

where $\vec{v} = (v_1, v_2)$ and $\vec{w} = (w_1, w_2)$. Thus, the vector (a, b) is a normal vector of L.

Similarly, a **normal vector** of a plane P in $\mathbb{R}^3$ is a vector that is perpendicular to P, and the **normal line** of P is the line perpendicular to P. If P is given by the equation $ax + by + cz = d$, then the vector (a, b, c) is a normal vector of P.

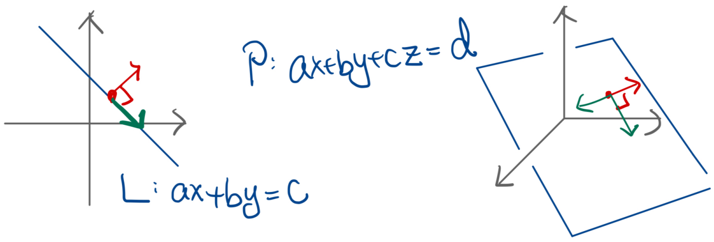


FIGURE 62. Normal vector.

Now consider a curve in xy-plane given an equation

$$C : \ f(x, y) = c.$$

Suppose $(x_0, y_0) \in C$ and $\nabla f(x_0, y_0) \neq 0$. If, near (x_0, y_0), the curve C is parametrized by a differentiable function $\vec{\gamma}(t) = (\gamma_1(t), \gamma_2(t))$ with $\vec{\gamma}(0) = (x_0, y_0)$, i.e.

$$f(\gamma_1(t), \gamma_2(t)) = c,$$

then it follows from the chain rule that

$$\nabla f(x_0, y_0) \cdot \vec{\gamma}'(0) = \frac{d}{dt} c = 0.$$

Thus, the gradient $\nabla f(x_0, y_0)$ is a normal vector of the tangent line of C at (x_0, y_0).

Proposition 12.30 ([3, (16.4.4) & (16.4.5)]). *Suppose $\nabla f(x_0, y_0) \neq 0$. The tangent line of the curve $C : \ f(x, y) = c$ at (x_0, y_0) is determined by the equation*

$$\nabla f(x_0, y_0) \cdot (x - x_0, y - y_0) = \frac{\partial f}{\partial x}(x_0, y_0)\,(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)\,(y - y_0) = 0.$$

*Furthermore, the **normal line**, i.e. the normal line of the tangent line, of C at (x_0, y_0) is determined by the equation*

$$\frac{\partial f}{\partial y}(x_0, y_0)\,(x - x_0) - \frac{\partial f}{\partial x}(x_0, y_0)\,(y - y_0) = 0.$$

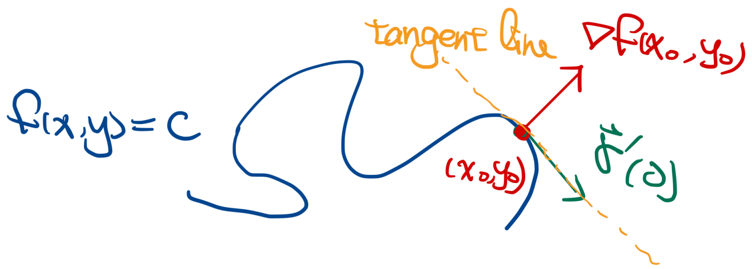


FIGURE 63. Gradient and tangent line.

Example 12.31. *Draw a picture for gradients, tangent lines and normal lines of the circle* $x^2 + y^2 = 1$.

For the surface case, consider

$$S : f(x, y, z) = c.$$

Suppose $\vec{v}_0 = (x_0, y_0, z_0) \in S$ with the property $\nabla f(\vec{v}_0) \neq 0$. The **tangent plane** of S at $\vec{v}_0$ is

$$T_{\vec{v}_0}S = \{\vec{v}_0 + \vec{\gamma}'(0) \mid \vec{\gamma} : \mathbb{R} \to S, \ \vec{\gamma}(0) = \vec{v}_0 = (x_0, y_0, z_0)\}.$$

Given any curve $\vec{\gamma}(t) = (\gamma_1(t), \gamma_2(t), \gamma_3(t))$ in S, we have

$$f(\gamma_1(t), \gamma_2(t), \gamma_3(t)) = c,$$

and hence

$$\nabla f(x_0, y_0, z_0) \cdot \vec{\gamma}'(0) = \frac{d}{dt}c = 0.$$

Therefore, the gradient $\nabla f(x_0, y_0, z_0)$ is a normal vector of the tangent plane of S at (x_0, y_0, z_0).

Proposition 12.32 ([3, (16.4.8) & (16.4.10)]). *Suppose* $\nabla f(x_0, y_0, z_0) \neq 0$. *The tangent plane of the surface* $S : f(x, y, z) = c$ *at* (x_0, y_0, z_0) *is determined by the equation*

$$\nabla f(x_0, y_0, z_0) \cdot (x - x_0, y - y_0, z - z_0) =$$
$$\frac{\partial f}{\partial x}(x_0, y_0, z_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0, z_0)(y - y_0) + \frac{\partial f}{\partial z}(x_0, y_0, z_0)(z - z_0) = 0.$$

Furthermore, the **normal line**, *i.e. the normal line of the tangent plane, of* S *at* (x_0, y_0, z_0) *can be parametrized as follows*

$$x = x_0 + \frac{\partial f}{\partial x}(x_0, y_0, z_0)\, t\,,$$
$$y = y_0 + \frac{\partial f}{\partial y}(x_0, y_0, z_0)\, t\,,$$
$$z = z_0 + \frac{\partial f}{\partial z}(x_0, y_0, z_0)\, t\,.$$

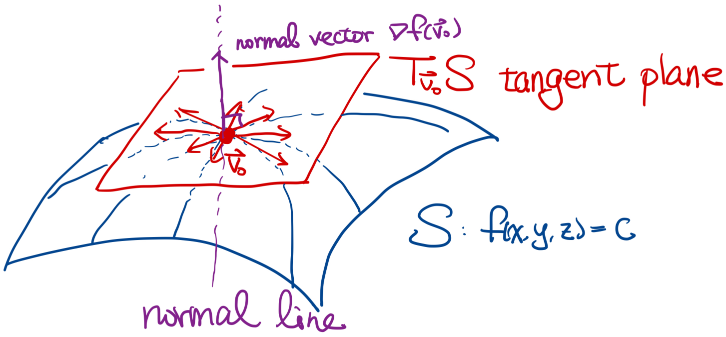


FIGURE 64. Gradient and tangent plane.

Example 12.33. *Draw pictures for gradients, tangent planes and normal lines of the sphere* $x^2 + y^2 + z^2 = 1$ *and the cone* $z = x^2 + y^2$.

Example 12.34. *At what point(s) of the surface*

$$z = 3xy - x^3 - y^3$$

is the tangent plane **horizontal** *(i.e. given by* $z = c$ *for some* c*)?* *ans:* $(0, 0, 0)$ *and* $(1, 1, 1)$.

12.6. **Extreme values.**

Definition 12.35 ([3, Definition 16.5.1])**.** *Suppose* f *is a function of several variables. The function* f *is said to have a* ***local maximum*** *at* $\vec{v}_0$ *if there exists* $\delta > 0$ *such that*

$$f(\vec{v}_0) \geq f(\vec{v}), \qquad \text{whenever } \left\| \vec{v} - \vec{v}_0 \right\| < \delta.$$

The function f *is said to have a* ***local minimum*** *at* $\vec{v}_0$ *if there exists* $\delta > 0$ *such that*

$$f(\vec{v}_0) \leq f(\vec{v}), \qquad \text{whenever } \left\| \vec{v} - \vec{v}_0 \right\| < \delta.$$

The local maxima and local minima of f *comprise the* ***local extreme values*** *of* f.

Similar to the case of function with one variable, we have the following

Theorem 12.36 ([3, Theorem 16.5.2])**.** *If* f *has a local extreme value at* $\vec{v}_0$*, then*

$$\nabla f(\vec{v}_0) = \vec{0} \qquad \text{or} \qquad \nabla f(\vec{v}_0) \text{ does not exist.}$$

Proof. Assume $\nabla f(\vec{v}_0) = (f_x(\vec{v}_0), f_y(\vec{v}_0))$ exists. We need to show that $\nabla f(\vec{v}_0) = 0$. For simplicity, we set $\vec{v}_0 = (x_0, y_0)$. The three-variable case is similar.

Since f has a local extreme value at (x_0, y_0), the function

$$g(x) = f(x, y_0)$$

has a local extreme value at x_0. Therefore,

$$f_x(\vec{v}_0) = g'(x_0) = 0.$$

Similarly $f_y(\vec{v}_0) = 0$. Thus $\nabla f(\vec{v}_0) = \vec{0}$. □

Definition 12.37. *The **critical points** of a function f are the points at which the gradient is zero or the gradient does not exist.*

*The points at which the gradient is zero are called **stationary points**. The stationary points that do not give rise to local extreme values are called **saddle points**.*

Example 12.38. *The function*

$$f(x, y) = 2x^2 + y^2 - xy - 7y$$

has gradient

$$\nabla f(x, y) = (4x - y)\,\vec{i} + (2y - x - 7)\,\vec{j}.$$

The only critical point of f is $(1, 4)$.

Now we compare the value f at $(1, 4)$ with the values of f at nearby points $(1 + h, 4 + k)$:

$$f(1, 4) = -14,$$
$$f(1 + h, 4 + k) = 2h^2 + k^2 - hk - 14.$$

The difference

$$\begin{aligned} f(1 + h, 4 + k) - f(1, 4) &= 2h^2 + k^2 - hk \\ &\geq h^2 + (h^2 - 2|h||k| + k^2) \\ &= h^2 + (|h| - |k|)^2 \geq 0. \end{aligned}$$

Thus, the function f has a local minimum -14 at $(1, 4)$.

Recall that if g is a smooth function of one variable and $g'(x_0) = 0$, then we have the second-derivative test:

$$\begin{aligned} &g \text{ has a local minimum at } x_0 \qquad \text{if} \qquad g''(x_0) > 0; \\ &g \text{ has a local maximum at } x_0 \qquad \text{if} \qquad g''(x_0) < 0. \end{aligned}$$

Following is a similar test for functions of two variables.

Theorem 12.39 ([3, Theorem 16.5.3])**.** *Suppose that f has continuous second-order partial derivatives near (x_0, y_0) and $\nabla f(x_0, y_0) = 0$. Set*

$$A = \frac{\partial^2 f}{\partial x^2}(x_0, y_0), \qquad B = \frac{\partial^2 f}{\partial x \partial y}(x_0, y_0), \qquad C = \frac{\partial^2 f}{\partial y^2}(x_0, y_0),$$

and form the discriminant $D = AC - B^2$.

(1) If $D < 0$, then (x_0, y_0) is a saddle point.
(2) If $D > 0$, then f has

$$\begin{aligned} &\textit{a local minimum at } (x_0, y_0) \textit{ if } A > 0, \\ &\textit{a local maximum at } (x_0, y_0) \textit{ if } A < 0. \end{aligned}$$

In Example 12.38, one can get the same conclusion by computing

$$A = 4, B = -1, C = 2, \text{ and } D = AC - B^2 = 7 > 0.$$

Remark 12.40. *The discriminant D is actually obtained by considering the matrix*

$$\begin{pmatrix} \frac{\partial^2 f}{\partial x^2}(x_0, y_0) & \frac{\partial^2 f}{\partial x \partial y}(x_0, y_0) \\ \frac{\partial^2 f}{\partial y \partial x}(x_0, y_0) & \frac{\partial^2 f}{\partial y^2}(x_0, y_0) \end{pmatrix} = \begin{pmatrix} A & B \\ B & C \end{pmatrix}.$$

The discriminant $D = AC - B^2$ is the determinant of this matrix. The criterion actually determines that this matrix is "indefinite," "positive definite" or "negative definite."

Example 12.41. *Find local minimum, local maximum and saddle point of $f(x, y)$.*

(1) $f(x, y) = x^2 + y^2$.
(2) $f(x, y) = -x^2 - y^2$.
(3) $f(x, y) = xy$.

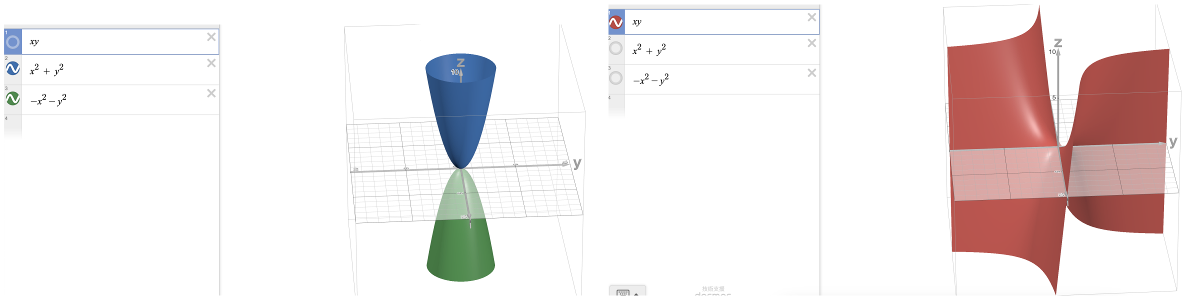


FIGURE 65. Local extreme values and saddle point.

Definition 12.42 ([3, Definition 16.6.1])**.** *Suppose f is a function of several variables. The function f is said to have an **absolute maximum** at $\vec{v}_0$ if*

$$f(\vec{v}_0) \geq f(\vec{v}), \qquad \text{whenever } f(\vec{v}) \text{ is defined.}$$

*The function f is said to have an **absolute minimum** at $\vec{v}_0$ if*

$$f(\vec{v}_0) \leq f(\vec{v}), \qquad \text{whenever } f(\vec{v}) \text{ is defined.}$$

*The absolute maxima and absolute minima of f comprise the **absolute extreme values** of f.*

Example 12.43. *Find the absolute extreme values taken by the function*

$$f(x, y) = x^2 + y^2 - 2x - 2y + 4$$

on the closed disk $D = \{(x, y) \mid x^2 + y^2 \leq 9\}$.

Solution. We solve this problem by the following steps.

Step 1: Find the critical points in the interior $\{x^2 + y^2 < 9\}$. By solving $\nabla f = 0$, one can conclude that $(1, 1)$ is the only critical point.

Step 2: Find the extreme values on the boundary $x^2 + y^2 = 9$. Let

$$\vec{\gamma}(t) = 3 \cos t \, \vec{i} + 3 \sin t \, \vec{j}.$$

This curve $\vec{\gamma}$ parametrizes the circle $x^2 + y^2 = 9$. Thus, the values of f on the boundary can be expressed by the function

$$F(t) = f(\vec{\gamma}(t)) = 13 - 6\cos t - 6\sin t.$$

of one variable. Since

$$F'(t) = 6\sin t - 6\cos t,$$

the critical points of F are $\frac{\pi}{4} + n\pi$. Thus, the extreme values of f on the boundary may occur at $\vec{\gamma}(\frac{\pi}{4} + n\pi) = (\frac{3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}), (-\frac{3\sqrt{2}}{2}, -\frac{3\sqrt{2}}{2})$.

Step 3: Compare the values of f at all the candidate points:

$$\begin{aligned} f(1,1) &= 2, \\ f(\frac{3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}) &= 13 - 6\sqrt{2} \approx 4.51, \\ f(-\frac{3\sqrt{2}}{2}, -\frac{3\sqrt{2}}{2}) &= 13 + 6\sqrt{2} \approx 21.49. \end{aligned}$$

Therefore, the absolute maximum is $13 + 6\sqrt{2}$, and the absolute minimum is 2. □

Remark 12.44. *In* **Step 3***, we implicitly assumed that f attains both a maximum and a minimum value on D. This assumption is justified by the following theorem: If f is a continuous function on a closed and bounded subset K of $\mathbb{R}^n$ — such as a closed disk $\{(x-a)^2 + (y-b)^2 \le r\}$ or a closed rectangle $\{a \le x \le b,\ c \le y \le d\}$ in $\mathbb{R}^2$ — then f attains both a maximum and a minimum value on K.*

Lagrange multiplier. About finding extreme values on the circle $x^2+y^2 = 9$, note that if $\vec{x}_0$ maximizes (or minimizes) $f(\vec{x})$ and if $\vec{\gamma}(t)$ is a smooth parametrization of the circle around $\vec{x}_0$ with $\vec{\gamma}(0) = \vec{x}_0$, then

$$\frac{d}{dt}\Big|_{t=0} f(\vec{\gamma}(t)) = \nabla f(\vec{x}_0) \cdot \vec{\gamma}'(0) = 0.$$

Therefore, it suffices to check the points (x, y) at which $\nabla f(x, y)$ is normal to the circle, i.e. parallel to the gradient of $x^2 + y^2$.

Explicitly, by solving (x, y) with the property:

$$\nabla f = (2x - 2, 2y - 2) \qquad \text{is parallel to} \qquad \nabla(x^2 + y^2) = (2x, 2y),$$

we have

$$(2x - 2)2y = (2y - 2)2x \qquad \Rightarrow \qquad x = y.$$

There are two points on $x^2 + y^2 = 9$ satisfying this property: $(\frac{3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2})$ and $(-\frac{3\sqrt{2}}{2}, -\frac{3\sqrt{2}}{2})$.

This method is summarized in the following remark.

Remark 12.45. *If $\vec{x}_0$ maximizes (or minimizes) $f(\vec{x})$ subject to the side condition $g(\vec{x}) = 0$, then*

$$\nabla f(\vec{x}_0) \quad \textit{and} \quad \nabla g(\vec{x}_0) \quad \textit{are parallel.}$$

Thus, if $\nabla g(\vec{x}_0) \neq \vec{0}$, then there exists a scalar λ such that

$$\nabla f(\vec{x}_0) = \lambda \nabla g(\vec{x}_0).$$

Such a scalar λ is called a ***Lagrange multiplier****.*

The method of Lagrange multipliers may be applied to extreme value problems with constraints.

Example 12.46. *Find the minimum value taken by the function*

$$f(x, y) = x^2 + (y - 2)^2$$

on the hyperbola $x^2 - y^2 = 1$.

Solution. Note that since $f(x, y)$ gives the square of the distance between $(0, 2)$ and (x, y), its minimum value exists and there is no maximum value.

Set

$$g(x, y) = x^2 - y^2 - 1.$$

We want to minimize

$$f(x, y) = x^2 + (y - 2)^2 \qquad \text{subject to the condition } g(x, y) = 0.$$

Solve (x, y) with the property

$$\nabla f = (2x, 2y - 4) \qquad \text{is parallel to} \qquad \nabla g = (2x, -2y). \tag{25}$$

Since the condition $g(x, y) = 0$ guarantees that $x \neq 0$, the condition (25) is equivalent to

$$\frac{2y - 4}{2x} = \frac{-2y}{2x}.$$

which implies that $y = 1$ and $x = \pm\sqrt{2}$. The value

$$f(\sqrt{2}, 1) = f(-\sqrt{2}, 1) = 3$$

is the desired minimum. □

13. Integration of functions of several variables

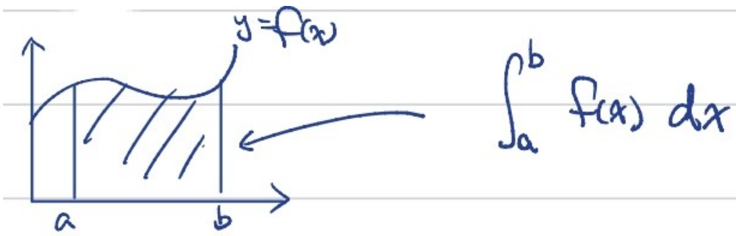
We will consider integration of functions of several variables. For simplicity, we will consider only good situations and omit theoretical discussions in this course. More complete discussions of integration can be found in the courses "Advanced Calculus" and "Real Analysis."

13.1. **Double integrals.**

Recall that the integration

$$\int_a^b f(x)\,dx$$

of a single-variable (non-negative) function $f(x)$ over $[a, b]$ is the area of the region bounded by the graph of $f(x)$, $x = a$, $x = b$, and the x-axis, and it can be defined by the limit of Riemann sums.



Parallelly, the integration

$$\iint_\Omega f(x, y)\,dxdy$$

of a two-variable (non-negative) function $f(x, y)$ over a region Ω is the volume of the solid bounded by $z = f(x, y)$, Ω and the cylinder generated by Ω.

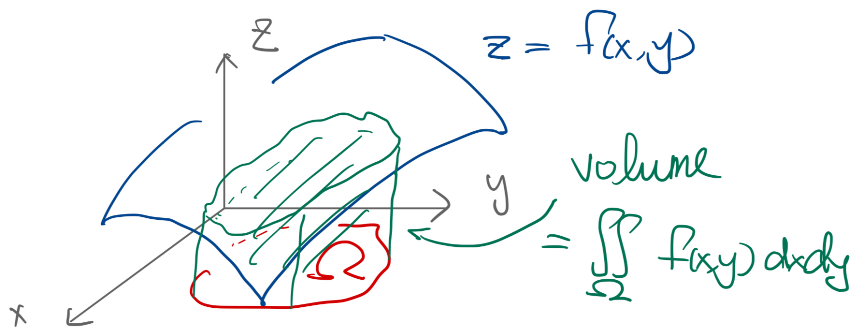


Figure 66. Double integral.

In particular, if $f(x, y)$ is the constant function 1, then the integral

$$\iint_\Omega dxdy = \iint_\Omega 1\,dxdy$$

gives the area of Ω.

An integral of the above type is called a **double integral**, and it can be defined by a limit of "two-dimensional" Riemann sums. The precise definition is complicated, and we skip it here. In practice, double integrals are computed by *repeated integrals*:

Theorem 13.1 ([2, Corollary 9.2.2] in Marsden's book)**.** *Let $\phi_1, \phi_2 : [a,b] \to \mathbb{R}$ be continuous maps such that $\phi_1(x) \le \phi_2(x)$ for all $x \in [a,b]$, let*

$$\Omega = \{(x,y) \in \mathbb{R}^2 \mid a \le x \le b, \phi_1(x) \le y \le \phi_2(x)\},$$

and let $f : \Omega \to \mathbb{R}$ be continuous. Then

$$\iint_\Omega f(x,y)\,dxdy = \int_a^b \Big(\int_{\phi_1(x)}^{\phi_2(x)} f(x,y)\,dy \Big)\,dx.$$

Also see (17.3.1) and (17.3.2) in the textbook [3].

There are two ways to compute a double integral $\iint_\Omega f(x,y)\,dxdy$:

(1) Integrate $f(x,y)$ as a function of x first, and then integrate the result as a function of y, i.e.

$$\iint_\Omega f(x,y)\,dxdy = \int_c^d \Big(\int_{\psi_1(y)}^{\psi_2(y)} f(x,y)\,dx \Big)\,dy.$$

(2) Or integrate $f(x,y)$ as a function of y first, and then integrate the result as a function of x, i.e.

$$\iint_\Omega f(x,y)\,dxdy = \int_a^b \Big(\int_{\phi_1(x)}^{\phi_2(x)} f(x,y)\,dy \Big)\,dx.$$

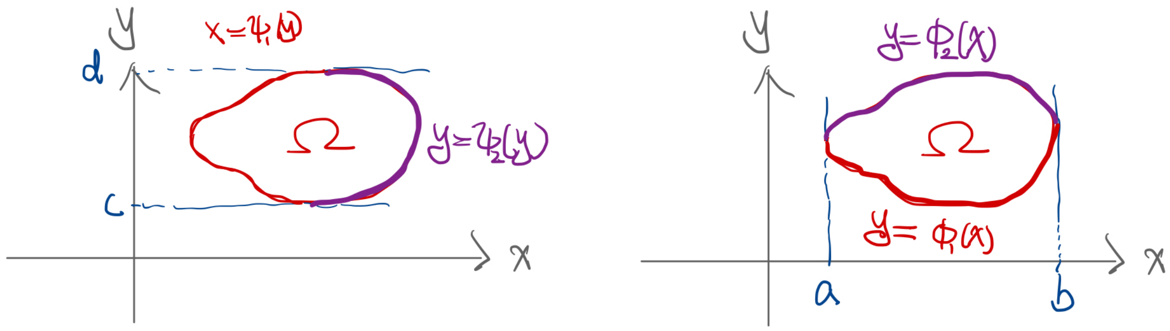


FIGURE 67. Integral region.

Example 13.2. *Evaluate the integrals.*

(1) $\iint_R (x+y-2)\,dxdy$, *where R is the rectangle: $1 \le x \le 4, 1 \le y \le 3$.* *ans:* 15

(2) $\iint_\Omega (xy - y^3)\,dxdy$, *where Ω is the region enclosed by $y = 0,\ y = 1,\ x = -1,\ x = y$.* *Compute it by two methods. ans:* $-\frac{23}{40}$

(3) $\iint_\Omega (x^{1/2} - y^2)\,dxdy$, *where Ω is the region, in the first quadrant, enclosed by $x = y^{1/2}$ and $x = y^2$.* *Compute it by two methods. ans:* $\frac{9}{70}$

Example 13.3. *Use double integration to calculate the area of the region Ω enclosed by*

$$y = x^2 \qquad \text{and} \qquad x + y = 2. \qquad ans: \frac{9}{2}.$$

Example 13.4. *Calculate the volume within the cylinder $x^2 + y^2 = 1$ between the planes $y + z = 2$ and $z = 0$.* *ans: 2π*

13.2. **Triple integrals.** Let f be a function on $\mathbb{R}^3$. Similar as double integrals, if T is a subset in $\mathbb{R}^3$, one can define the *triple integral*

$$\iiint_T f(x, y, z)\, dxdydz \tag{26}$$

by a limit of Riemann sums. Geometrically, such an integral computes a "4-dimensional volume." Alternatively, if one consider $f(x, y, z)$ as a density function, then the triple integral (26) computes the weight of T. In particular, if $f(x, y, z)$ is the constant function 1, then the triple integral

$$\iiint_T dxdydz = \iiint_T 1\, dxdydz$$

gives the volume of T.

Similar as double integrals, triple integrals can be computed by repeated integrals.

Example 13.5. *Compute the volume of the tetrahedron T shown in Figure 68.*

Solution. The volume is

$$\begin{aligned}\iiint_T dxdydz &= \int_0^1 \int_0^{1-z} \int_0^{1-y-z} dxdydz \\ &= \int_0^1 \int_0^{1-z} (1 - y - z)\, dydz \\ &= \int_0^1 (1 - z)(1 - z) - \frac{1}{2}(1 - z)^2\, dz \\ &= -\frac{1}{6}(1 - z)^3 \Big|_0^1 = \frac{1}{6}.\end{aligned}$$

□

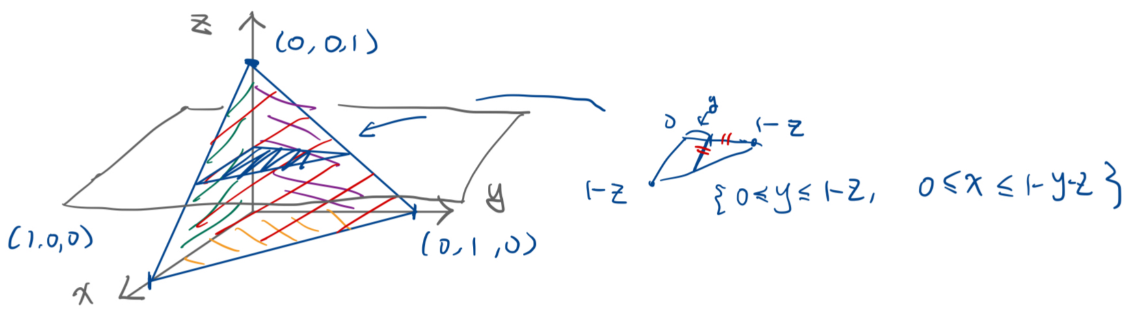


Figure 68. Tetrahedron.

Example 13.6. *Integrate $f(x, y, z) = xy$ over the first-octant solid T bounded by the coordinate planes and the ellipsoid*

$$x^2 + \frac{y^2}{4} + \frac{z^2}{9} = 1.$$

Solution. The solid T can be rephrased as

$$T = \{(x, y, z) \in \mathbb{R}^3 \mid 0 \le z \le 3,\ 0 \le y \le \phi(z),\ 0 \le x \le \psi(y, z)\},$$

where

$$\phi(z) = 2\sqrt{1 - \frac{z^2}{9}} \qquad \text{and} \qquad \psi(y, z) = \sqrt{1 - \frac{y^2}{4} - \frac{z^2}{9}}.$$

Thus,

$$\iiint_T xy\, dxdydz = \int_0^3 \int_0^{\phi(z)} \int_0^{\psi(y,z)} xy\, dxdydz.$$

By a straightforward computation, the values of this integral is $\frac{4}{5}$. □

13.3. **Changing variables and Jacobians.** Recall that we have a changing-variable formula for integration (integration by substitution): If f is a continuous single-variable function, and u is differentiable, then

$$\int_a^b f(u(x)) \cdot u'(x)\, dx = \int_{u(a)}^{u(b)} f(u) du.$$

For the case of functions of two variables, the formula is summarized in the following theorem.

Theorem 13.7 ([3, (17.10.2)])**.** *Suppose that*

$$x = x(u, v) \qquad \textit{and} \qquad y = y(u, v)$$

are smooth functions that map a region $(u, v) \in \Gamma$ to a region $(x, y) \in \Omega$ bijectively, and the ***Jacobian***

$$J(u, v) = \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{pmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}$$

is nonzero on Γ. Then

$$\iint_\Omega f(x, y)\, dxdy = \iint_\Gamma f(u, v)\, |J(u, v)|\, dudv, \tag{27}$$

where $f(u, v) = f(x(u, v), y(u, v))$.

Geometrically, one can consider $dxdy$ as a infinitesimal unit area, and the transfer from from (x, y) to (u, v) changes the infinitesimal area by the Jacobian. In a simple case, one can see this phenomena by observing that the area of a parallelogram is given by the absolute value of the determinant. Draw a picture.

Example 13.8. *Evaluate*

$$\iint_\Omega (x + y)^2\, dxdy$$

where Ω is the parallelogram bounded by the lines

$$x + y = 0, \qquad x + y = 1, \qquad 2x - y = 0, \qquad 2x - y = 3.$$

Solution. Set

$$u = x + y, \qquad v = 2x - y,$$

that is,

$$x = \frac{u+v}{3}, \qquad y = \frac{2u-v}{3}.$$

This transformation gives a one-to-one correspondence between the region Ω and the region Γ, where

$$\Gamma = \{(u,v) \in \mathbb{R}^2 \mid 0 \le u \le 1, 0 \le v \le 3\}.$$

Since the Jacobian $J(u,v)$ is

$$J(u,v) = \det\begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{pmatrix} = -\frac{1}{3},$$

we have

$$\iint_\Omega (x+y)^2\, dxdy = \iint_\Gamma \Big(\frac{u+v}{3} + \frac{2u-v}{3}\Big)^2 |J(u,v)|\, dudv$$
$$= \frac{1}{3}\int_0^3 \int_0^1 u^2\, dudv = \frac{1}{3}.$$

□

Let

$$x = r\cos\theta, \qquad y = r\sin\theta$$

be the transformation between the rectangular coordinates and the polar coordinates. The Jacobian of this transformation is

$$J(r,\theta) = \det\begin{pmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{pmatrix} = r.$$

By Theorem 13.7,

$$\iint_\Omega f(x,y)\, dxdy = \iint_\Gamma f(r\cos\theta, r\sin\theta)\, r\, drd\theta.$$

Example 13.9. *The area of the region Ω, in the first quadrant, bounded by*

$$x^2+y^2 = 1, \qquad x^2+y^2 = 2^2, \qquad x = 0, \qquad y = 0,$$

is

$$\iint_\Omega dxdy = \int_0^{\pi/2}\int_1^2 r\, drd\theta = \frac{\pi}{2}\cdot\frac{2^2-1^2}{2} = \frac{3\pi}{4}.$$

For the case of functions of three variables, we have the following parallel theorem.

Theorem 13.10 ([3, Page 934])**.** *Suppose that*

$$x = x(u,v,w), \qquad y = y(u,v,w), \qquad z = z(u,v,w)$$

are smooth functions that map a solid $(u,v,w) \in \Gamma$ to a solid $(x,y,z) \in T$ bijectively, and the ***Jacobian***

$$J(u,v,w) = \det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \\ \frac{\partial x}{\partial w} & \frac{\partial y}{\partial w} & \frac{\partial z}{\partial w} \end{pmatrix}$$
$$= \frac{\partial x}{\partial u}\cdot\det\begin{pmatrix} \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \\ \frac{\partial y}{\partial w} & \frac{\partial z}{\partial w} \end{pmatrix} - \frac{\partial x}{\partial v}\cdot\det\begin{pmatrix} \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial y}{\partial w} & \frac{\partial z}{\partial w} \end{pmatrix} + \frac{\partial x}{\partial w}\cdot\det\begin{pmatrix} \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{pmatrix}$$

is nonzero on Γ*. Then*

$$\iiint_T f(x,y,z)\,dxdydz = \iiint_\Gamma f(u,v,w)\,|J(u,v,w)|\,dudvdw, \tag{28}$$

where $f(u,v,w) = f(x(u,v,w), y(u,v,w), z(u,v,w))$.

Example 13.11. *Consider the transformation*

$$x = r\sin\phi\cos\theta, \qquad y = r\sin\phi\sin\theta, \qquad z = r\cos\phi$$

between the rectangular coordinates and the spherical coordinates*. Draw a picture. The Jacobian is*

$$\begin{aligned} J(r,\phi,\theta) &= \det\begin{pmatrix} \sin\phi\cos\theta & \sin\phi\sin\theta & \cos\phi \\ r\cos\phi\cos\theta & r\cos\phi\sin\theta & -r\sin\phi \\ -r\sin\phi\sin\theta & r\sin\phi\cos\theta & 0 \end{pmatrix} \\ &= \sin\phi\cos\theta(r^2\sin^2\phi\cos\theta) - r\cos\phi\cos\theta(-r\sin\phi\cos\theta\cos\phi) \\ &\quad - r\sin\phi\sin\theta(-r\sin^2\phi\sin\theta - r\cos^2\phi\sin\theta) \\ &= r^2\sin^3\phi + r^2\sin\phi\cos^2\phi \\ &= r^2\sin\phi. \end{aligned}$$

Thus, the volume of the ball

$$B = \{(x,y,z)\in\mathbb{R}^3 \mid x^2+y^2+z^2 \le 1\}$$

is

$$\begin{aligned} \iiint_B dxdydz &= \int_0^{2\pi}\int_0^{\pi}\int_0^1 r^2\sin\phi\,drd\phi d\theta \\ &= \frac{1^3}{3}\int_0^{2\pi}(-\cos\phi)\Big|_0^{\pi} d\theta \\ &= \frac{4}{3}\pi. \end{aligned}$$

Remark 13.12 (Techniques for Computation)**.** *Recall that when we integrate single-variable functions by substitution, we substitute by the rule*

$$du = u'(x)dx.$$

Jacobians also can be obtained by similar computation rules:

Step 1. Substitute dx, dy, dz *by*

$$\begin{aligned} dx &= \frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv + \frac{\partial x}{\partial w}dw, \\ dy &= \frac{\partial y}{\partial u}du + \frac{\partial y}{\partial v}dv + \frac{\partial y}{\partial w}dw, \\ dz &= \frac{\partial z}{\partial u}du + \frac{\partial z}{\partial v}dv + \frac{\partial z}{\partial w}dw. \end{aligned}$$

Step 2. Simplify the result by the rules:

$$d\xi d\eta = -d\eta d\xi \qquad \text{and} \qquad d\xi d\xi = 0,$$

for $\xi, \eta \in \{x, y, z\}$.

Step 3. Equation 28 reads

$$\iiint_T f(x,y,z)\,dxdydz = \iiint_\Gamma f(u,v,w)\,|dxdydz|.$$

Warning: *The assumptions of bijectivity and nonzero Jacobian are still necessary by this technique. If the assumptions are not satisfied, then Equation 28 might be wrong.*

For example, Example 13.9 can be solved by the computation:

$$\begin{aligned}\iint_\Omega dxdy &= \iint_\Gamma |(\cos\theta dr - r\sin\theta d\theta)(\sin\theta dr + r\cos\theta d\theta)| \\ &= \iint_\Gamma |\cos\theta\cdot r\cos\theta\, drd\theta - r\sin\theta\cdot\sin\theta\, d\theta dr| \\ &= \int_0^{\pi/2}\int_1^2 r\,drd\theta.\end{aligned}$$

13.4. **Green's theorem — a two-dimensional fundamental theorem of calculus.** Recall that the most important theorem for computing integrals is the fundamental theorem of calculus:

$$\int_a^b f'(x)\,dx = f(b) - f(a).$$

To obtain a high-dimensional analogue of this theorem, observe that the left-hand-side can be considered as an integration of f over the closed interval $[a, b]$, and the right-hand-side can be considered as an integration of f over the boundary points a and b with proper signs. In other words, the fundamental theorem of calculus can be considered as a way to reduce a one-dimensional integral to a zero-dimensional integral (integral over boundaries). This observation will lead us to see

- Green's theorem — a two-dimensional analogue of the fundamental theorem of calculus,
- divergence theorem — a three-dimensional analogue of the fundamental theorem of calculus.

The boundary of a two-dimensional region is generally a curve. To order to describe Green's theorem, we need

(1) integration over a curve (the boundary of a region),
(2) reasonable assumptions on the integral region.

Definition 13.13 ([3, Definition 18.1.3])**.** *Let*

$$C: \quad \vec{\gamma} = \vec{\gamma}(t) = (\gamma_1(t), \gamma_2(t)), \qquad t \in [a, b],$$

be a smooth parametrized curve. Suppose $P(x, y)$ and $Q(x, y)$ are two smooth function on $\mathbb{R}^2$. The ***line integral***

$$\int_C P(x,y)\,dx + Q(x,y)\,dy \tag{29}$$

over C is defined to be

$$\begin{aligned}\int_C P(x,y)\,dx + Q(x,y)\,dy &= \int_a^b P(\gamma_1(t),\gamma_2(t))\,d\gamma_1 + Q(\gamma_1(t),\gamma_2(t))\,d\gamma_2\\ &= \int_a^b \Big(P(\gamma_1(t),\gamma_2(t))\,\gamma_1'(t) + Q(\gamma_1(t),\gamma_2(t))\,\gamma_2'(t)\Big)\,dt\\ &= \int_a^b \Big(P(\vec{\gamma}(t))\,\vec{\imath} + Q(\vec{\gamma}(t))\,\vec{\jmath}\Big)\cdot\vec{\gamma}'(t)\,dt.\end{aligned}$$

Remark 13.14. *The line integral (29) can be interpreted as the integration of the "vector field"*

$$P(x,y)\,\vec{\imath} + Q(x,y)\,\vec{\jmath}$$

over the curve C. Draw a picture of a vector field. See [3, Section 18.1].

Note that a curve can be parametrized in different ways. Actually, one can choose any parametrization with the same direction.

Theorem 13.15 ([3, Theorem 18.1.3])**.** *The line integral*

$$\int_C P(x,y)\,dx + Q(x,y)\,dy = \int_a^b \Big(P(\vec{\gamma}(t))\,\vec{\imath} + Q(\vec{\gamma}(t))\,\vec{\jmath}\Big)\cdot\vec{\gamma}'(t)\,dt$$

is invariant by every direction-preserving *change of parameter. In other words, if*

$$\phi : [c,d] \to [a.b]$$

is an increasing smooth function such that $\phi'(s) > 0$ for all $s \in (c,d)$, and if

$$\vec{\alpha}(s) = \vec{\gamma}(\phi(s)), \qquad s \in [c,d],$$

which is another parametrization of C with the same direction, then

$$\int_a^b \Big(P(\vec{\gamma}(t))\,\vec{\imath} + Q(\vec{\gamma}(t))\,\vec{\jmath}\Big)\cdot\vec{\gamma}'(t)\,dt = \int_c^d \Big(P(\vec{\alpha}(s))\,\vec{\imath} + Q(\vec{\alpha}(s))\,\vec{\jmath}\Big)\cdot\vec{\alpha}'(s)\,ds.$$

Proof. By the chain rule $(t = \phi(s))$,

$$\begin{aligned}\int_a^b \Big(P(\vec{\gamma}(t))\,\vec{\imath} + Q(\vec{\gamma}(t))\,\vec{\jmath}\Big)\cdot\vec{\gamma}'(t)\,dt &= \int_c^d \Big(P(\vec{\gamma}(\phi(s)))\,\vec{\imath} + Q(\vec{\gamma}(\phi(s)))\,\vec{\jmath}\Big)\cdot\vec{\gamma}'(\phi(s))\phi'(s)\,ds\\ &= \int_c^d \Big(P(\vec{\alpha}(s))\,\vec{\imath} + Q(\vec{\alpha}(s))\,\vec{\jmath}\Big)\cdot\vec{\alpha}'(s)\,ds\end{aligned}$$

as desired. □

Nevertheless, the direction of the parametrized curve is important in the computation of line integral. Changing direction actually gives a minus sign to the line integral.

Theorem 13.16. *If*

$$\psi : [c,d] \to [a.b]$$

is a decreasing smooth function such that $\psi'(s) < 0$ for all $s \in (c,d)$, and if

$$\vec{\beta}(s) = \vec{\gamma}(\psi(s)), \qquad s \in [c,d],$$

which is another parametrization of C with the reversed direction*, then*

$$\int_a^b \left(P(\vec{\gamma}(t))\,\vec{\imath} + Q(\vec{\gamma}(t))\,\vec{\jmath}\right) \cdot \vec{\gamma}'(t)\,dt = -\int_c^d \left(P(\vec{\beta}(s))\,\vec{\imath} + Q(\vec{\beta}(s))\,\vec{\jmath}\right) \cdot \vec{\beta}'(s)\,ds.$$

Proof. Again, apply the chain rule. Note that $\psi(c) = b$ and $\psi(d) = a$ in this setting. □

Example 13.17. *Let C be the curve parametrized by*

$$\vec{\gamma}(t) = \sin t\,\vec{\imath} + \cos t\,\vec{\jmath}, \qquad t \in [0, \pi].$$

Note that

$$\vec{\beta}(s) = \vec{\gamma}(\pi - s) = \sin s\,\vec{\imath} - \cos s\,\vec{\jmath}, \qquad s \in [0, \pi]$$

is another parametrization of C with the reversed direction. Let us denote by C_γ the curve C with the γ-direction and by C_β the curve C with the β-direction. Then we have

$$\begin{aligned}\int_{C_\gamma} y\,dx - x\,dy &= \int_0^\pi (\cos t\,\vec{\imath} - \sin t\,\vec{\jmath}) \cdot (\cos t\,\vec{\imath} - \sin t\,\vec{\jmath})\,dt \\ &= \int_0^\pi 1\,dt = \pi,\end{aligned}$$

and

$$\begin{aligned}\int_{C_\beta} y\,dx - x\,dy &= \int_0^\pi (-\cos s\,\vec{\imath} - \sin s\,\vec{\jmath}) \cdot (\cos s\,\vec{\imath} + \sin s\,\vec{\jmath})\,ds \\ &= \int_0^\pi (-1)\,dt = -\pi,\end{aligned}$$

Definition 13.18. *A curve in $\mathbb{R}^2$*

$$C: \quad \vec{\gamma} = \vec{\gamma}(t), \qquad t \in [a, b],$$

*is called **closed** if $\vec{\gamma}(a) = \vec{\gamma}(b)$, and it is called **simple** if it does not intersect itself: $a < t_1 \neq t_2 < b$ implies $\vec{\gamma}(t_1) \neq \vec{\gamma}(t_2)$. A **Jordan curve** is a curve in $\mathbb{R}^2$ which is both closed and simple. A closed region Ω whose total boundary is a Jordan curve is called a **Jordan region**.*

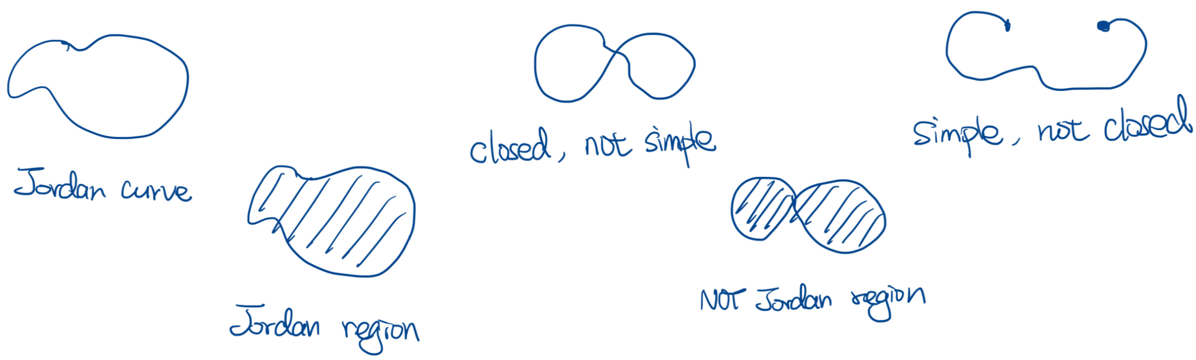


FIGURE 69. Jordan region.

Theorem 13.19 (Green's Theorem, [3, Theorem 18.5.1])**.** *Let Ω be a Jordan region with a piecewise-smooth boundary C. If P and Q are smooth functions on Ω (i.e. smooth in a neighborhood of Ω),* *then*

$$\iint_{\Omega} \left(\frac{\partial Q}{\partial x}(x,y) - \frac{\partial P}{\partial y}(x,y)\right) dxdy = \oint_C P(x,y)\,dx + Q(x,y)\,dy, \tag{30}$$

where the integral on the right is the line integral taken over C in the counter-clockwise direction.

Example 13.20. *Use Green's theorem to evaluate*

$$\oint_C (3x^2 + y)\,dx + (2x + y^3)\,dy$$

where C is the circle $x^2 + y^2 = 3^2$. *ans:* 9π

Example 13.21. *Evaluate*

$$\oint_C e^x \sin y\,dx + e^x \cos y\,dy$$

where C is the closed curve consisting of the semicircle $y = \sqrt{1-x^2}$ and the line segment connecting $(-1,0)$ and $(1,0)$. *ans:* 0

Example 13.22. *Let C be a Jordan curve that does not pass through the origin $(0,0)$. Show that*

$$\oint_C -\frac{y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy = \begin{cases} 0 & \text{if } C \text{ dose not enclose the origin,} \\ 2\pi & \text{if } C \text{ dose enclose the origin.} \end{cases}$$

Solution. Assume C enclose the origin. We need a geometric fact: there exists a circle

$$C_a : \quad x^2 + y^2 = a^2$$

that is inside the Jordan curve C. Consider the auxiliary curves as shown in Figure 70.

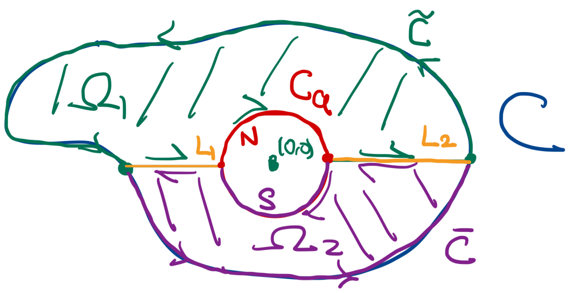


FIGURE 70. Auxiliary curves.

Let $P = -\dfrac{y}{x^2+y^2}$ and $Q = \dfrac{x}{x^2+y^2}$. Since

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) - \frac{\partial}{\partial y}\left(-\frac{y}{x^2+y^2}\right) = 0, \qquad \forall (x,y) \neq (0,0),$$

it follows from Green's theorem that

$$\begin{aligned} 0 &= \oint_{\partial\Omega_1} -\frac{y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy \\ &= \int_{\tilde{C}} (P\,dx + Q\,dy) + \int_{L_{1,r}} (P\,dx + Q\,dy) + \int_N (P\,dx + Q\,dy) + \int_{L_{2,r}} (P\,dx + Q\,dy) \end{aligned}$$

and

$$
\begin{aligned}
0 &= \oint_{\partial\Omega_2} -\frac{y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy \\
&= \int_{\bar{C}} (P\,dx + Q\,dy) + \int_{L_{1,l}} (P\,dx + Q\,dy) + \int_{S} (P\,dx + Q\,dy) + \int_{L_{2,l}} (P\,dx + Q\,dy),
\end{aligned}
$$

where $\partial\Omega_i$ is the boundary of Ω_i, $L_{i,r}$ is the line segment L_i equipped with the right direction, and $L_{i,l}$ is the line segment L_i equipped with the left direction. Since

$$\int_{L_{i,l}} P\,dx + Q\,dy = -\int_{L_{i,r}} P\,dx + Q\,dy,$$

we have

$$
\begin{aligned}
0 &= \oint_{\partial\Omega_1} P\,dx + Q\,dy + \oint_{\partial\Omega_2} P\,dx + Q\,dy \\
&= \int \tilde{C} P\,dx + Q\,dy + \int \hat{C} P\,dx + Q\,dy + \int N P\,dx + Q\,dy + \int S\,P\,dx + Q\,dy \\
&= \oint_{C} P\,dx + Q\,dy - \oint_{C_a} P\,dx + Q\,dy.
\end{aligned}
$$

By the parametrization

$$\vec{\gamma}(t) = a\cos t\,\vec{i} + a\sin t\,\vec{j}, \qquad t \in [0, 2\pi]$$

of C_a, we have

$$
\begin{aligned}
\oint_{C} P\,dx + Q\,dy = \oint_{C_a} P\,dx + Q\,dy \\
&= \int_0^{2\pi} P(a\cos t, a\sin t)\,(-a\sin t) + Q(a\cos t, a\sin t)\,(a\cos t)\,dt \\
&= \int_0^{2\pi} -\frac{a\sin t}{a^2}\,(-a\sin t) + \frac{a\cos t}{a^2}\,(a\cos t)\,dt \\
&= 2\pi.
\end{aligned}
$$

If C does not enclose the origin, one can apply the Green's theorem to show that

$$\oint_{C} P\,dx + Q\,dy = 0$$

since P and Q are smooth on C and inside C, and since $\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y} = 0$. □

Remark 13.23. *The formula $\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y}$ can be rephrased as*

$$\iint_{\Omega} d(P\,dx + Q\,dy) = \oint_{\partial\Omega} P\,dx + Q\,dy,$$

where $\partial\Omega = C$ is the boundary of Ω, and $d(P\,dx + Q\,dy)$ is defined to be

$$\begin{aligned} d(P\,dx + Q\,dy) &= dP\,dx + dQ\,dy \\ &= \left(\frac{\partial P}{\partial x}\,dx + \frac{\partial P}{\partial y}\,dy\right)dx + \left(\frac{\partial Q}{\partial x}\,dx + \frac{\partial Q}{\partial y}\,dy\right)dy \\ &= \frac{\partial P}{\partial y}\,dydx + \frac{\partial Q}{\partial x}\,dxdy \\ &= \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dxdy. \end{aligned}$$

Here we follow the computation rules in Remark 13.12.

Remark 13.24. *Green's theorem also can be applied to the study of electric fields. See* [3, Project 18.9].

13.5. **Divergence theorem — a three-dimensional fundamental theorem of calculus.** We will introduce a three-dimensional fundamental theorem of calculus — divergence theorem. Green's theorem, divergence theorem and other theorems of this type are all known as *Stoke's theorem*. See Theorem 5.4.2, Theorem 18.5.1, Theorem 18.9.2 and Theorem 18.10.1 in [3].

We need surface integrals to describe the divergence theorem.

Definition 13.25. *Let*

$$S: \quad \vec{\gamma}(u,v) = \gamma_1(u,v)\,\vec{\imath} + \gamma_2(u,v)\,\vec{\jmath} + \gamma_3(u,v)\,\vec{k}, \qquad (u,v) \in \Omega \subset \mathbb{R}^2$$

be a smooth parametrized surface. Define the ***surface integral*** *of H over S to be*

$$\iint\limits_S H(x,y,z)\,d\sigma = \iint\limits_\Omega H(\gamma_1(u,v),\gamma_2(u,v),\gamma_3(u,v))\left\|\vec{N}(u,v)\right\|dudv,$$

where

$$\begin{aligned} \vec{N}(u,v) &= \det\begin{pmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \frac{\partial\gamma_1}{\partial u} & \frac{\partial\gamma_2}{\partial u} & \frac{\partial\gamma_3}{\partial u} \\ \frac{\partial\gamma_1}{\partial v} & \frac{\partial\gamma_2}{\partial v} & \frac{\partial\gamma_3}{\partial v} \end{pmatrix} \\ &= \left(\frac{\partial\gamma_2}{\partial u}\frac{\partial\gamma_3}{\partial v} - \frac{\partial\gamma_3}{\partial u}\frac{\partial\gamma_2}{\partial v}\right)\vec{\imath} - \left(\frac{\partial\gamma_1}{\partial u}\frac{\partial\gamma_3}{\partial v} - \frac{\partial\gamma_3}{\partial u}\frac{\partial\gamma_1}{\partial v}\right)\vec{\jmath} + \left(\frac{\partial\gamma_1}{\partial u}\frac{\partial\gamma_2}{\partial v} - \frac{\partial\gamma_2}{\partial u}\frac{\partial\gamma_1}{\partial v}\right)\vec{k}. \end{aligned}$$

Remark 13.26. *Sometime a surface S can be expressed as a disjoint union of subsets $S_1, \cdots, S_k$ such that each S_i is parametrized by a smooth function $\vec{\gamma}(u,v)$. In this case, we define*

$$\iint\limits_S H(x,y,z)\,d\sigma = \iint\limits_{S_1} H(x,y,z)\,d\sigma + \cdots + \iint\limits_{S_k} H(x,y,z)\,d\sigma.$$

Example 13.27. *Let*

$$S^2 = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$$

be the unit sphere. Note that

$$S^2 = S^2_{\geq 0} \cup S^2_{<0},$$

where

$$S^2_{\geq 0} = S^2 \cap \{z \geq 0\} \qquad \text{and} \qquad S^2_{<0} = S^2 \cap \{z < 0\}.$$

The functions

$$\vec{\gamma}(u,v) = u\,\vec{\imath} + v\,\vec{\jmath} + \sqrt{1-u^2-v^2}\,\vec{k}, \qquad \forall u,v \in \overline{D} = \{u^2+v^2 \le 1\}$$

and

$$\vec{\beta}(u,v) = u\,\vec{\imath} + v\,\vec{\jmath} - \sqrt{1-u^2-v^2}\,\vec{k}, \qquad \forall u,v \in D = \{u^2+v^2 < 1\}$$

parametrize $S^2_{\ge 0}$ *and* $S^2_{<0}$*, respectively. Note that*

$$\begin{aligned}\vec{N}_\gamma(u,v) &= \det\begin{pmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 1 & 0 & -u\Big/\sqrt{1-u^2-v^2} \\ 0 & 1 & -v\Big/\sqrt{1-u^2-v^2}\end{pmatrix} \\ &= \left(u\Big/\sqrt{1-u^2-v^2}\right)\vec{\imath} + \left(v\Big/\sqrt{1-u^2-v^2}\right)\vec{\jmath} + \vec{k}\end{aligned}$$

and

$$\begin{aligned}\vec{N}_\beta(u,v) &= \det\begin{pmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 1 & 0 & u\Big/\sqrt{1-u^2-v^2} \\ 0 & 1 & v\Big/\sqrt{1-u^2-v^2}\end{pmatrix} \\ &= \left(-u\Big/\sqrt{1-u^2-v^2}\right)\vec{\imath} - \left(v\Big/\sqrt{1-u^2-v^2}\right)\vec{\jmath} + \vec{k}.\end{aligned}$$

Thus, the surface integral of the constant function 1 *over* S^2 *is*

$$\begin{aligned}\iint\limits_{S^2} 1\,d\sigma &= \iint\limits_{S^2_{\ge 0}} 1\,d\sigma + \iint\limits_{S^2_{<0}} 1\,d\sigma \\ &= \iint\limits_{\overline{D}} \frac{1}{\sqrt{1-u^2-v^2}}\,dudv + \iint\limits_{D} \frac{1}{\sqrt{1-u^2-v^2}}\,dudv.\end{aligned}$$

Substituting

$$u = r\cos\theta \qquad \text{and} \qquad v = r\sin\theta \qquad (\text{Jacobian is } r),$$

we have

$$\iint\limits_{\overline{D}} \frac{1}{\sqrt{1-u^2-v^2}}\,dudv = \int_0^{2\pi}\int_0^1 \frac{1}{\sqrt{1-r^2}}\,r\,drd\theta = -2\pi(1-r^2)^{1/2}\Big|_{r=0}^{1} = 2\pi.$$

Therefore,

$$\iint\limits_{S^2} 1\,d\sigma = 2\pi + 2\pi = 4\pi$$

which is the surface area of the unit sphere S^2*. (In general, the surface area of a sphere of radius r is* $4\pi r^2$*.)*

Remark 13.28. *In general, the surface area of a smooth surface S is given by the surface integral*

$$\iint\limits_{S} 1\,d\sigma.$$

Definition 13.29 ([3, (18.8.2)])**.** *Let*

$$\vec{v}(x,y,z) = v_1(x,y,z)\,\vec{\imath} + v_2(x,y,z)\,\vec{\jmath} + v_3(x,y,z)\,\vec{k}$$

*be a **vector field** on $\mathbb{R}^3$, i.e. a triple of smooth functions on $\mathbb{R}^3$. The **divergence** of $\vec{v}$ is the smooth function*

$$\operatorname{div} \vec{v} = \nabla \cdot \vec{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}.$$

The divergence of a vector field is the sum of the speed of v_1 in the x-axis direction, v_2 in the y-axis direction and v_3 in the z-axis direction. It roughly measures how fast a vector field leaving the center point.

Theorem 13.30 (Divergence Theorem, [3, Theorem 18.9.2])**.** *Let T be a solid bounded by a closed piecewise smooth surface S. If $\vec{v} = \vec{v}(x, y, z)$ is a smooth vector field on T, then*

$$\iiint_T \operatorname{div} \vec{v}\, dxdydz = \iint_S \vec{v} \cdot \vec{n}\, d\sigma, \tag{31}$$

where $\vec{n} = \vec{n}(x, y, z)$ is the outer unit normal vector *of S.*

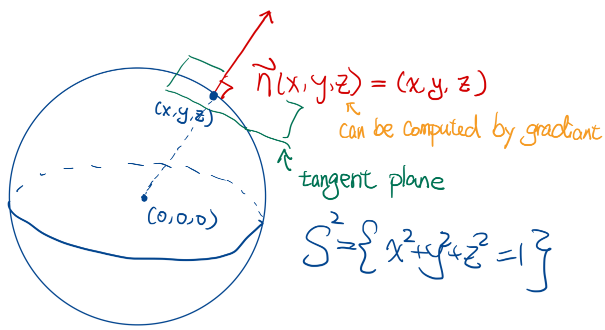


FIGURE 71. Outer unit normal vector of sphere.

Example 13.31. *Let $B = \{x^2 + y^2 + z^2 \leq 1\}$ whose boundary is the unit sphere $S^2 = \{x^2 + y^2 + z^2 = 1\}$. Consider the vector field*

$$\vec{v} = (x, y, z).$$

Then

$$\operatorname{div} \vec{v} = 1 + 1 + 1 = 3, \qquad \vec{v} \cdot \vec{n} = (x, y, z) \cdot (x, y, z) = 1,$$

and by the divergence theorem,

$$\iiint_B 3\, dxdydz = \iint_{S^2} 1\, d\sigma = 4\pi.$$

Thus, the volume of the unit ball B is

$$\iiint_B 1\, dxdydz = \frac{4}{3}\pi.$$

This is another way to compute the volume of the unit ball. See Example 13.11.

Remark 13.32. *In fact, the divergence theorem also can be formulated in a way similar to Equation (30):*

$$\iiint_T \operatorname{div} \vec{v}\, dxdydz = \iint_S v_1(x, y, z)\, dydz + v_2(x, y, z)\, dzdx + v_3(x, y, z)\, dxdy,$$

where the "orientation" of S is chosen in a such way that it gives the standard orientation of $\mathbb{R}^3$ with the outer normal vector of S. Here, if

$$S: \quad \vec{\gamma}(u,v) = \gamma_1(u,v)\,\vec{\imath} + \gamma_2(u,v)\,\vec{\jmath} + \gamma_3(u,v)\,\vec{k}, \qquad (u,v) \in \Omega \subset \mathbb{R}^2$$

is a smooth parametrization of S, then

$$\iint_S v_1(x,y,z)\,dydz + v_2(x,y,z)\,dzdx + v_3(x,y,z)\,dxdy$$
$$= \iint_\Omega v_1(\gamma_1(u,v),\gamma_2(u,v),\gamma_3(u,v))\,d\gamma_2 d\gamma_3 + v_2(\gamma_1(u,v),\gamma_2(u,v),\gamma_3(u,v))\,d\gamma_3 d\gamma_1$$
$$+ v_3(\gamma_1(u,v),\gamma_2(u,v),\gamma_3(u,v))\,d\gamma_1 d\gamma_2.$$

One can find more details by searching for "differential form" and "Stoke's theorem."

13.6. **Stoke's theorem — a Green's theorem in a 3-dimensional space.** Recall Green's theorem Equation (30):

$$\iint_\Omega \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = \oint_C P\,dx + Q\,dy.$$

In the formula, the integral region Ω lies in $\mathbb{R}^2$, but what if it is a surface in $\mathbb{R}^3$. To formulate a more general version of Green's theorem, we introduce the following operation of vector fields.

Definition 13.33 ([3, (18.8.3)])**.** *Let*

$$\vec{v}(x,y,z) = v_1(x,y,z)\,\vec{\imath} + v_2(x,y,z)\,\vec{\jmath} + v_3(x,y,z)\,\vec{k}$$

*be a vector field on $\mathbb{R}^3$. The **curl** of $\vec{v}$ is the vector field*

$$\operatorname{curl}\vec{v} = \nabla \times \vec{v} = \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right)\vec{\imath} + \left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right)\vec{\jmath} + \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)\vec{k}.$$

Example 13.34. *If $\vec{v} = -y\,\vec{\imath} + x\,\vec{\jmath}$, then $\operatorname{div}\vec{v} = 0$ and $\operatorname{curl}\vec{v} = 2\,\vec{k}$. Draw a picture.*

Theorem 13.35 (Stokes' Theorem, [3, Theorem 18.10.1])**.** *Let S be a smooth oriented surface with a (piecewise) smooth bounding curve C in $\mathbb{R}^3$. If $\vec{v}$ is a smooth vector field on $\mathbb{R}^3$, then*

$$\iint_S \operatorname{curl}\vec{v}\cdot\vec{n}\,d\sigma = \int_C \vec{v}(r)\cdot dr, \tag{32}$$

*where $\vec{n} = \vec{n}(x,y,z)$ is a smooth unit normal vector (i.e. **orientation**) of S and the line integral is taken in the positive sense with respect to $\vec{n}$.*

In particular, if $C = \emptyset$ (such as $S = \{x^2 + y^2 + z^2 = 1\}$), then $\displaystyle\iint_S \operatorname{curl}\vec{v}\cdot\vec{n}\,d\sigma = 0.$

Example 13.36. *Let*

$$\vec{v} = z^2\,\vec{\imath} - 2x\,\vec{\jmath} + y^3\,\vec{k},$$

and S be the upper half of the unit sphere $x^2 + y^2 + z^2 = 1$. Then $\vec{n} = x\,\vec{\imath} + y\,\vec{\jmath} + z\,\vec{k}$ is a unit normal vector of S. The curl of $\vec{v}$ is

$$\operatorname{curl}\vec{v} = 3y^2\,\vec{\imath} + 2z\,\vec{\jmath} - 2\,\vec{k}.$$

Therefore,

$$\iint_S \operatorname{curl}\vec{v}\cdot\vec{n}\,d\sigma = \iint_S 3xy^2 + 2yz - 2z\,d\sigma.$$

We apply the parametrization $\vec{\gamma}$ in Example 13.27:

$$\begin{aligned}\iint_S 3xy^2 + 2yz - 2z\,d\sigma &= \iint_{\overline{D}} \left(3uv^2 + 2v\sqrt{1-u^2-v^2} + 2\sqrt{1-u^2-v^2}\right)\left\|\vec{N}_\gamma\right\| dudv \\ &= \iint_{\overline{D}} \frac{3uv^2}{\sqrt{1-u^2-v^2}}\,dudv + \iint_{\overline{D}} 2v\,dudv - \iint_{\overline{D}} 2\,dudv.\end{aligned}$$

The first term vanishes since the integrand is odd respect to u, and the second term also vanishes since the integrand is odd with respect to v. Consequently,

$$\iint_S \operatorname{curl}\vec{v}\cdot\vec{n}\,d\sigma = -\iint_{\overline{D}} 2\,dudv = -2\pi.$$

On the other hand, the boundary C of $\overline{D}$ can be parametrized by

$$C: \qquad \cos\theta\,\vec{\imath} + \sin\theta\,\vec{\jmath}, \quad \theta \in [0, 2\pi],$$

and

$$\begin{aligned}\int_C \vec{v}(r)\cdot dr &= \int_0^{2\pi} (-2\cos\theta\,\vec{\jmath} + (\sin\theta)^3\,\vec{k})\cdot(-\sin\theta\,\vec{\imath} + \cos\theta\,\vec{\jmath})\,d\theta \\ &= \int_0^{2\pi} -2\cos^2\theta\,d\theta = -2\pi.\end{aligned}$$

Note that the parametrization is chosen to be counterclockwise since the normal vector $\vec{n}$ points upward.

Remark 13.37. *The Green's theorem, divergence theorem and Stoke's theorem can be unified by the theory of differential forms. In particular, consider $\vec{v}$ as the one-form $v_1dx + v_2dy + v_3dz$, then its exterior differential gives the curl:*

$$\begin{aligned}&d(v_1dx + v_2dy + v_3dz) \\ &\quad= \left(\frac{\partial v_1}{\partial x}dx + \frac{\partial v_1}{\partial y}dy + \frac{\partial v_1}{\partial z}dz\right)dx + \left(\frac{\partial v_2}{\partial x}dx + \frac{\partial v_2}{\partial y}dy + \frac{\partial v_2}{\partial z}dz\right)dy + \left(\frac{\partial v_3}{\partial x}dx + \frac{\partial v_3}{\partial y}dy + \frac{\partial v_3}{\partial z}dz\right)dz \\ &\quad= \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right)dydz + \left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right)dzdx + \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)dxdy.\end{aligned}$$

Part 2. Advanced Calculus

Appendix A. Applications of integral

A.1. Area and arc length.

Remark A.1 ([1, Page 369, Section 5.6], [3, Section 6.1])**.** *If f and g are continuous with $f(x) \geq g(x)$ for all $x \in [a, b]$, then the area A of the region between the curves $y = f(x)$ and $y = g(x)$ from a to b is the integral*

$$A = \int_a^b (f(x) - g(x))\, dx.$$

Example A.2. *Find the area of the region bounded by the graphs of $f(x) = x + 2$ and $g(x) = x^2$.*

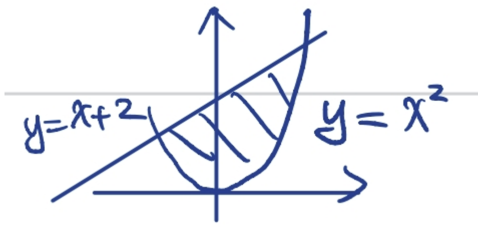


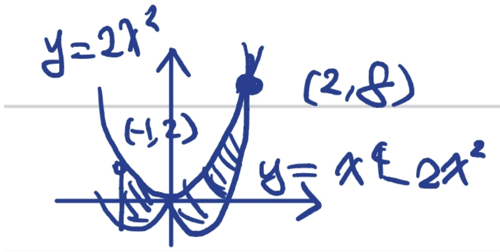
Figure 72. Region bounded by the graphs of $f(x) = x + 2$ and $g(x) = x^2$

Solution. Since the curves $y = f(x)$ and $y = g(x)$ intersect at the points $(-1, 1)$ and $(2, 4)$, the area is

$$\int_{-1}^2 (x + 2) - x^2\, dx = (2x + \frac{1}{2}x^2 - \frac{1}{3}x^3)\Big|_{-1}^2 = 4 + 2 - \frac{8}{3} + 2 - \frac{1}{2} - \frac{1}{3} = \frac{9}{2}.$$

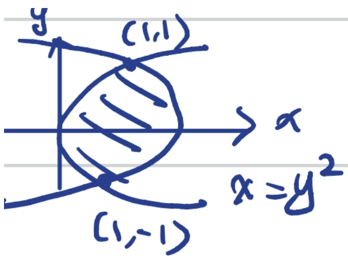
□

Example A.3. *Find the area of the region bounded by the curves $y = 2x^2$, $y = x^4 - 2x^2$ and the line $x = -1$.*



Solution. Area $= \int_{-1}^2 2x^2 - (x^4 - 2x^2)\, dx = \int_{-1}^2 (4x^2 - x^4)\, dx = (\frac{4}{3}x^3 - \frac{1}{5}x^5)\Big|_{-1}^2 = \frac{27}{5}.$ □

Example A.4. *Find the area of the region bounded by the curves $x = 3 - 2y^2$ and $x = y^2$.*



Solution. Area $= \int_{-1}^1 -y^2 + (3 - 2y^2)\, dy = 3y - y^3\Big|_{-1}^1 = 4.$ □

Remark A.5 ([1, Page 404, Section 6.3])**.** *If f' is continuous on $[a,b]$, then the **(arc) length** L of the curve $y = f(x)$ from $(a, f(a))$ to $(b, f(b))$ is*

$$L = \int_a^b \sqrt{1 + (f'(x))^2}\, dx.$$

(By checking a graph, one can see the Riemann sum of this integral is an approximation of the arc length.)

Example A.6. *Find the length of the graph of*

$$y = \frac{2\sqrt{2}}{3} x^{3/2} - 1, \qquad 0 \le x \le 1.$$

Solution. The length is $\displaystyle\int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \frac{1}{3}(3\sqrt{3} - 1).$ □

Remark A.7 ([1, Page 411, Section 6.4])**.** *If the function $f(x) \ge 0$ is continuously differentiable on $[a,b]$, the area A of the surface generated by revolving the graph of $y = f(x)$ about the x-axis is*

$$A = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_a^b 2\pi f(x) \sqrt{1 + (f'(x))^2}\, dx.$$

Example A.8. *Find the lateral surface area of the cone with radius r and height h.*

Solution. Method I. Recall from basic mathematics that the lateral surface area is

$$\pi(r^2 + h^2) \cdot \frac{2\pi r}{2\pi\sqrt{r^2 + h^2}} = \pi r\sqrt{r^2 + h^2}.$$

Method II. By the calculus formula, the lateral surface area is

$$\int_0^h 2\pi \frac{r}{h} x \sqrt{1 + \left(\frac{r}{h}\right)^2}\, dx = \pi \frac{r\sqrt{h^2 + r^2}}{h^2} x^2 \Big|_0^h = \pi r\sqrt{h^2 + r^2}.$$

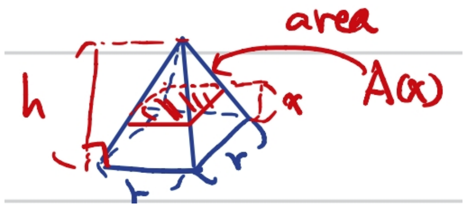
□

A.2. **Volume.**

Remark A.9 ([1, Page 385, Section 6.1], [3, Equation 6.2.1])**.** *The **volume** V of a solid with integrable cross-sectional area $A(x)$ from $x = a$ to $x = b$ is*

$$V = \int_a^b A(x)\, dx.$$

Example A.10. *Find the volume of the pyramid with height h and width r.*



Solution. The area of cross-section in the pyramid is $A(x) = (\frac{h-x}{h} \cdot r)^2$. Thus the volume of pyramid is

$$\int_0^h A(x)\,dx = \int_0^h (\frac{r}{h})^2 \cdot (h-x)^2\,dx = (\frac{r}{h}) \cdot \frac{-1}{3}(h-x)^3 \Big|_0^h = \frac{1}{3}r^2 h.$$

□

Theorem A.11 (Volume of a Solid of Revolution, [1, Page 387, Section 6.1], [3, Equation 6.2.3])**.** *Let f be an integrable function. The volume of the solid of revolution given by rotating $y = f(x)$ about the x-axis from a to b is*

$$\int_a^b A(x)\,dx = \int_a^b \pi(f(x))^2\,dx.$$

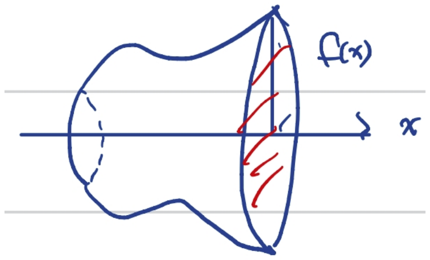


Figure 73. Solid of revolution given by rotating $y = f(x)$ about the x-axis

Example A.12. *Find the volume of cone with radius r and height h.*

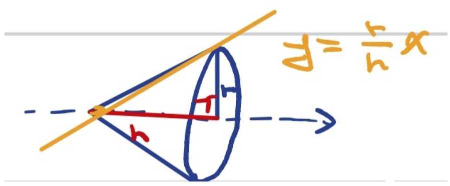


Figure 74. Solid generated by revolving $y = \frac{r}{h}x$ about the x-axis

Solution. To solve this problem, we can see it as the volume of the solid of revolution given by rotating $y = \frac{r}{h}x$ about the x-axis from 0 to h. Thus, the volume is

$$\int_0^h \pi(\frac{r}{h}x)^2\,dx = \frac{\pi r^2}{h^2}\int_0^h x^2\,dx = \frac{\pi r^2 h}{3}.$$

□

Example A.13. *Find the volume of the solid generated by revolving the region between $y = x^2$ and $y = 2x$ (i) about x-axis, and (ii) about y-axis.*

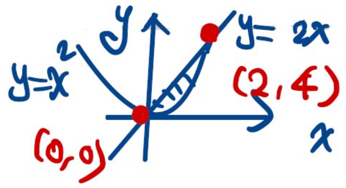


FIGURE 75. Region between $y = x^2$ and $y = 2x$

Solution. Note that the curves $y = 2x$ and $y = x^2$ intersect at the points $(0, 0)$ and $(2, 4)$.

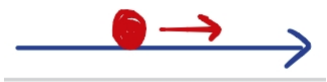
(i) $\int_0^2 (\pi(2x)^2) - (\pi(x^2)^2)\, dx = \pi(\frac{4}{3}x^3 - \frac{1}{5}x^5)\Big|_0^2 = \frac{64}{15}\pi.$

(ii) $\int_0^4 \pi(\sqrt{y})^2 - \pi(\frac{y}{2})^2\, dy = \pi(\frac{1}{2}y^2 - \frac{1}{12}y^3)\Big|_0^4 = \frac{8}{3}\pi.$

□

A.3. **Applications to physics.**

Example A.14. *Suppose an object move with velocity $v(t) = 2 - 3t + t^2$ (m/s), and its initial position is at 2 m ($t = 0$). Find the position of the object at time 4 seconds.*



Solution. Let $x(t)$ be the position of the object at time t seconds. Then

$$x(t) = \int_0^t v(s)\, ds + 2 = \int_0^t 2 - 3s + s^2\, ds + 2 = 2t - \frac{3}{2}t^2 + \frac{1}{3}t^3 + 2.$$

□

Example A.15. *Suppose an object move along the x-axis with acceleration $a(t)$, with velocity $v(t)$ and with position $x(t)$ at time t seconds. Assume $a(t) = 2t - 2$, $v(1) = -4$ and $x(0) = 5$. Find the function $x(t)$ of position at time t seconds.*

Solution. The function of velocity is

$$v(t) = \int_1^t a(s)\, ds - 4 = \int_1^t 2s - 2\, ds - 4 = s^2 - 2s|_1^t - 4 = t^2 - 2t - 3.$$

The function of position is

$$x(t) = \int_0^t v(s)\, ds + 5 = \int_0^t s^2 - 2s - 3\, ds + 5 = \frac{1}{3}t^3 - t^2 - 3t + 5.$$

□

Appendix B. Differential equations

A **differential equation** is an equation which contains unknown functions and their derivative. For example,

$$y'(x) = y(x) \qquad \text{and} \qquad y''(x) = x^2 + y(x)$$

are differential equations.

The **order** of a differential equation is the highest order of derivative in the equation. For example,

1st order: $y'(x) = x$, $xy'(x) = y(x)$,

2nd order: $y''(x) = x$, $xy''(x) - y'(x) + y(x) = 0$.

An **ordinary differential equation (ODE)** is a differential equation which consists of functions of one variable. A **partial differential equation (PDE)** is a differential equation which consists of functions of many variables and partial derivatives.

Remark B.1. *The solutions of the differential equation $y'(x) = f(x)$ are*

$$\int f(x)\,dx = \int_0^x f(t)\,dt + C,$$

where C is an arbitrary constant. Thus, a differential equation may have infinitely many solutions.

The **general solution** of a differential equation consists of all the solutions of this equations. A **particular solution** of a differential equation a certain special solution of this equation.

In general, higher order means more difficult, and an arbitrary differential equation is extremely difficult.

B.1. Exponential growth and logistic growth.

As a motivating question, we consider **Malthusian growth model**. Let $P(t)$ be the population at time t, and r be the population growth rate. Malthus suggested the population satisfies the equation

$$\frac{dP}{dt} = rP. \tag{33}$$

Theorem B.2 ([3, Theorem 7.6.1])**.** *If*

$$f'(t) = kf(t), \qquad \forall\, t \in (a, b),$$

then there exists $C \in \mathbb{R}$ such that $f(t) = Ce^{kt}$, $\forall\, t \in (a, b)$.

Proof. By assumption, we have $f'(t) - kf(t) = 0$, and thus

$$(e^{-kt} f(t))' = e^{-kt} f'(t) - e^{-kt} kf(t) = 0$$

which implies $f(t) = Ce^{kt}$. □

Thus, the equation (33) implies the population at time t is

$$P(t) = P(0)e^{rt}$$

which is a typical example of exponential growth functions.

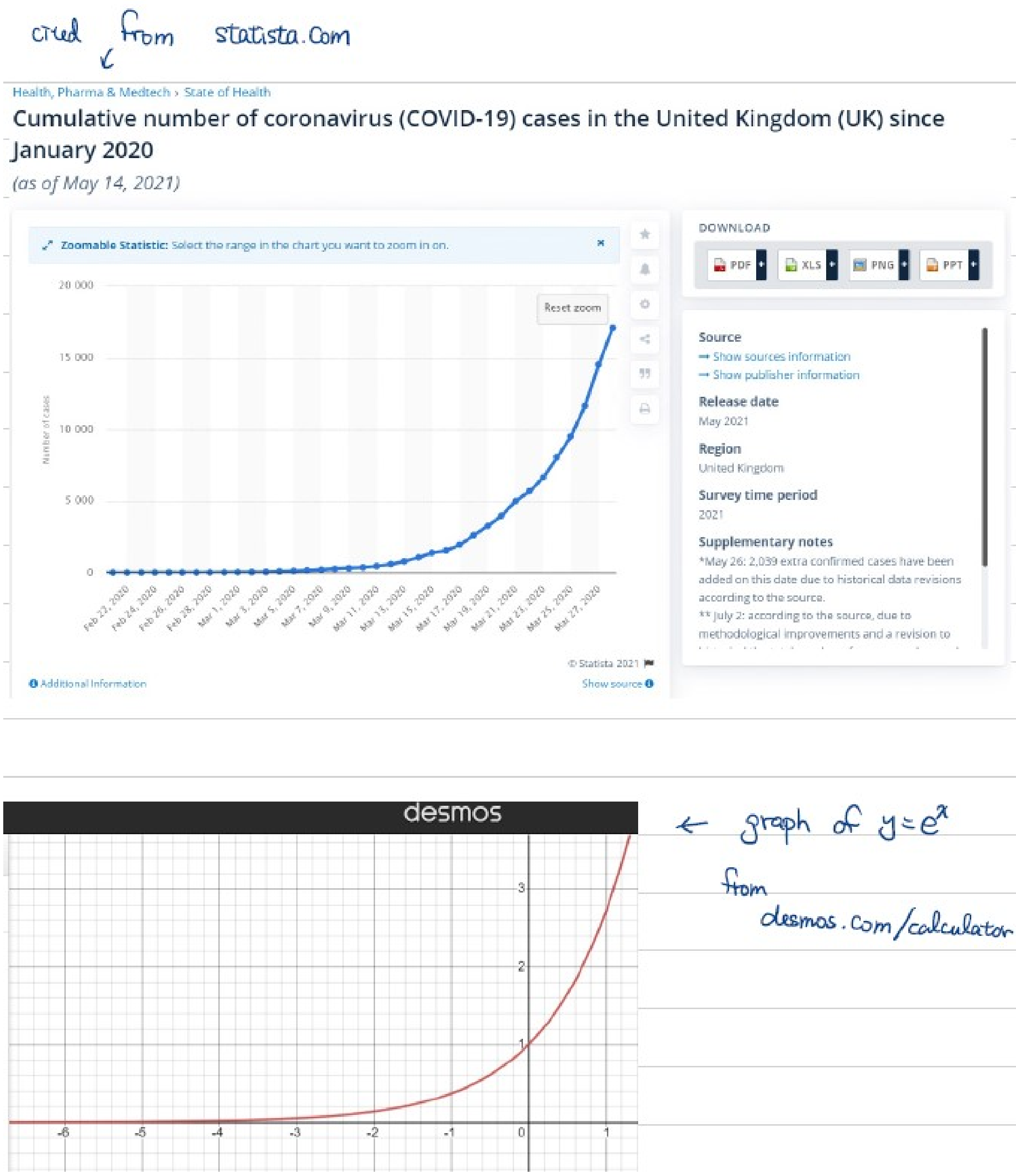


FIGURE 76. Cumulative number of COVID-19 cases in UK and exponential map

Example B.3. *Here are other examples for exponential growth/delay.*

(1) *Radioactive decay. Let $A(t)$ be the amount of a substance at time t, and k be the decay constant which is negative. A physical law says $A'(t) = kA(t)$, and thus $A(t) = A(0)e^{kt}$. The number T with the property $A(T) = \frac{1}{2}A(0)$ is called the half-life of this substance. By our formula,* $T = -\frac{\ln 2}{k}$.

(2) *Continuous compounding formula. Let $A(t)$ be the amount of money at time t (year), and r be the annual interest. Then the continuous compounding formula suggests an approximation of $A(t)$: $A'(t) = rA(t)$, and thus $A(t) = A(0)e^{rt}$. For example, if $A(0) = 100000, r = 1\%$, then $A(t) = 10^5e^{0.01t}$, and $A(10) \approx 110517$. If you need 150000 in 10 years, then* $A(10) = 10^5e^{r\cdot 10} = 150000$, *and* $r = \frac{1}{10}\ln(\frac{3}{2}) \approx 0.041$. *So you need about 4% annual interest.*

Malthusian growth model doesn't count the factor of capacity of environment, so it is usually not accurate in the long term. To improve it, Verhulst (a Belgian mathematician) designed the **logistic growth model**

$$\frac{dP}{dt} = kP(M - P),$$

where M is the capacity of environment. To study this equation, we introduce separable equations.

Definition B.4 ([3, (9.2.1)]). *A differential equation is said to be* **separable** *if it can be written as the form*

$$p(x) + q(y)y' = 0.$$

For such an equation, we integrate the both sides and get

$$\int p(x)\,dx + \int q(y)\,dy = \int p(x) + q(y)y'\,dx = C$$

for some constant C. Then we can solve y from the integrated equation.

Example B.5. *Solve the ODE $x + yy' = 0$.*

Solution. Since $x + yy' = 0$, we have $\int x\,dx + \int yy'\,dx = C$ for some C. Thus, $\frac{x^2}{2} + \frac{y^2}{2} = C$ which implies $x^2 + y^2 = C'$ for some constant C'. □

Example B.6. *Solve the ODE $y + yy' = xy - y'$ with the initial value $y(2) = 1$.*

Solution. By the equation, we have $(y + 1)y' = y(x - 1)$, and thus $\frac{y+1}{y}y' = x - 1$. By integrating the equation,

$$\int_2^x \frac{y(t)+1}{y(t)} \cdot y'(t)\,dt = \int_2^x t - 1\,dt = \frac{x^2}{2} - x.$$

The left-hand-side is $\int_2^x \frac{y(t)+1}{y(t)} \cdot y'(t)\,dt = \int_{y(2)}^{y(x)} \frac{y+1}{y}\,dy = y + \ln|y|\Big|_1^{y(x)} = y(x) + \ln|y(x)| - 1$. Thus,

$$y + \ln|y| - \frac{x^2}{2} + x = 1.$$

□

Example B.7. *Solve the ODE $y' = xe^{y-x}$.*

Solution. Since $y' = xe^{y-x}$, we have $e^{-y}y' = xe^{-x}$ and $\int e^{-y}y'\,dx = \int e^{-y}\,dy = \int xe^{-x}\,dx + C$. Thus,

$$-e^{-y} = -xe^{-x} + \int e^{-x}\,dx + C = -xe^{-x} - e^{-x} + C$$

which implies $y = -\ln(xe^{-x} + e^{-x} - C)$. □

Example B.8 ([3, (9.2.3)]). *Solve Verhulst's population model (logistic equation)*

$$\frac{dy}{dt} = ky(M - y)$$

where $y(t)$ is the number of people at time t, M is the maximum number of people which can be accommodated.

Solution. Since $\dfrac{y'}{y(M-y)} = k$, we have

$$\int \frac{y'}{y(M-y)}\,dt = \int \frac{1}{y(M-y)}\,dy = \int k\,dt = kt + C,$$

for some constant C. The left-hand-side is

$$\frac{1}{M}\int \frac{1}{y} + \frac{1}{M-y}\,dy = \frac{1}{M}(\ln|y| - \ln|M-y|) = \frac{1}{M}\cdot \ln\left|\frac{y}{M-y}\right|.$$

Thus,

$$\left|\frac{y}{M-y}\right| = e^{Mkt+MC} = \frac{y}{M-y} \qquad (\text{assume } M > y > 0),$$

and $Me^{Mkt+C'} = (1+e^{Mkt+C'})y$ which implies

$$y = \frac{Me^{C'}\cdot e^{Mkt}}{1+e^{C'}\cdot e^{Mkt}} = \frac{Me^{C'}}{e^{-Mkt}+e^{C'}} = \frac{C''M}{C''+e^{-Mkt}},$$

for $C'' = e^{C'} > 0$.

Suppose $y(0) = R$, $0 < R < M$. Then $y(0) = \dfrac{C''M}{C''+1} = R$ which implies $C'' = \dfrac{R}{M-R}$. Therefore,

$$y(t) = \frac{MR}{R+(M-R)e^{-Mkt}}$$

which is called a **logistic function**. □

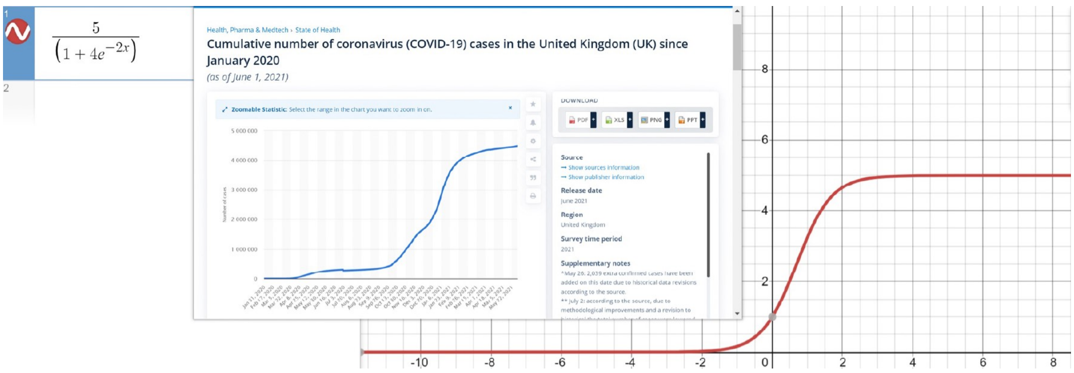


FIGURE 77. Cumulative number of COVID-19 cases in UK and logistic function

B.2. **First order linear equation.** .

Recall that in Theorem 6.28, we solved $y' = ky$ by a multiplier e^{-kt}: $(e^{-kt}y)' = e^{-kt}(y' - ky) = 0$. This multiplier method applies to the **first order linear equations** which are of the form

$$y' + p(x)y = q(x).$$

Suppose $H(x) = \int p(x)\,dx$, i.e. $H(x)$ is any function such that $H'(x) = p(x)$. Then

$$\begin{aligned}(e^{H(x)}y)' &= e^{H(x)}y' + H'(x)e^{H(x)}y \\ &= e^{H(x)}\cdot(y' + p(x)y) = e^{H(x)}\cdot q(x).\end{aligned}$$

Thus, $e^{H(x)}y = \int_a^x q(t)\cdot e^{H(t)}\,dt + C$, and

$$y = e^{-H(x)}\cdot\Big(\int_a^x q(t)\cdot e^{H(t)}\,dt + C\Big).$$

See [3, Section 9.1].

Example B.9. *Solve the ODE $y' + y = 1$.*

Solution. Multiplying $e^{\int 1\,dx} = e^x$ to the both sides of the equation, we have

$$(e^x y)' = e^x y' + e^x y = e^x.$$

Thus, $e^x y = e^x + C$, and $y = 1 + C\cdot e^{-x}$ for some constant C. □

Example B.10. *Solve the ODE $y' + 2xy = x$ with the initial value $y(0) = 2$.*

Solution. Multiplying $e^{\int 2x\,dx} = e^{x^2}$ to the both sides of the equation, we have

$$(e^{x^2}y)' = e^{x^2}y' + 2xe^{x^2}y = xe^{x^2}.$$

Thus,

$$e^{x^2}y(x) - e^{0^2}y(0) = \int_0^x te^{t^2}\,dt = \frac{1}{2}e^{x^2} - \frac{1}{2},$$

and $y(x) = \frac{1}{2} + \frac{3}{2}e^{-x^2}$. □

Example B.11. *Solve the ODE $xy' - 2y = 3x^4$ with the initial value $y(-1) = 2$.*

Solution. We rewrite the equation as $y' - \frac{2}{x}y = 3x^3$ and multiply the both sides of the equation by $e^{\int -\frac{2}{x}\,dx} = e^{-2\ln|x|} = \frac{1}{x^2}$. As a result,

$$\Big(\frac{1}{x^2}\cdot y\Big)' = \frac{1}{x^2}\cdot y' - \frac{2}{x^3}\cdot y = 3x,$$

and thus

$$\int_{-1}^x 3t\,dt = \frac{3}{2}t^2\Big|_{t=-1}^x = \frac{3}{2}(x^2-1) = \int_{-1}^x \Big(\frac{1}{t^2}\cdot y(t)\Big)'\,dt = \frac{1}{x^2}\cdot y(x) - y(-1) = \frac{1}{x^2}\cdot y(x) - 2.$$

Simplifying the equation, we have $y(x) = \frac{3}{2}x^4 + \frac{1}{2}x^2$. □

Example B.12 (Newton's law of cooling, [3, (9.1.3)])**.** *Suppose it's $15°C$ and there is a coffee in the room. Let $T(t)$ be the temperature of coffee at time t (minutes). We know that $T(0) = 85°C$ and $T(3) = 65°C$. Then how many minutes do you expect to wait for the coffee to cool down to $40°C$?*

Solution. By Newton's law of cooling, we have the differential equation

$$\frac{dT}{dt} = -k(T - 15),$$

where k is a constant, $T(0) = 85$ and $T(3) = 65$. Applying the multiplier $e^{\int k\,dt} = e^{kt}$, we have

$$(e^{kt}T)' = e^{kt}T' + ke^{kt}T = 15ke^{kt},$$

which implies $e^{kt}T = 15e^{kt} + C$, and $T(t) = 15 + Ce^{-kt}$. Since $85 = T(0) = 15 + C$ and $65 = T(3) = 15 + Ce^{-3k}$, we have $C = 70$ and $k = -\frac{1}{3} \cdot \ln \frac{5}{7} \approx 0.11$. Therefore, if $T \approx 15 + 70e^{-0.11t} = 40$, then $t = \frac{1}{-0.11} \ln \frac{25}{70} \approx 9.4$ (minutes). □

B.3. **Homogeneous linear differential equation with constant coefficients.** In this section, we study differential equations of the form

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1y' + a_0y = 0,$$

where $a_0, a_1, \cdots a_{n-1}$ are constant. See [3, Section 9.3].

We start with the first order case. The first order homogeneous linear differential equation with constant coefficients is of the form

$$y' + ay = 0$$

where $a \in \mathbb{R}$. In this case, since $y' = -ay$, one has $y = C \cdot e^{-ax}$ for some $C \in \mathbb{R}$.

Example B.13. *Solve the following ODE.*

(1) $y'' = 0$. *(2)* $y'' - y = 0$. *(3)* $y'' + y = 0$.

Solution. For (1), since $(y')' = 0$, one has $y' = C_1$ for some $C_1 \in \mathbb{R}$, and $y = \int C_1\,dx = C_1x + C_2$, for some $C_2 \in \mathbb{R}$. In other words, the general solution is

$$y = C_1x + C_2. \tag{34}$$

For (2), let $z(x) = y'(x)$. Then $\begin{cases} y' = z, \\ z' = y'' = y, \end{cases}$ which implies $\begin{cases} (y+z)' = y + z, \\ (y-z)' = z - y = -(y-z). \end{cases}$ Therefore,

$$y + z = C_1 \cdot e^x, \qquad \text{and} \qquad y - z = C_2 \cdot e^{-x}$$

for some $C_1, C_2 \in \mathbb{R}$. Consequently, the general solution is $y = \frac{C_1 \cdot e^x + C_2 \cdot e^{-x}}{2}$, or equivalently

$$y = C_1e^x + C_2e^{-x}. \tag{35}$$

A similar method for (3) also works but requires more knowledge about complex functions. Here, we skip the computation, and the general solution is

$$y = C_1 \cos x + C_2 \sin x. \tag{36}$$

□

Theorem B.14 ([3, Theorem 9.3.6])**.** *The second order homogeneous linear differential equation with constant coefficients is of the form*

$$y'' + ay' + by = 0 \tag{37}$$

where $a, b \in \mathbb{R}$. To solve (37), we consider the **characteristic equation**

$$r^2 + ar + b = 0. \tag{38}$$

- *If* (38) *has two different real roots* $r = r_1, r_2$*, then the general solution is*
$$y = C_1 e^{r_1 x} + C_2 e^{r_2 x}.$$
- *If* (38) *has exactly one real root* $r = \alpha$*, then the general solution is*
$$y = C_1 e^{\alpha x} + C_2 x e^{\alpha x} = (C_1 + C_2 x)e^{\alpha x}.$$
- *If* (38) *has two nonreal roots* $r_1 = \alpha \pm i\beta$*, then the general solution is*
$$y = C_1 e^{\alpha x}\cos\beta x + C_2 e^{\alpha x}\sin\beta x = (C_1\cos\beta x + C_2\sin\beta x)e^{\alpha x}.$$

Example B.15. *Solve the following equations.*

(1) $y'' = 0$.
(2) $y'' - y = 0$.
(3) $y'' + y = 0$.
(4) $y'' + 2y' - 15y = 0$ *with initial value* $y(0) = 0$ *and* $y'(0) = -1$.
(5) $y'' + 4y' + 4y = 0$.
(6) $y'' + y' + 3y = 0$.

Solution. We solve them by the method of characteristic equation.

(1) The characteristic equation
$$r^2 = 0$$
has exactly one real root $r = 0$, and thus $y = (C_1 + C_2 x)e^{0\cdot x} = (C_1 + C_2 x)$.

(2) The characteristic equation
$$r^2 - 1 = 0$$
has two real roots $r = \pm 1$, and thus $y = C_1 e^x + C_2 e^{-x}$.

(3) The characteristic equation
$$r^2 + 1 = 0$$
has two nonreal roots $r = \pm i$, and thus $y = (C_1\cos x + C_2\sin x)e^{0\cdot x} = (C_1\cos x + C_2\sin x)$.

(4) $y = \frac{1}{8}e^{-5x} - \frac{1}{8}e^{3x}$.

(5) $y = C_1 e^{-2x} + C_2 x e^{-2x}$.

(6) $y = e^{-\frac{x}{2}}\Big(C_1\cos(\frac{\sqrt{11}}{2}x) + C_2\sin(\frac{\sqrt{11}}{2}x)\Big)$.

□

Theorem B.16. *The n-th order homogeneous linear differential equation with constant coefficients is of the form*
$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0 \tag{39}$$
where any $a_k \in \mathbb{R}$*. To solve* (39)*, we consider the characteristic equation*
$$r^n + a_{n-1}r^{n-1} + \cdots + a_1 r + a_0 = 0. \tag{40}$$

- *If* r *is a real root with multiplicity* k*, then*
$$e^{rx}, xe^{rx}, \cdots, x^{k-1}e^{rx}$$

are k linearly independent solutions.

- *If $\alpha + \beta i$ is a complex root with multiplicity k, then*

$$e^{\alpha x}\cos\beta x, xe^{\alpha x}\cos\beta x, \cdots, x^{k-1}e^{\alpha x}\cos\beta x$$

$$e^{\alpha x}\sin\beta x, xe^{\alpha x}\sin\beta x, \cdots, x^{k-1}e^{\alpha x}\sin\beta x$$

are $2k$ linearly independent solutions.

The general solution is any linear combination of n linearly independent solutions.

Example B.17. *Solve the ODE $y^{(4)} - 10y'' + 25y = 0$.*

Solution. Solving the characteristic equation $r^4 - 10r^2 + 25 = 0$, we have $r = \pm\sqrt{5}$ with multiplicity 2 for each. Thus, the general solution is

$$y = C_1 e^{-\sqrt{5}x} + C_2 x\, e^{-\sqrt{5}x} + C_3 e^{\sqrt{5}x} + C_4 x\, e^{\sqrt{5}x}$$

where C_1, C_2, C_3, C_4 are constants. □

Linear system approach. Back to the equation (37), we introduce another function $z(x) = y'(x)$. Then $z' = y'' = -ay' - by = -az - by$, the equation (37) is equivalent to

$$\begin{cases} y' = z, \\ z' = -by - az. \end{cases}$$

Or equivalently,

$$\begin{pmatrix} y' \\ z' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -b & -a \end{pmatrix}\begin{pmatrix} y \\ z \end{pmatrix}.$$

In this way, one can use matrix techniques, such as diagonalization, to solve (37).

Example B.18. *Solve the ODE $y'' + 2y' - 15y = 0$ by matrixes.*

Solution. Let $z = y'$. Then $z' = y'' = -2y' + 15y = 15y - 2z$, and

$$\begin{pmatrix} y' \\ z' \end{pmatrix} = \begin{pmatrix} z \\ 15y - 2z \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 15 & -2 \end{pmatrix}\begin{pmatrix} y \\ z \end{pmatrix}.$$

Note that

$$\begin{pmatrix} 0 & 1 \\ 15 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 3 & -5 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & -5 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 3 & -5 \end{pmatrix}^{-1} = \frac{1}{8}\begin{pmatrix} 1 & 1 \\ 3 & -5 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & -5 \end{pmatrix}\begin{pmatrix} 5 & 1 \\ 3 & -1 \end{pmatrix}.$$

Thus,

$$\frac{1}{8}\begin{pmatrix} 5 & 1 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} y' \\ z' \end{pmatrix} = \frac{1}{8}\begin{pmatrix} 3 & 0 \\ 0 & -5 \end{pmatrix}\begin{pmatrix} 5 & 1 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} y \\ z \end{pmatrix},$$

and

$$\begin{cases} (\frac{5y+z}{8})' = 3\cdot(\frac{5y+z}{8}) \\ (\frac{3y-z}{8})' = -5\cdot(\frac{3y-z}{8}) \end{cases} \Rightarrow \begin{cases} \frac{5y+z}{8} = C_1 e^{3x} \\ \frac{3y-z}{8} = C_2 e^{-5x} \end{cases} \Rightarrow y = C_1 e^{3x} + C_2 e^{-5x}.$$

□

The advantage of the matrix method is that one can prove that the results are the only solutions of the equation.

Example B.19. *Let $y = y(x)$ be a differentiable function defined on an open interval. Prove that if $y'' + y = 0$, then $y = C_1 \cos x + C_2 \sin x$.*

Proof. Let $z = y'$. Then

$$\begin{pmatrix} y' \\ z' \end{pmatrix} = \begin{pmatrix} z \\ -y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix}.$$

Multiply the equation by $\begin{pmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{pmatrix}$:

$$\begin{aligned} \begin{pmatrix} 0 \\ 0 \end{pmatrix} &= \begin{pmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{pmatrix} \begin{pmatrix} y' \\ z' \end{pmatrix} - \begin{pmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix} \\ &= \begin{pmatrix} (y' \cos x - z' \sin x) - (z \cos x + y \sin x) \\ (y' \sin x + z' \cos x) - (z \sin x - y \cos x) \end{pmatrix} \\ &= \begin{pmatrix} (y \cos x - z \sin x)' \\ (y \sin x + z \cos x)' \end{pmatrix}. \end{aligned}$$

Thus there exist $C_1, C_2 \in \mathbb{R}$ such that

$$\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} (y \cos x - z \sin x) \\ (y \sin x + z \cos x) \end{pmatrix} = \begin{pmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix}$$

and

$$\begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} C_1 \cos x + C_2 \sin x \\ -C_1 \sin x + C_2 \cos x \end{pmatrix},$$

as desired. □

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