# Advanced Calculus I — Homework 9 (Fall 2026)

Unless otherwise specified, Euclidean spaces have their usual metrics, and subsets have the induced metrics. Write $\mathrm{cl}(A)$ for the closure of $A$ in the ambient space under consideration.

On finite products of metric spaces $(M_i, d_i)$, use the sum metric

$$d_\Sigma((x_1,\dots,x_n),(y_1,\dots,y_n)) := \sum_{i=1}^{n} d_i(x_i,y_i).$$

All vector spaces are over $\mathbb{R}$.

## Examples of continuous maps

1. Prove that each of the following maps is continuous. Normed spaces carry the metric $d_V(x,y) = \|x-y\|$.
   (a) $q : \left\{(x,y)\in\mathbb{R}^2 \,\middle|\, y\neq 0\right\} \to \mathbb{R}, \quad q(x,y) = x/y.$
   (b) Let $(M_i,d_i)$, $1\le i\le n$, be nonempty metric spaces. Fix $a_i\in M_i$ for $1\le i\le n$ and an index $1\le j\le n$. Consider
   $$\iota_j : M_j \to M_1\times\cdots\times M_n, \qquad \iota_j(x) = (a_1,\dots,a_{j-1},x,a_{j+1},\dots,a_n).$$
   (c) For a metric space $(M,d)$, consider $d : M\times M\to\mathbb{R}, \quad (x,y)\mapsto d(x,y)$.
   (d) For a normed vector space $V$, consider $N : V\to\mathbb{R}, \quad N(x) = \|x\|$.
   (e) Let $V$ be an inner product space, with the induced norm $\|x\| = \sqrt{\langle x,x\rangle}$. Consider $P : V\times V\to\mathbb{R}, \quad P(x,y) = \langle x,y\rangle$.

## Restriction, extension, and gluing

2. Let $(M,d_M)$ and $(N,d_N)$ be metric spaces, and let $A\subseteq M$ carry the induced metric.
   (a) If $f : M\to N$ is continuous, prove that its restriction $f|_A : A\to N$ is continuous.
   (b) A **continuous extension** of a map $g : A\to N$ to $M$ is a continuous map $\widetilde{g} : M\to N$ such that $\widetilde{g}|_A = g$. Give an example of a bounded continuous function $g : (0,1)\to\mathbb{R}$ that has no continuous extension to $\mathbb{R}$, and justify your answer.

3. Let $M$ and $N$ be metric spaces, and suppose $M = A\cup B$. Let $f_A : A\to N$ and $f_B : B\to N$ be continuous with respect to the induced metrics, and suppose that $f_A = f_B$ on $A\cap B$. Define
   $$f(x) = \begin{cases} f_A(x), & x\in A,\\ f_B(x), & x\in B.\end{cases}$$
   (a) Show that $f$ is well-defined. Prove that $f$ is continuous if both $A$ and $B$ are open in $M$.
   (b) Prove that $f$ is continuous if both $A$ and $B$ are closed in $M$.
   (c) Give a counterexample showing that $f$ need not be continuous for an arbitrary cover $M = A\cup B$, even when $f_A$ and $f_B$ are continuous and agree on $A\cap B$.

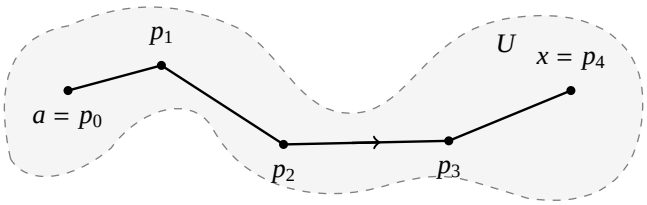
## Connectedness and polygonal paths

4. Let $C \subseteq \mathbb{R}^n$ be nonempty and connected, let $c_0 \in C$, and let $f : C \to \mathbb{R}$ be continuous. Suppose that

$$f(c_0) = \frac{1}{2}, \qquad f(x) \neq 0, \quad f(x) \neq 1 \quad \text{for every } x \in C.$$

   Prove that $0 < f(x) < 1$ for every $x \in C$.

5. Let $U \subseteq \mathbb{R}^n$ be nonempty, open, and connected. A **polygonal path** in $U$ is a continuous path obtained by concatenating finitely many line segments contained in $U$; constant paths are allowed.




An example in $\mathbb{R}^2$: follow the line segments from $a = p_0$ through $p_1, p_2, p_3$ to $x = p_4$, staying inside $U$.

   Fix $a \in U$, and define

$$E := \{x \in U \,\big|\, \text{there is a polygonal path in } U \text{ from } a \text{ to } x\}.$$

   (a) Prove that $E$ is open in $U$.

   (b) Prove that $E$ is closed in $U$ by showing that $U \setminus E$ is open in $U$.

   (c) Deduce that any two points of $U$ can be joined by a polygonal path in $U$.

## Closed sets and continuous maps

6. Let $A$ be a nonempty subset of $\mathbb{R}^n$, and let $f : A \to \mathbb{R}^m$ be continuous. Prove that, for every subset $B \subseteq A$,

$$f(\mathrm{cl}(B) \cap A) \subseteq \mathrm{cl}(f(B)).$$

   Here the closures are taken in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively.

7. Recall that $C^1((a,b))$ denotes the space of real-valued **continuously differentiable functions** on $(a,b)$: a function $f$ belongs to $C^1((a,b))$ if $f$ is differentiable at every point of $(a,b)$ and its derivative $f'$ is continuous on $(a,b)$.

   Let $f_1, f_2 \in C^1((a,b))$, and write $g_j = f_j'$ for $j = 1, 2$. Suppose that $(c,d) \subseteq (a,b)$ is a nonempty open interval such that, for every $y \in (c,d)$, either $f_1(y) \neq f_2(y)$ or $g_1(y) \neq g_2(y)$. Define

$$H := \{x \in (a,b) \,|\, f_1(x) = f_2(x)\}.$$

   (a) Prove that $H$ is relatively closed in $(a,b)$.

   (b) Prove that no point of $(c,d)$ is an accumulation point of $H$.

   (c) Prove that $H \cap (c,d)$ is at most countable.

8. Let $K \subseteq \mathbb{R}^n$ be compact, and let $f : K \to \mathbb{R}^m$ be a map. Its **graph** is

$$\Gamma_f := \{(x, f(x)) \mid x \in K\} \subseteq \mathbb{R}^n \times \mathbb{R}^m.$$

   (a) Suppose that $f$ is continuous. Prove that $\Gamma_f$ is compact.

   (b) Suppose that $\Gamma_f$ is compact. Prove that $f$ is continuous.

## Open maps, closed maps, and homeomorphisms

Let $(M, d_M)$ and $(N, d_N)$ be metric spaces. A map $p : M \to N$ is a **closed map** if $p(F)$ is closed in $N$ for every closed subset $F$ of $M$. It is an **open map** if $p(U)$ is open in $N$ for every open subset $U$ of $M$.

9. Let $p : (M, d_M) \to (N, d_N)$ be a map.

   (a) Suppose that $p$ is a closed map. Prove that

$$\mathrm{cl}(p(A)) \subseteq p(\mathrm{cl}(A)) \qquad \text{for every } A \subseteq M.$$

   (b) Conversely, suppose that this inclusion holds for every $A \subseteq M$. Prove that $p$ is a closed map.

10. Let $q : (M, d_M) \to (N, d_N)$ be a bijection.

   (a) Give an example in which $q$ is continuous but $q^{-1}$ is not continuous, and justify your answer.

   (b) Prove that the following statements are equivalent:

      (i) $q$ is continuous and is an open map.

      (ii) $q$ is continuous and is a closed map.

      (iii) For every $A \subseteq M$, $q(\mathrm{cl}(A)) = \mathrm{cl}(q(A))$.

      (iv) Both $q$ and $q^{-1}$ are continuous.

      A bijection satisfying (iv) is called a **homeomorphism**.

   (c) Suppose that $M$ is compact and $q$ is continuous. Prove that $q^{-1}$ is continuous, and hence that $q$ is a homeomorphism.