Advanced Calculus I — Homework 8 (Fall 2026)

Unless otherwise specified, $\mathbb{R}^n$ is equipped with its Euclidean metric. An unsubscripted boundary is taken in the ambient space.

Connectedness and the subspace topology

1. Let S be a subset of a metric space (M, d), with the subspace metric $d_S = d|_{S\times S}$.
   (a) Prove that the following conditions are equivalent:
      (i) There exist open sets $U, V \subseteq M$ such that
      $$U \cap S \neq \varnothing, \qquad V \cap S \neq \varnothing, \qquad S \subseteq U \cup V, \qquad U \cap V \cap S = \varnothing.$$
      (ii) $S = A \cup B$ for some disjoint, nonempty subsets A, B open in (S, d_S).

      In (i), must U and V be disjoint outside S? Justify your answer.
   (b) Deduce that S is connected if and only if the only subsets of (S, d_S) that are both open and closed are $\varnothing$ and S. Why does this criterion depend only on d_S?

2. Let $A \subseteq S \subseteq M$, where (M, d) is a metric space. Write $\mathrm{cl}_S A$ for the closure in S and $\mathcal{L}_M(A)$ for the set of accumulation points of A in M.
   (a) Prove that
   $$\mathrm{cl}_S A = S \cap \mathrm{cl}_M A.$$
   Deduce that A is relatively closed in S if and only if $S \cap \mathcal{L}_M(A) \subseteq A$.
   (b) Characterize $\mathrm{int}_S A$ and $\partial_S A$ using balls in S. For $S = [0, 1]$ and $A = [0, 1/2)$, compute $\partial_S A$ and $\partial_{\mathbb{R}} A$.
   (c) Suppose that A is connected and $A \subseteq B \subseteq \mathrm{cl}_M A$. Prove that B is connected.

Examples of connected and disconnected sets

3. Show that $\mathbb{Q}^2$ is disconnected.

4. Let $I \subseteq \mathbb{R}$ be an interval with its usual subspace metric. Prove that I is connected by showing that a subset $U \subseteq I$ that is both relatively open and relatively closed must be $\varnothing$ or I. Your proof should apply to intervals with or without endpoints and to unbounded intervals. Do not use path-connectedness or the intermediate value theorem.

5. Let
   $$\Gamma = \{(x, \sin(1/x)) \mid 0 < x \leq 1\}, \qquad J = \{0\} \times [-1, 1].$$
   Let F be a nonempty subset of J. In Homework 7, you showed that $\Gamma \cup F$ is not path-connected. Prove that $\Gamma \cup F$ is connected.

6. For each question, justify your answer.
   (a) Let $F \subseteq \mathbb{R}^2$ be closed, and suppose every point of F is isolated in F. Is $\mathbb{R}^2 \setminus F$ connected?
   (b) Show that $\mathbb{R} \setminus \mathbb{Q}$ and $(\mathbb{R} \setminus \mathbb{Q}) \cup \{0\}$ are both disconnected.
   (c) Is $(\mathbb{R}^2 \setminus \mathbb{Q}^2) \cup \{(0,0)\}$ disconnected?

Unions, boundaries, and connected components

7. Let $(C_i)_{i \in I}$ be a nonempty family of connected subsets of a metric space (M, d). Suppose that

$$\bigcap_{i \in I} C_i \neq \varnothing.$$

   Prove that $\bigcup_{i \in I} C_i$ is connected.

8. Let $A \subseteq \mathbb{R}^2$.
   (a) Prove that

$$\mathbb{R}^2 = \operatorname{int}(A) \cup \partial A \cup \operatorname{int}(\mathbb{R}^2 \setminus A)$$

   and that the three sets on the right are pairwise disjoint.
   (b) Let $C \subseteq \mathbb{R}^2$ be connected. Suppose $C \cap A \neq \varnothing$ and $C \setminus A \neq \varnothing$. Prove that $C \cap \partial A \neq \varnothing$.

9. Let $s_0 \in S \subseteq \mathbb{R}^n$. The **connected component** of s_0 in S is

$$K = C_S(s_0) := \bigcup \{A \subseteq S \mid s_0 \in A \text{ and } A \text{ is connected}\}.$$

   (a) Prove that K is connected.
   (b) Prove that K is relatively closed in S.
   (c) If S is open in $\mathbb{R}^n$, prove that $S \cap \partial K = \varnothing$.
   (d) If S is closed in $\mathbb{R}^n$, prove that $\partial K = K \cap \partial S$. Deduce that the boundary of a closed set is the union of the boundaries of its connected components.