Advanced Calculus I — Homework 7 (Fall 2026)

Unless otherwise specified, $\mathbb{R}^n$ is equipped with its Euclidean metric. Interiors, closures, and boundaries are taken in the ambient space.

Compactness

1. Let $(K_n)_{n=1}^{\infty}$ be a sequence of compact subsets of a metric space (M, d). Suppose that

$$K_1 \cap \cdots \cap K_n \neq \varnothing \qquad \text{for every } n \geq 1.$$

Prove that

$$\bigcap_{n=1}^{\infty} K_n \neq \varnothing.$$

2. Let F be a nonempty bounded subset of $\mathbb{R}^n$. Define

$$d(x, F) := \inf \{ \| x - f \|_2 \,\big|\, f \in F \}, \qquad F_1 := \{ x \in \mathbb{R}^n \,\big|\, d(x, F) \leq 1 \}.$$

Prove that F_1 is compact.

3. Let G_1 and G_2 be nonempty compact subsets of $\mathbb{R}^n$, and define

$$G := \{ x + y \,|\, x \in G_1,\ y \in G_2 \}.$$

   (a) Prove that G is bounded.

   (b) Is G compact? Justify your answer.

4. Let U and K be nonempty subsets of $\mathbb{R}^n$ such that U is bounded and open, K is closed, and $K \subseteq U$. For $A \subseteq \mathbb{R}^n$ and $\varepsilon > 0$, define the *ε-neighborhood* of A by

$$D(A; \varepsilon) := \{ x \in \mathbb{R}^n \,\big|\, \| x - a \|_2 < \varepsilon \text{ for some } a \in A \}.$$

   (a) Prove that $\partial U \cap K = \varnothing$.

   (b) Prove that there exists $\varepsilon > 0$ such that

$$D(K; \varepsilon) \cap D(\partial U; \varepsilon) = \varnothing.$$

Path-connectedness

5. Let (M, d) be a metric space and let $a < b$. Recall that a function $\gamma : [a, b] \to M$ is a **continuous path** if, for every $t_0 \in [a, b]$ and every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$t \in [a, b], \quad |t - t_0| < \delta \quad \Longrightarrow \quad d(\gamma(t), \gamma(t_0)) < \varepsilon.$$

The points $\gamma(a)$ and $\gamma(b)$ are its endpoints. A subset $C \subseteq M$ is *path-connected* if, for every $x, y \in C$, there is a continuous path $\gamma : [0, 1] \to C$ with $\gamma(0) = x$ and $\gamma(1) = y$, where C has the induced metric.

Let (M, d) be a metric space, and let $a < c < b$. Suppose $f_1 : [a, c] \to M$ and $f_2 : [c, b] \to M$ are continuous paths such that $f_1(c) = f_2(c)$. Define

$$f(t) = \begin{cases} f_1(t), & t \in [a, c], \\ f_2(t), & t \in [c, b]. \end{cases}$$

Prove directly from the epsilon–delta definition that f is well-defined and continuous on $[a, b]$.

6. Let $r > 0$, and let $D(0, r)$ be the open ball of radius r centered at 0 in $\mathbb{R}^n$. Define

$$S(0, r) := \{x \in \mathbb{R}^n \,\big|\, \|x\|_2 = r\}.$$

Let B be a nonempty subset of $S(0, r)$, and set

$$C := B \cup D(0, r).$$

Prove that C is path-connected.

7. Determine whether each of the following sets, equipped with the induced Euclidean metric, is path-connected. Justify each answer by constructing paths between arbitrary points or proving that no path can join a suitable pair of points. You may use the intermediate value theorem for continuous real-valued functions.

   (a) $\mathbb{R} \setminus \{0\}$.

   (b) $\mathbb{R}^2 \setminus \{(0, 0)\}$.

   (c) $\left\{(x, y) \in \mathbb{R}^2 \,\big|\, xy = 0\right\}$.

   (d) $\left\{(x, y) \in \mathbb{R}^2 \,\big|\, xy = 1\right\}$.

8. Define

$$\Gamma := \left\{\left(x, \sin\frac{1}{x}\right) \,\middle|\, 0 < x \leq 1\right\}, \qquad J := \{(0, y) \,|\, -1 \leq y \leq 1\}.$$

Let F be a nonempty subset of J, and set $C := \Gamma \cup F$. Is C path-connected? Justify your answer.