

Advanced Calculus I — Homework 6 (Fall 2026)

1. Let (x_n) be a sequence in a metric space (M, d) converging to $x \in M$. Prove that

$$K = \{x\} \cup \{x_n \mid n \in \mathbb{N}\}$$

is compact.

2. Let K_1 and K_2 be disjoint nonempty compact subsets of a metric space (M, d) . Prove that there are disjoint open sets $U_1, U_2 \subseteq M$ such that

$$K_1 \subseteq U_1, \quad K_2 \subseteq U_2.$$

3. A metric space (M, d) is called **locally compact** if every point $x \in M$ has an open neighborhood U whose closure $\text{cl}(U)$ is compact.

- (a) Prove that every compact metric space is locally compact.
- (b) Prove that $\mathbb{R}^d$ with its Euclidean metric is locally compact.

4. In $\mathbb{Q}$ with the usual metric, prove that no open neighborhood V of 0 has compact closure in $\mathbb{Q}$. Thus $\mathbb{Q}$ is not locally compact.

Hint: Choose $r > 0$ with $(-r, r) \cap \mathbb{Q} \subseteq V$, then choose an irrational number in $(-r, r)$ and a sequence of rational numbers in $(-r, r)$ converging to it.

5. Let K be a compact subset and F a closed subset of a metric space (M, d) , with $K \cap F = \emptyset$.

- (a) Suppose (M, d) is locally compact, and let $U \subseteq M$ be open with $K \subseteq U$. Prove that there exists an open set V such that

$$K \subseteq V \subseteq \text{cl}(V) \subseteq U$$

and $\text{cl}(V)$ is compact.

- (b) Without assuming local compactness, prove that there are disjoint open sets $W_K, W_F \subseteq M$ such that

$$K \subseteq W_K, \quad F \subseteq W_F.$$

6. Let K_1 and K_2 be closed subsets of a metric space (M, d) . Suppose there are sequences of distinct points $(x_n) \subseteq K_1$ and $(y_n) \subseteq K_2$ such that

$$d(x_n, y_n) \longrightarrow 0.$$

- (a) Must $K_1 \cap K_2$ be nonempty? Justify your answer with a proof or a counterexample.
- (b) Prove that $K_1 \cap K_2 \neq \emptyset$ if both K_1 and K_2 are compact.

7. Let (M_1, d_1) and (M_2, d_2) be metric spaces. Equip $M = M_1 \times M_2$ with the metric

$$d((a_1, a_2), (b_1, b_2)) = d_1(a_1, b_1) + d_2(a_2, b_2).$$

Let $K_1 \subseteq M_1$ and $K_2 \subseteq M_2$ be nonempty, and set $K = K_1 \times K_2$. Prove that K is compact in (M, d) if and only if K_1 and K_2 are compact in their respective metric spaces.

8. Let (K, d) be a compact metric space and let $f : K \rightarrow K$ satisfy

$$d(f(x), f(y)) = d(x, y) \quad \text{for all } x, y \in K.$$

- (a) Prove that $f(K)$ is a closed subset of K .

(b) Prove that f is surjective.

Hint: Given $z \in K$, set $z_0 = z$ and define $z_n = f(z_{n-1})$ for $n \geq 1$. Take a convergent subsequence (z_{n_j}) and consider the gaps $m_j = n_{j+1} - n_j$.

9. Recall from Homework 1 that

$$\ell_2 = \left\{ c = (c_n)_{n=1}^{\infty} \mid c_n \in \mathbb{R}, \sum_{n=1}^{\infty} |c_n|^2 < \infty \right\}$$

is a normed vector space with norm

$$\|c\|_2 = \left(\sum_{n=1}^{\infty} |c_n|^2 \right)^{1/2} \quad \text{and metric} \quad d_2(c, d) = \|c - d\|_2.$$

For each $n \in \mathbb{N}$, let $e_n \in \ell_2$ be the sequence whose n -th coordinate is 1 and whose other coordinates are 0.

Determine whether each of the following subsets of ℓ_2 is compact. Prove your answer.

(a) For a fixed $N \in \mathbb{N}$, the set

$$K_N = \{c = (c_n) \in \ell_2 \mid \|c\|_2 \leq 1, \quad c_n = 0 \text{ for all } n > N\}.$$

(b) The closed unit ball $B = \{c \in \ell_2 \mid \|c\|_2 \leq 1\}$.

(c) The set $A = \{e_n/n \mid n \in \mathbb{N}\}$.

(d) The set $K = \{0\} \cup \{e_n/n \mid n \in \mathbb{N}\}$.

(e) The set

$$H = \left\{ c = (c_n) \in \ell_2 \mid |c_n| \leq \frac{1}{n} \text{ for every } n \in \mathbb{N} \right\}.$$

Hint: Given a sequence in H , use successive subsequences and a diagonal argument to obtain a subsequence for which every coordinate converges. To prove convergence in the ℓ_2 norm, use the estimate

$$\sum_{n>N} |c_n - d_n|^2 \leq 4 \sum_{n>N} \frac{1}{n^2} \quad \text{for all } c, d \in H.$$