

## Advanced Calculus I — Homework 5 (Fall 2026)

### Boundary

1. Let  $S \subseteq \mathbb{R}$  be a nonempty bounded set containing at least two distinct points. Show that  $S$  has at least two boundary points.

### Isolated points

2. Let  $(M, d)$  be a metric space and  $S \subseteq M$ . Recall that a point  $x_0 \in S$  is **isolated** (or an **isolated point**) in  $S$  if  $D(x_0, \delta) \cap S = \{x_0\}$  for some  $\delta > 0$ . Show that if  $M$  has no isolated points and  $x_0$  is isolated in  $S$ , then  $x_0 \in \partial S$ .
3. Let  $(M, d)$  be a metric space, and  $Q \subseteq F$  be dense in  $F$ . Prove that  $F$  has no isolated points if and only if  $Q$  has no isolated points.
4. Let  $F$  be a closed subset of  $\mathbb{R}^n$ . Suppose that:
  - (i)  $F$  is an infinite set, and
  - (ii) every point of  $F$  is isolated in  $F$ .

Prove that  $F$  is unbounded.

5. A set  $P \subseteq \mathbb{R}^d$  is called **perfect** if it is non-empty, closed, and contains no isolated points. Let  $Q \subseteq \mathbb{R}^d$  be nonempty.
  - (a) If  $Q$  has no isolated points, show that  $\text{cl}(Q)$  is perfect.
  - (b) If  $Q$  is perfect and  $D \subseteq Q$  is dense in  $Q$ , show that  $D$  has no isolated points.

### Dense and nowhere dense sets

6. Let  $(M, d)$  be a metric space. A subset  $T \subseteq M$  is called **nowhere dense** in  $M$  if its closure  $\text{cl}(T)$  has empty interior (that is,  $\text{cl}(T)$  contains no open ball  $D(x, r) = \{y \in M \mid d(x, y) < r\}$  for any  $x \in M$  and  $r > 0$ ).
  - (a) Show that  $\mathbb{Z}$  is nowhere dense in  $\mathbb{R}$  with respect to the standard metric  $d(x, y) = |x - y|$ .
  - (b) Prove that  $T$  is nowhere dense in  $M$  if and only if its complement  $M \setminus \text{cl}(T)$  is dense in  $M$ .
  - (c) Let  $d_1$  and  $d_2$  be two equivalent metrics on  $M$ . Prove that  $T$  is dense (respectively, nowhere dense) in  $(M, d_1)$  if and only if it is dense (respectively, nowhere dense) in  $(M, d_2)$ .

### Cauchy sequences and completion

7. Prove that every convergent sequence in a metric space is a Cauchy sequence.
8. Let  $(M, d)$  be a metric space. A **completion** of  $(M, d)$  is a pair  $(\widehat{M}, \iota)$  consisting of a complete metric space  $(\widehat{M}, \hat{d})$  and a map  $\iota : M \rightarrow \widehat{M}$  such that:
  - (i)  $\iota$  is an **isometry** (i.e.,  $\hat{d}(\iota(x), \iota(y)) = d(x, y)$  for all  $x, y \in M$ ), and
  - (ii)  $\iota(M)$  is dense in  $\widehat{M}$ .
  - (a) Show that if  $(M, d)$  is already complete, then  $(M, \text{id}_M)$  is a completion of  $M$ .
  - (b) If  $(\widehat{M}_1, \iota_1)$  and  $(\widehat{M}_2, \iota_2)$  are completions of  $M$ , show that there exists a unique bijective isometry  $F : \widehat{M}_1 \rightarrow \widehat{M}_2$  such that  $F \circ \iota_1 = \iota_2$ .
9. Consider the rational numbers  $\mathbb{Q}$  equipped with the standard metric  $d(x, y) = |x - y|$ . Show that  $(\mathbb{R}, \iota)$ , where  $\iota : \mathbb{Q} \hookrightarrow \mathbb{R}$  is the inclusion map, is a completion of  $(\mathbb{Q}, d)$ .

10. Let  $(M, d)$  be a metric space. This exercise details the explicit construction of a completion of  $(M, d)$ .

Let  $C$  denote the set of all Cauchy sequences in  $M$ . Define a relation  $\sim$  on  $C$  by

$$(x_n) \sim (y_n) \iff \lim_{n \rightarrow \infty} d(x_n, y_n) = 0.$$

(a) Prove that  $\sim$  is an equivalence relation on  $C$ .

(b) Let  $\widehat{M} = C / \sim$  denote the set of equivalence classes, and denote the equivalence class of  $(x_n)$  by  $[(x_n)]$ . Define

$$\hat{d}([(x_n)], [(y_n)]) = \lim_{n \rightarrow \infty} d(x_n, y_n).$$

Prove that this limit always exists in  $\mathbb{R}$  for any Cauchy sequences  $(x_n)$  and  $(y_n)$  in  $M$ .

(c) Prove that  $\hat{d}$  is a well-defined metric on  $\widehat{M}$ .

(d) Define  $\iota : M \rightarrow \widehat{M}$  by  $\iota(x) = [(x, x, x, \dots)]$ . Show that  $\iota$  is an isometry and that  $\iota(M)$  is dense in  $\widehat{M}$ .

(e) Prove that  $(\widehat{M}, \hat{d})$  is complete, thereby concluding that  $(\widehat{M}, \iota)$  is a completion of  $(M, d)$ .