

Advanced Calculus I — Homework 4 (Fall 2026)

Accumulation points

- Find the accumulation points of each of the following subsets of $\mathbb{R}^2$:
 - $\{(m, n) \mid m, n \in \mathbb{Z}\}$
 - $\{(p, q) \mid p, q \in \mathbb{Q}\}$
 - $\{(m/n, 1/n) \mid m, n \in \mathbb{Z}, n \neq 0\}$
 - $\{(1/n + 1/m, 0) \mid n, m \in \mathbb{Z}, n \neq 0, m \neq 0\}$
- Let S be a subset of a metric space (M, d) . Is the set $\mathcal{L}(S)$ of accumulation points of S in M always closed? Justify your answer.

Cartesian products and closed sets

Let (M_1, d_1) and (M_2, d_2) be metric spaces.

- For any $(x_1, x_2), (y_1, y_2) \in M_1 \times M_2$, define

$$d((x_1, x_2), (y_1, y_2)) = d_1(x_1, y_1) + d_2(x_2, y_2).$$

Show that d is a metric on $M_1 \times M_2$.

- Let $A_1 \subseteq M_1$ and $A_2 \subseteq M_2$ be nonempty sets.
 - Show that $A_1 \times A_2$ is open in $M_1 \times M_2$ if and only if A_1 is open in M_1 and A_2 is open in M_2 .
 - Show that $A_1 \times A_2$ is closed in $M_1 \times M_2$ if and only if A_1 is closed in M_1 and A_2 is closed in M_2 .
- Let $A_1 \subseteq M_1$ and $A_2 \subseteq M_2$ be nonempty sets, and let $\text{pr}_1 : M_1 \times M_2 \rightarrow M_1$ be the projection map defined by $\text{pr}_1(x_1, x_2) = x_1$. Prove or disprove each of the following:
 - $\mathcal{L}(A_1 \times A_2) = \mathcal{L}(A_1) \times \mathcal{L}(A_2)$.
 - If U is open in $M_1 \times M_2$, then $\text{pr}_1(U) = \{\text{pr}_1(z) \in M_1 \mid z \in U\}$ is open in M_1 .
 - If F is closed in $M_1 \times M_2$, then $\text{pr}_1(F) = \{\text{pr}_1(z) \in M_1 \mid z \in F\}$ is closed in M_1 .

Interior, closure, and boundary

- Show that for any subset A of a metric space M :
 - $\text{int}(A) = A \setminus \partial(A)$.
 - $\text{cl}(A) = M \setminus \text{int}(M \setminus A)$.
- Determine whether each of the following statements is true or false.
 - $\text{int}(\text{cl}(A)) = \text{int}(A)$.
 - $\text{cl}(A) \cap A = A$.
 - $\text{cl}(\text{int}(A)) = A$.
 - $\partial(\text{cl}(A)) = \partial(A)$.
 - If A is open, then $\partial(A) \subseteq M \setminus A$.
- Prove the following properties for subsets A, B of a metric space M :
 - $\text{cl}(\text{cl}(A)) = \text{cl}(A)$.
 - $\text{cl}(A \cup B) = \text{cl}(A) \cup \text{cl}(B)$.
 - $\text{cl}(A \cap B) \subseteq \text{cl}(A) \cap \text{cl}(B)$.

9. Prove the following properties for subsets A, B of a metric space M :
- (a) $\partial(A) = \partial(M \setminus A)$.
 - (b) $\partial(\partial(A)) \subseteq \partial(A)$.
 - (c) $\partial(A \cup B) \subseteq \partial(A) \cup \partial(B) \subseteq \partial(A \cup B) \cup A \cup B$.
 - (d) $\partial(\partial(\partial(A))) = \partial(\partial(A))$.
10. An open subset U of a metric space (M, d) is called **regular open** if $U = \text{int}(\text{cl}(U))$. A closed subset F of (M, d) is called **regular closed** if $F = \text{cl}(\text{int}(F))$. Note that the empty set $\emptyset$ is both regular open and regular closed.
- (a) Construct a nonempty regular open set and a nonempty regular closed set in $\mathbb{R}$.
 - (b) Suppose F is closed in $\mathbb{R}^n$. Prove that $\text{int}(F)$ is regular open.
 - (c) Suppose U is open in $\mathbb{R}^n$. Prove that $\text{cl}(U)$ is regular closed.
 - (d) Let F_1 and F_2 be closed subsets of $\mathbb{R}^n$. The condition $\text{int}(F_1) \subseteq \text{int}(F_2)$ does not generally imply that $F_1 \subseteq F_2$ (construct a counterexample). Assuming F_1 and F_2 are both regular closed sets in $\mathbb{R}^n$, prove that $F_1 \subseteq F_2$ if and only if $\text{int}(F_1) \subseteq \text{int}(F_2)$.
 - (e) Prove that if F_1 and F_2 are regular closed sets in $\mathbb{R}^n$, then $F_1 \cup F_2$ is also regular closed.
[Note: The dual statement is: *If U and V are regular open subsets of $\mathbb{R}^n$, then $U \cap V$ is also regular open.*]