Advanced Calculus I — Homework 3 (Fall 2026)

Open sets in metric spaces

1. Let F be a finite subset of $\mathbb{R}^n$. Show that $\mathbb{R}^n \setminus F$ is open in $\mathbb{R}^n$.
2. Let $(x_n)_{n=1}^{\infty}$ be a sequence in a metric space (M, d), and let $x_* \in M$. Prove that $\lim_{n\to\infty} x_n = x_*$ in (M, d) if and only if for every open set U in (M, d) containing x_*, there exists $N \in \mathbb{N}$ such that $x_n \in U$ for all $n \geq N$.
3. Let (M, d) be a metric space, and V be a nonempty subset of M.
   (a) Prove that $(V, d|_{V\times V})$ is a metric space.
   (b) Prove that A is open in $(V, d|_{V\times V})$ if and only if there exists an open set U in (M, d) such that $A = V \cap U$.
   (c) Prove that B is closed in $(V, d|_{V\times V})$ if and only if there exists a closed set F in (M, d) such that $B = V \cap F$.
   (d) Prove or disprove that a subset $U \subseteq V$ is open in $(V, d|_{V\times V})$ if and only if U is open in (M, d).
4. Let d and d' be equivalent metrics on a nonempty set M. Prove the following:
   (a) A subset $U \subseteq M$ is open in (M, d) if and only if it is open in (M, d').
   (b) A subset $F \subseteq M$ is closed in (M, d) if and only if it is closed in (M, d').
5. Let A be a subset of $(\mathbb{R}^n, d_e)$, where d_e is the Euclidean metric. For $\delta > 0$, define the **δ-neighborhood** of A as follows:
$$D(A, \delta) = \{x \in \mathbb{R}^n \mid d_e(x, a) < \delta \text{ for some } a \in A\}.$$
   Prove that $D(A, \delta)$ is an open set in $\mathbb{R}^n$.
6. In $\mathbb{R}^2$, consider the sets
$$\mathcal{E} = \{(x, y) \in \mathbb{R}^2 \mid x^2 + 2xy + 3y^2 < 1\},$$
$$\mathcal{H} = \{(x, y) \in \mathbb{R}^2 \mid xy < 1\}.$$
   Prove that $\mathcal{E}$ and $\mathcal{H}$ are open sets in $\mathbb{R}^2$.
7. Given a nonempty open set U in $\mathbb{R}$, and a nonempty subset S in $\mathbb{R}$, define
$$U \cdot S = \{u \cdot s \mid u \in U, s \in S\}.$$
   (a) Prove or disprove that $U \cdot S$ is always an open set in $\mathbb{R}$.
   (b) Assume that S is also open. Can we conclude that $U \cdot S$ is open?
8. Let A be a subset of a metric space (M, d). Let $\operatorname{int}(A)$ denote the interior of A. Prove or disprove the following statements:
   (a) $\operatorname{int}(\operatorname{int}(A)) = \operatorname{int}(A)$.
   (b) Let A_1 and A_2 be two subsets of $(\mathbb{R}, d_e)$. For the product set $A_1 \times A_2$ in $(\mathbb{R}^2, d_e)$, we have the relation: $\operatorname{int}(A_1 \times A_2) = \operatorname{int}(A_1) \times \operatorname{int}(A_2)$.
   (c) Define the projection $P_j : \mathbb{R}^n \to \mathbb{R} : (x_1, \ldots, x_n) \mapsto x_j$. Let U be a subset of $\mathbb{R}^n$ which is open in $\mathbb{R}^n$. Then $P_j(U)$ is an open subset of $\mathbb{R}$.
   (d) Let P_j be the projection defined in part (c). Then for any subset S of $\mathbb{R}^n$, $\operatorname{int}(P_j(S)) = P_j(\operatorname{int}(S))$.

Topological spaces

A **topological space** is an ordered pair (X, τ), where X is a set and τ is a collection of subsets of X (called a **topology** on X) satisfying the following three axioms:

(i) The empty set $\varnothing$ and the whole set X belong to τ:
$$\varnothing \in \tau \quad \text{and} \quad X \in \tau.$$

(ii) The union of an arbitrary collection of sets in τ belongs to τ:

$$\text{If } \{U_\alpha\}_{\alpha\in I} \subseteq \tau, \text{ then } \bigcup_{\alpha\in I} U_\alpha \in \tau.$$

(iii) The intersection of any finite collection of sets in τ belongs to τ:

$$\text{If } U_1, U_2, \ldots, U_n \in \tau, \text{ then } \bigcap_{i=1}^{n} U_i \in \tau.$$

The elements of τ are called the **open sets** in (X, τ).

Note that if (M, d) is a metric space and τ is the collection of all open sets in (M, d), then (M, τ) is a topological space.

9. Let X be an infinite set, and define

$$\tau := \{U \subseteq X \mid X \setminus U \text{ is finite}\} \cup \{\varnothing\}.$$

   (a) Prove that (X, τ) is a topological space.

   (b) Does there exist a metric d on X such that $U \in \tau$ if and only if U is open in (X, d)? Prove your answer.

10. Let (X, τ) be a topological space, and let $\sim$ be an equivalence relation on X. Let

$$\pi : X \to X/\sim, \qquad \pi(x) = [x]$$

be the canonical projection map sending an element $x \in X$ to its equivalence class $[x]$. Define the **quotient topology** τ' on $X/\sim$ by

$$U \in \tau' \iff \pi^{-1}(U) \in \tau.$$

   (a) Prove that $(X/\sim, \tau')$ is a topological space.

   (b) Consider the closed interval $[0, 1] \subseteq \mathbb{R}$ equipped with the standard **subspace topology** (i.e. A is open in $[0, 1]$ if and only if there exists an open set U in $(\mathbb{R}, d_e)$ such that $A = [0, 1] \cap U$). Define the relation $\sim$ on $[0, 1]$ by

$$\sim := \{(x, x) \mid x \in [0, 1]\} \cup \{(1, 0), (0, 1)\}.$$

   Prove that $\sim$ is an equivalence relation on $[0, 1]$.

   (c) Let τ' be the quotient topology on $[0, 1]/\sim$. Prove that there exists a bijection $f : [0, 1]/\sim \to S^1$ such that

$$U \in \tau' \iff f(U) \text{ is open in } (S^1, d),$$

   where $S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ and $d = d_e|_{S^1\times S^1}$ is the subspace metric induced by the Euclidean metric on $\mathbb{R}^2$.