Advanced Calculus I — Homework 2 (Fall 2026)

Norm and inner product

Recall that an **inner product space** is a real vector space V together with a map $\langle -,- \rangle : V \times V \to \mathbb{R}$ satisfying the following properties for all $x, y, z \in V$ and $\alpha \in \mathbb{R}$:

(i) **Symmetry:** $\langle x, y \rangle = \langle y, x \rangle$.

(ii) **Linearity in the first argument:** $\langle \alpha x + y, z \rangle = \alpha \langle x, z \rangle + \langle y, z \rangle$.

(iii) **Positive-definiteness:** $\langle x, x \rangle \geq 0$, and $\langle x, x \rangle = 0$ if and only if $x = 0$.

1. Let $(V, \langle -,- \rangle)$ be an inner product space. For $x \in V$, define $\| x \| = \sqrt{\langle x, x \rangle}$. Show that $(V, \| - \|)$ is a normed vector space.

2. In an inner product space, prove the following statements:

   (a) $2 \| x \|^2 + 2 \| y \|^2 = \| x + y \|^2 + \| x - y \|^2$ (*parallelogram law*).

   (b) $\| x + y \| \| x - y \| \leq \| x \|^2 + \| y \|^2$.

   (c) $4 \langle x, y \rangle = \| x + y \|^2 - \| x - y \|^2$ (*polarization identity*).

   Interpret these results geometrically in terms of the parallelogram formed by x and y.

3. Let $(V, \| - \|)$ be a normed vector space. Prove that there exists an inner product $\langle -,- \rangle$ on V such that $\| x \| = \sqrt{\langle x, x \rangle}$ for all $x \in V$ if and only if the parallelogram law $2 \| x \|^2 + 2 \| y \|^2 = \| x + y \|^2 + \| x - y \|^2$ holds for all $x, y \in V$.

Countability

4. A complex number z is said to be **algebraic** if there exist integers $a_0, a_1, \ldots, a_n$, not all zero, such that

$$a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n = 0.$$

   (a) Prove that the set of all algebraic numbers is countable.

   (b) Prove that there exist real numbers that are not algebraic (i.e., transcendental numbers).

5. Is the set of all irrational real numbers countable? Prove your answer.

6. Let $S_1 = \{a_1, a_2, a_3\}$ with $a_j \neq a_k$ for $j \neq k$, and let S_2 be an infinitely countable set of real numbers.

   (a) Suppose $(a_n^{(j)})_{n=1}^{\infty}$ for $j = 1, 2, 3$ are three convergent sequences such that $\lim_{n \to \infty} a_n^{(j)} = a_j$ for each j. Define a single sequence $(b_n)_{n=1}^{\infty}$ using these three sequences such that a_1, a_2, and a_3 are cluster points of $(b_n)_{n=1}^{\infty}$.

   (b) Prove that there exists a sequence $(x_n)_{n=1}^{\infty}$ such that every point in S_2 is a cluster point of $(x_n)_{n=1}^{\infty}$.

Supremum and infimum

7. For nonempty sets $A, B \subset \mathbb{R}$, let $A + B = \{x + y \mid x \in A \text{ and } y \in B\}$. Show that $\sup(A + B) = \sup(A) + \sup(B)$.

8. For nonempty sets $A, B \subset \mathbb{R}$ with $A \cap B \neq \emptyset$, determine which of the following statements are true. Prove the true statements and provide counterexamples for those that are false:

   (a) $\sup(A \cap B) \leq \inf\{\sup(A), \sup(B)\}$.

   (b) $\sup(A \cap B) = \inf\{\sup(A), \sup(B)\}$.

   (c) $\sup(A \cup B) \geq \sup\{\sup(A), \sup(B)\}$.

   (d) $\sup(A \cup B) = \sup\{\sup(A), \sup(B)\}$.

9. Let $S = \{(x, y) \in \mathbb{R}^2 \mid xy > 1\}$ and $B = \{d((x, y), (0, 0)) \mid (x, y) \in S\}$. Find $\inf(B)$ and prove your answer.

Limits superior and inferior

10. Determine whether each of the following statements is true or false. Prove the true statements, and provide a counterexample for each false statement:
    (a) If $(a_n)_{n=1}^{\infty}$ is bounded above, then $\overline{\lim\limits_{n\to\infty}}\, a_n$ is finite.
    (b) If
    $$-\infty < \underline{\lim\limits_{n\to\infty}}\, b_n \le \overline{\lim\limits_{n\to\infty}}\, b_n < +\infty,$$
    then $(b_n)_{n=1}^{\infty}$ is a bounded sequence.
11. Let $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ be two bounded sequences. It is a known fact that
    $$\overline{\lim_{n\to\infty}}(a_n + b_n) \le \overline{\lim_{n\to\infty}}\, a_n + \overline{\lim_{n\to\infty}}\, b_n,$$
    $$\underline{\lim_{n\to\infty}}\, a_n + \underline{\lim_{n\to\infty}}\, b_n \le \underline{\lim_{n\to\infty}}(a_n + b_n).$$
    (a) Construct two bounded sequences (a_n) and (b_n) such that
    $$\overline{\lim_{n\to\infty}}(a_n + b_n) < \overline{\lim_{n\to\infty}}\, a_n + \overline{\lim_{n\to\infty}}\, b_n.$$
    (b) Suppose (a_n) is convergent. How can the inequalities above be simplified or refined?

Sequences of sets

Let $(S_n)_{n=1}^{\infty}$ be a sequence of subsets of $\mathbb{R}^m$. Define the **limit superior** and **limit inferior** of $(S_n)_{n=1}^{\infty}$ respectively as
$$\overline{\lim_{n\to\infty}}\, S_n = \bigcap_{n=1}^{\infty}\left(\bigcup_{k=n}^{\infty} S_k\right), \qquad \underline{\lim_{n\to\infty}}\, S_n = \bigcup_{n=1}^{\infty}\left(\bigcap_{k=n}^{\infty} S_k\right).$$
If $\overline{\lim\limits_{n\to\infty}}\, S_n = \underline{\lim\limits_{n\to\infty}}\, S_n$, we say that the sequence $(S_n)_{n=1}^{\infty}$ converges to the limit $\lim\limits_{n\to\infty} S_n = \overline{\lim\limits_{n\to\infty}}\, S_n$.

For a subset $S \subseteq \mathbb{R}^m$, let $S^c = \mathbb{R}^m \setminus S$ denote its complement.

12. Let $(S_n)_{n=1}^{\infty}$ be a sequence of subsets of $\mathbb{R}^m$. Prove that
    $$\left[\overline{\lim_{n\to\infty}}\, S_n\right]^c = \underline{\lim_{n\to\infty}}\, S_n^c.$$
13. Let $A, B \subseteq \mathbb{R}^m$. Define a sequence of subsets $(S_n)_{n=1}^{\infty}$ by
    $$S_n = \begin{cases} A, & \text{if } n \text{ is odd,} \\ B, & \text{if } n \text{ is even.} \end{cases}$$
    Find the limit superior and limit inferior of $(S_n)_{n=1}^{\infty}$.
14. Given a sequence $(S_n)_{n=1}^{\infty}$ of subsets of $\mathbb{R}^m$ such that $S_{n+1} \subseteq S_n$ for all $n \ge 1$, prove that $\lim\limits_{n\to\infty} S_n$ exists and equals $\bigcap_{n=1}^{\infty} S_n$.
15. Let $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ be two real sequences satisfying:
    (i) $0 < a_n < 1 < b_n$ for all n.
    (ii) $a_n > a_{n+1}$ for all n, with $\lim\limits_{n\to\infty} a_n = 0$.
    (iii) $b_n > b_{n+1}$ for all n, with $\lim\limits_{n\to\infty} b_n = 1$.

    Find $\underline{\lim\limits_{n\to\infty}}\,[a_n, b_n]$ and $\overline{\lim\limits_{n\to\infty}}\,[a_n, b_n]$. Is the sequence of intervals $([a_n, b_n])_{n=1}^{\infty}$ convergent?
16. Let $(c_n)_{n=1}^{\infty}$ and $(d_n)_{n=1}^{\infty}$ be bounded sequences with $c_n < d_n$ for all n.
    (a) Suppose that $\overline{\lim\limits_{n\to\infty}}\, c_n < \underline{\lim\limits_{n\to\infty}}\, d_n$. Are the limit superior and limit inferior of the interval sequence $([c_n, d_n])_{n=1}^{\infty}$ nonempty sets? Justify your assertion.
    (b) Express the sets $\underline{\lim\limits_{n\to\infty}}\,[c_n, d_n]$ and $\overline{\lim\limits_{n\to\infty}}\,[c_n, d_n]$ in terms of the limits superior and inferior of (c_n) and (d_n).