Advanced Calculus I — Homework 1 (Fall 2026)

1. Let $f : [0, +\infty) \to [0, +\infty)$ be a bijective (i.e., one-to-one and onto), differentiable, strictly increasing function satisfying $f(0) = 0$. Let $g(y) = f^{-1}(y)$ denote the inverse function of f. Define

$$\Phi(z) = \int_0^z f(t)\,dt \quad \text{and} \quad \Psi(z) = \int_0^z g(s)\,ds.$$

   (a) Prove that for $c > 0$, $cf(c) = \Phi(c) + \Psi(f(c))$.

   (b) For $a, b \geq 0$, prove that
   $$ab \leq \Phi(a) + \Psi(b),$$
   and prove that equality holds if and only if $b = f(a)$.

   (c) By taking $f(x) = x^{p-1}$ with $p > 1$, prove that $ab \leq \frac{a^p}{p} + \frac{b^{p'}}{p'}$, where $\frac{1}{p} + \frac{1}{p'} = 1$.

Recall the following definitions:

(i) Let d and d' be two metrics defined on a set M. We say that d' is **equivalent** to d, denoted by $d \sim d'$, if there exist constants $\alpha, \beta > 0$ such that

$$\alpha d(x, y) \leq d'(x, y) \leq \beta d(x, y) \quad \forall x, y \in M.$$

(ii) A **relation** $\sim$ on a set S is a subset of $S \times S$. We write $x \sim y$ if $(x, y) \in \sim$. An **equivalence relation** is a relation satisfying:

- **Reflexivity:** $x \sim x \quad \forall x \in S$.
- **Symmetry:** $x \sim y \iff y \sim x \quad \forall x, y \in S$.
- **Transitivity:** $x \sim y$ and $y \sim z \implies x \sim z \quad \forall x, y, z \in S$.

2. Prove that the equivalence of metrics on a set M is an equivalence relation.

3. Let d and d' be equivalent metrics on a set M. Suppose $(x_n)_{n=1}^\infty$ is a sequence in M and $x_* \in M$. Prove that $\lim_{n\to\infty} d(x_n, x_*) = 0$ if and only if $\lim_{n\to\infty} d'(x_n, x_*) = 0$. In other words, $x_n \to x_*$ in (M, d) if and only if $x_n \to x_*$ in (M, d').

4. On $\mathbb{R}^n$, we can define many metrics; the following are three examples: For $x, y \in \mathbb{R}^n$, where $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$, and $p \geq 1$, define

$$d_2(x, y) = \sqrt{\sum_{j=1}^n (x_j - y_j)^2} \qquad \text{(The Euclidean metric)},$$

$$d_p(x, y) = \left(\sum_{j=1}^n |x_j - y_j|^p \right)^{1/p} \qquad \text{(The } p\text{-norm metric)},$$

$$d_\infty(x, y) = \max\{|x_j - y_j| : j = 1, \ldots, n\} \qquad \text{(The sup norm metric)}.$$

   (a) Prove that these three metrics are equivalent.

   (b) For $x, y \in \mathbb{R}^n$, is it true that $\lim_{p\to+\infty} d_p(x, y) = d_\infty(x, y)$? Justify your assertion.

5. Let V be a nonempty subset of $\mathbb{R}^n$. Let d_e denote the Euclidean metric on $\mathbb{R}^n$. For $x, y \in V$, define

$$d(x, y) = \frac{d_e(x, y)}{1 + d_e(x, y)}$$

   (a) Prove that $(V, d_e|_{V\times V})$ is a metric space.

   (b) Prove that (V, d) is also a metric space.

   (c) Let $x_*, x_m \in V$ for $m \in \mathbb{N}$. Prove that $x_m \to x_*$ in $(V, d_e|_{V\times V})$ if and only if $x_m \to x_*$ in (V, d).

   (d) Prove or disprove that d_e is equivalent to d.

6. Recall that if $(x_n)_{n=1}^{\infty}$ is a bounded sequence in $\mathbb{R}$, then $(x_n)_{n=1}^{\infty}$ has a convergent subsequence (the Bolzano–Weierstrass theorem). Let (V, d) be a metric space. Is it true that any bounded sequence in V has a convergent subsequence? Justify your assertion.

7. Let ℓ_2 denote the set of infinite sequences $c = (c_n)_{n=1}^{\infty}$ such that $\sum_{n=1}^{\infty} |c_n|^2 < +\infty$. For $c, d \in \ell_2$, define

$$\| c \|_2 = \sqrt{\sum_{n=1}^{\infty} |c_n|^2}.$$

Follow these steps to prove that $(\ell_2, \| - \|_2)$ is a normed vector space:

   (a) Prove that $\| c \|_2 = 0$ if and only if $c_n = 0$ for all n; and $\| rc \|_2 = |r| \cdot \| c \|_2$ for any $c \in \ell_2$ and any $r \in \mathbb{R}$.

   (b) Prove that for $c, d \in \ell_2$, the infinite series $\sum_{n=1}^{\infty} c_n d_n$ is convergent.

   (c) For $c, d \in \ell_2$, we can define the inner product:

$$\langle c, d \rangle = \sum_{n=1}^{\infty} c_n d_n.$$

   Prove that $|\langle c, d \rangle| \leq \| c \|_2 \cdot \| d \|_2$.

   (d) Prove that if $c, d \in \ell_2$ and $r \in \mathbb{R}$, then $c + rd \in \ell_2$. In other words, ℓ_2 is a vector space over $\mathbb{R}$.

   (e) Prove that $\| c + d \|_2 \leq \| c \|_2 + \| d \|_2$ for all $c, d \in \ell_2$.

8. Let $(c_n)_{n=1}^{\infty}$ be a sequence of real numbers. Let C_0 be the set consisting of all convergent real sequences $(c_n)_{n=1}^{\infty}$ whose limits are equal to 0. For $c = (c_n) \in C_0$, the set $\{|c_n| \mid n = 1, 2, \cdots\}$ is bounded; define

$$\| c \|_{\infty} = \sup\{|c_n| \mid n = 1, 2, \dots\}.$$

Let ℓ_2 be the vector space defined in the previous problem.

   (a) Prove that $(C_0, \| - \|_{\infty})$ is a normed vector space.

   (b) Show that ℓ_2 is a subspace of C_0.

   (c) On ℓ_2, are the two norms $\| - \|_{\infty}$ and $\| - \|_2$ equivalent? That is, do there exist two positive constants α and β such that

$$\alpha \| c \|_{\infty} \leq \| c \|_2 \leq \beta \| c \|_{\infty}$$

   for all $c \in \ell_2$?