Advanced Calculus I — Homework 10 (Fall 2026)

Unless otherwise specified, Euclidean spaces have their usual metrics, and subsets have the induced metrics. Write $\mathrm{cl}(A)$ for the closure of A in the ambient space under consideration.

Uniform continuity and boundedness

1. Let $f : (0,1) \to \mathbb{R}$ be uniformly continuous. Can there exist a sequence $(x_n)_{n\geq 1}$ in $(0,1)$ such that $f(x_n) = n$ for every $n \geq 1$? Justify your answer.

2. Let $S \subseteq \mathbb{R}^n$ be nonempty and bounded, and let $f : S \to \mathbb{R}$ be uniformly continuous. Prove that $\mathrm{cl}(f(S))$ is compact.

Pointwise and uniform convergence

3. For $n \in \mathbb{N}$, define $f_n : (0,1) \to \mathbb{R}$ by
$$f_n(x) = \frac{1}{nx+1}.$$
   (a) Prove that $f_n(x) \to 0$ for every $x \in (0,1)$.
   (b) Prove that (f_n) does not converge uniformly to 0 on $(0,1)$.

4. For $n \in \mathbb{N}$, define $f_n : [0,1] \to \mathbb{R}$ by
$$f_n(x) = n^2 x(1-x)^n.$$
   (a) Prove that $f_n(x) \to 0$ for every $x \in [0,1]$.
   (b) Let $x_n = 1/(n+1)$. Prove that $f_n(x_n) \to +\infty$.

5. Let $g_n : [0,1] \to \mathbb{R}$ be continuous functions converging uniformly to g. Suppose that $x_n \in [0,1]$ and $x_n \to x_* \in [0,1]$.
   (a) Prove that $g_n(x_n) \to g(x_*)$.
   (b) Apply this result to show that the sequence in Question 4 does not converge uniformly on $[0,1]$.

6. Let $a < b$, and let $f_n : (a,b) \to \mathbb{R}$ be uniformly continuous functions converging uniformly to f. Must f be uniformly continuous on (a,b)? Justify your answer.

Uniform convergence of series

7. Let (a_n) be a sequence of real numbers such that $\sum_{n=1}^{\infty} |a_n| < \infty$. Prove that
$$\sum_{n=1}^{\infty} a_n e^{-nx}$$
converges uniformly on $[0, +\infty)$.

8. Does the series

$$\sum_{n=1}^{\infty} \frac{1}{(1+x)^n}$$

converge uniformly on $[1, 2]$? Justify your answer.

9. Let (a_n) be a sequence of real numbers such that $\sum_{n=1}^{\infty} a_n^2 < \infty$. Prove that

$$\sum_{n=1}^{\infty} \frac{a_n x^n}{n}$$

converges uniformly on $[0, 1]$.