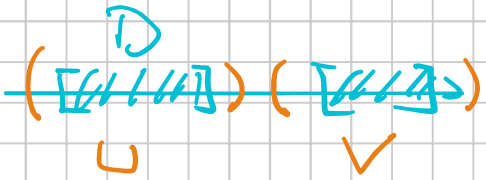
Advanced Calculus, week 9, Fall 2026

Connectedness



Def (Def 3.5.1)

Let $D \subseteq (M,d)$. The set D is said to be disconnected if $\exists\ U, V \underset{\text{open}}{\subset} M$ s.t.

(a) $U \cap D \neq \emptyset$, $V \cap D \neq \emptyset$

(b) $D \subseteq U \cup V$

(c) $U \cap V \cap D = \emptyset$ ← $(D \cap U) \cap (D \cap V)$

The pair (U, V) is called a separation (or disconnection) of D.

Let $C \subseteq (M,d)$. If C is not disconnected then we say C is connected.

Thm

Let $D \subseteq (M,d)$. Then D is disconnected

$\Leftrightarrow \exists\ U_D \subset D$ s.t.

(i) $U_D \neq \emptyset$, $U_D \neq D$

(ii) U_D is both relatively open and relatively closed in D.

$\exists\ U \underset{\text{open}}{\subset} M$ s.t. $U_D = U \cap D$

$\exists\ F \underset{\text{closed}}{\subset} M$ s.t. $U_D = D \cap F$.

Pf

"⇒" Suppose D is disconnected. Then

$\exists\ U, V$ open in M s.t.

(a) $U \cap D \neq \emptyset$, $V \cap D \neq \emptyset$

(b) $D \subseteq U \cup V$

(c) $U \cap V \cap D = \emptyset$

$\Rightarrow (U \cap D) \cap (V \cap D) = \emptyset$

Define

$U_D = U \cap D$

$\Rightarrow$ (i) $U_D = U \cap D \overset{(a)}{\neq} \emptyset$

$D \setminus U_D \overset{(c)}{=} D \cap U_D^c \supseteq D \cap (V \cap D)$

$= V \cap D \overset{(a)}{\neq} \emptyset$

$\Rightarrow U_D \neq D$

(ii) $U_D = U \cap D$ is relatively open

By (c), $U \cap V \cap D = \emptyset \Rightarrow V^c \supseteq U \cap D = U_D$

($V \cap (U \cap D)$)

$\Rightarrow V^c \cap D \supseteq U_D \cap D = U_D$

Conversely, if $x \in D \cap V^c$, then $x \in D, x \notin V$

$\because D \subseteq U \cup V \ \therefore x \in U \Rightarrow x \in U_D$

$\Rightarrow D \cap V^c \subseteq U_D$

$\Rightarrow U_D = D \cap V^c$ is relatively closed!

(V^c closed in M)

"$\Leftarrow$" Assume $\exists\ U_D \subseteq D$, $U_D \neq \emptyset, \neq D$ and U_D is both relatively open & closed relatively

$\Rightarrow \exists\ U \subseteq_{\text{open}} M,\ F \subseteq_{\text{closed}} M$ s.t.

$$D \cap U = U_D, \qquad D \cap F = U_D$$

Let $V = F^c = M \setminus F$ which is open in M.

(a) $U \cap D = U_D \neq \emptyset$

$$V \cap D = F^c \cap D = D \setminus (D \cap F)$$
$$= D \setminus U_D \quad (\because U_D \neq D) \quad \neq \emptyset$$

(b) $U \cup V = U \cup F^c \supseteq D$

$\forall x \in D$, $x \in U_D \Rightarrow x \in U$

or

$x \notin U_D \Rightarrow x \in (U_D)^c$

$U_D = F \cap D$, $(U_D)^c = (F \cap D)^c$

$x \in F^c$ $\quad (\because x \in D)$ $\quad \subseteq F^c \cup D^c$

(c) $U \cap V \cap D = U \cap F^c \cap D$

$= (U \cap D) \cap (F^c \cap D)$ $\quad = D \setminus (D \cap F)$

$= U_D \cap (D \setminus U_D)$ $\quad = D \setminus U_D$

$= \emptyset$

$\Rightarrow D$ is disconnected. #

Thm

Let $C \subseteq (M, d)$. Then

C is connected

$\Leftrightarrow$ The only subsets of C that are relatively open and relatively closed are $\emptyset$ and C.

Example

$\mathbb{Q} \subseteq \mathbb{R}$ is disconnected

pf

Choose an irrational number r (e.g. $r = \sqrt{2}$)

Define

$$U = (-\infty, r)$$

$$V = (r, \infty)$$

which are open in $\mathbb{R}$.

Then

(a) $U \cap \mathbb{Q} \neq \emptyset$ $\quad V \cap \mathbb{Q} \neq \emptyset$

since $\mathbb{Q}$ is dense in $\mathbb{R}$

(b) $\mathbb{Q} \subseteq U \cup V = \mathbb{R} - \{r\}$

(c) $U \cap V \cap \mathbb{Q} = \emptyset$ ($U \cap V = \emptyset$)

$\Rightarrow$ $\mathbb{Q}$ is disconnected. ∎

Example

$[0, 1]$ is connected

More generally, any (closed) interval is connected

pf

Suppose $[0,1]$ is disconnected, i.e.
$\exists\ U, V \subseteq_{\text{open}} \mathbb{R}$ s.t.

(a) $U \cap [0,1] \neq \emptyset$, $V \cap [0,1] \neq \emptyset$

(b) $[0,1] \subset U \cup V$

(c) $U \cap V \cap [0,1] = \emptyset$.

By (b), $0 \in U$ or $0 \in V$.

Without loss of generality, assume $0 \in U$.

Consider

$$A = \{a \in [0,1] \mid [0,a] \subseteq U\} \subseteq U$$

$\Rightarrow$ $0 \in A$ (so $A \neq \emptyset$)

$A \subseteq [0,1]$

$\Rightarrow$ $0 \leq \alpha := \sup A \leq 1$

① Note that $U \cap [0,1]$ is relatively closed

$\Rightarrow \exists\ F \subset_{\text{closed}} \mathbb{R}$ s.t. $F \cap [0,1] = U \cap [0,1]$

which is closed in $\mathbb{R}$

② $A \subseteq U \cap [0,1] \subset_{\text{closed}} \mathbb{R}$

$\Rightarrow \alpha = \sup(A) \in cl(A) \subseteq U \cap [0,1]$

$\Rightarrow \alpha \in U$

③ Claim: $\alpha = 1$

If $\alpha < 1$, since $\alpha \in U \subseteq_{\text{open}} \mathbb{R}$,

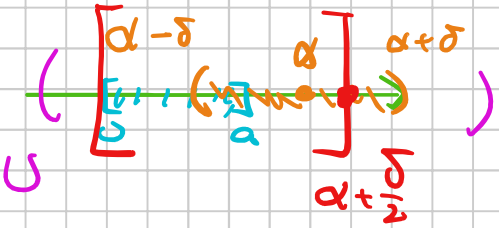
$\exists\ \delta > 0$ s.t.

- $\alpha + \delta < 1$
- $(\alpha - \delta,\ \alpha + \delta) \subseteq U$

Since $\alpha = \sup A$, $\exists\ a \in A$ s.t.

$$\alpha - \delta < a \leq \alpha$$

$$\Rightarrow [0, a] \subseteq U$$



$$\Rightarrow [0,\ \alpha + \tfrac{\delta}{2}] \subseteq U$$

$\Rightarrow$ $\alpha + \frac{\delta}{2} \in A$ $\Rightarrow$ α is NOT an upper bound of A ($\rightarrow\leftarrow$)

$\alpha = \sup(A)$

So $\alpha = 1$

④ $\alpha = 1 \in U \Rightarrow \delta > 0$ s.t.

$$(\alpha - \delta,\ \alpha + \delta) \subseteq U$$

$\exists\ a \in A$ s.t. $a > \alpha - \delta \Rightarrow [0, a] \subseteq U$

$\Rightarrow [0, \alpha] \subseteq U \Rightarrow [0, 1] \subseteq U$.

⑤ $V \cap [0,1] = [0,1] \setminus ([0,1] \cap U) = [0,1]$

$[0,1] \cap V \cap U = \phi$, $U \cup V \supseteq [0,1]$

$= \phi$ $\rightarrow\leftarrow$ to (a)

So $[0,1]$ is connected. #

Example Show that the only connected subsets of $\mathbb{R}$ are $\emptyset$, $\{a\}$, intervals. I

$\forall a<b$, $a,b \in I$, we have $a<c<b \Rightarrow c \in I$

Pf

Assume $C \subseteq \mathbb{R}$ is connected, not $\emptyset$, $\{a\}$

That is, $\exists\ x \neq y$, $x, y \in C$

If C is not an interval, then

$\exists\ a < r < b$ s.t. $a, b \in C$, $r \notin C$

Then $((-\infty, r), (r, \infty))$ is a separation of C:

(a) $a \in C \cap (-\infty, r) \neq \emptyset$ $\quad C \cap (r, \infty) \neq \emptyset$ (b)

(b) $C \subseteq (-\infty, r) \cup (r, \infty) = \mathbb{R} - \{r\}$

(c) $C \cap (-\infty, r) \cap (r, \infty) = \emptyset$

$\Rightarrow$ C is disconnected ($\lightning$)

So C is an interval #

Prop

If $C_1, C_2 \subseteq (M, d)$ are connected and $C_1 \cap C_2 \neq \emptyset$, then $C_1 \cup C_2$ is connected.

pf

Assume $C_1 \cup C_2$ is NOT cnt.

$\Rightarrow \exists\ U, V \subseteq_{\text{open}} M$ s.t.

(a) $U \cap (C_1 \cup C_2) \neq \phi$, $V \cap (C_1 \cup C_2) \neq \phi$

(b) $C_1 \cup C_2 \subseteq U \cup V$

(c) $U \cap V \cap (C_1 \cup C_2) = \phi$

① Then $C_1 \subseteq C_1 \cup C_2 \overset{(b)}{\subseteq} U \cup V$

$U \cap V \cap C_1 \overset{(c)}{=} \phi$

Since C_1 is cnt., we have

$U \cap C_1 = \phi$ or $V \cap C_1 = \phi$

$\Rightarrow$ either $C_1 \subseteq V$ or $C_1 \subseteq U$

Similarly, either $C_2 \subseteq V$ or $C_2 \subseteq U$

② Since $C_1 \cap C_2 \neq \phi$, $\exists\ x \in C_1 \cap C_2$

By (b), $x \in U$ or $x \in V$.

Assume $x \in U$ (the other case is similar)

- $x \in C_1 \Rightarrow C_1 \cap U \neq \phi \overset{①}{\Rightarrow} C_1 \subseteq U$
- $x \in C_2 \Rightarrow C_2 \cap U \neq \phi \overset{①}{\Rightarrow} C_2 \subseteq U$

$\Rightarrow C_1 \cup C_2 \subseteq U$

$\Rightarrow (C_1 \cup C_2) \cap V = \phi$ $\not\rightarrow\leftarrow$ to (a)

So $C_1 \cup C_2$ is connected. #

Remark (exer.)

By the same argument, if $\{C_i\}_{i\in\Lambda}$ is a family of cpt. sets with

$$\bigcap_{i\in\Lambda} C_i \neq \emptyset$$

then $\bigcup_{i\in\Lambda} C_i$ is cpt.

Recall

Let $S \subseteq (M,d)$. Then (S, d_S) ($d_S = d|_{S\times S}$) is also a metric sp. A subset $U \subseteq S$ is (relatively) open in S

$\Leftrightarrow \exists\ \tilde{U} \subseteq_{\text{open}} M$ s.t. $U = S \cap \tilde{U}$

$\Leftrightarrow$ U is open in (S, d_S)

Notation:

$U \subseteq_{\text{open}} S \Leftrightarrow U$ is (relatively) open in S

$F \subseteq_{\text{closed}} S \Leftrightarrow F$ is (relatively) closed in S

Intrinsic definition of connectedness

A metric space (M,d) is disconnected if $\exists\ U \neq^{*}\emptyset, V \neq^{*}\emptyset \subseteq_{\text{open}} M$ s.t.

$$M = U \cup V \quad , \quad U \cap V = \emptyset$$

If no such pair exists, M is connected

Remark: This is the special case $D = M$ of Tuesday's definition

For $D \subseteq (M, d)$, (D, d_D) is a metric sp.

We say D is (dis)connected if (D, d_D) is (dis)connected, i.e. $\exists\ U, V \neq \emptyset$ open in D s.t.

$D = U \cup V$, $U \cap V = \emptyset$

$D = D \cap (\tilde{U} \cup \tilde{V})$ $\Leftrightarrow \tilde{U} \cup \tilde{V} \supseteq D$

$D \cap \tilde{U} \cap \tilde{V} \neq \emptyset$

$\exists\ \tilde{U}, \tilde{V}$ open in M s.t. $U = D \cap \tilde{U} \neq \emptyset$, $V = D \cap \tilde{V} \neq \emptyset$

Prop

A subset $D \subseteq M$ is disconnected in the sense of today's definition $\Leftrightarrow$ D is disconnected in the sence of Tuesday's definition

pf: exer.

Remark

S is disconnected $\Leftrightarrow \exists\ U, V \neq \emptyset$ open in S s.t. $S = U \cup V$, $U \cap V = \emptyset$

$\Rightarrow U = S \setminus V$ (V open) is closed in S

$\Rightarrow \exists\, \emptyset \neq U \neq S$ is open, closed in S.

Conversely, if $\exists\, \emptyset \neq U \subsetneq_{\text{open closed}} S$, then

$$V := S \setminus U \subsetneq_{\text{open}} S$$

$\Rightarrow S = U \cup V$, $U \cap V = \emptyset \Rightarrow S$ is disconnected.

Recall

- Intervals are connected
- If $\{C_i\}_{i\in\Lambda}$ is a family of connected set s.t. $\bigcap_{i\in\Lambda} C_i \neq \emptyset$, then $\bigcup_{i\in\Lambda} C_i$ is cnt.

Lemma

If $f:[a,b] \to (M,d)$ is a continuous path, then $f([a,b]) = \{f(x) \in M \mid x \in [a,b]\}$ is connected.

pf

Step1: "$U \subsetneq_{\text{open}} M \Rightarrow f^{-1}(U) = \{x \in [a,b] \mid f(x) \in U\}$ is open in $[a,b]$"

Given $t \in f^{-1}(U) \Rightarrow f(t) \in U \subsetneq_{\text{open}} M$

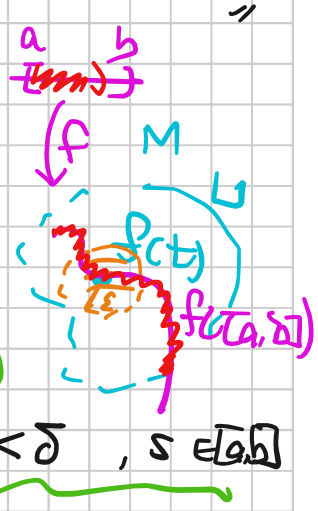
$\Rightarrow \exists\, \varepsilon > 0$ s.t. $D_M(f(t),\varepsilon) \subseteq U$

f conti. at t $\Rightarrow \exists\, \delta > 0$ s.t.

$d(f(t), f(s)) < \varepsilon$ when $|t-s| < \delta$, $s \in [a,b]$

$\Leftrightarrow f(s) \in D_M(f(t),\varepsilon)$

$\Leftarrow s \in D(t,\delta)$



$\Rightarrow D_{[a,b]}(t,\delta) \subseteq f^{-1}(D_M(f(t),\varepsilon))$

$\subseteq f^{-1}(U)$

$\Rightarrow f^{-1}(U) \underset{\text{open}}{\subseteq} [a,b]$.

Step2: $f([a,b])$ is connected

If not, $\exists U, V \underset{\text{open}}{\subseteq} f([a,b])$ s.t. $U \cap V = \phi$, $U \cup V = f([a,b])$

($U \neq \phi$, $V \neq \phi$)

$\Rightarrow f^{-1}(U), f^{-1}(V) \underset{\text{open}}{\subseteq} [a,b]$ ($\neq \phi$, $\neq \phi$ by Step1)

$f^{-1}(U) \cap f^{-1}(V) = \phi$ ($x \in f^{-1}(U) \cap f^{-1}(V) \Rightarrow f(x) \in U \cap V = \phi$ ↯)

$f^{-1}(U) \cup f^{-1}(V) = (f^{-1}(f([a,b])) = [a,b]$

$\{x \in [a,b] \mid f(x) \in U\} = [a,b] \cap U$

$\forall x \in [a,b]$, $f(x) \in f([a,b]) = U \cup V$

$\Rightarrow f(x) \in U$ or $\in V$

$\Rightarrow x \in f^{-1}(U)$ or $f^{-1}(V) \Rightarrow x \in f^{-1}(U) \cup f^{-1}(V)$

So $[a,b] \subseteq f^{-1}(U) \cup f^{-1}(V) \subseteq [a,b]$

$\Rightarrow [a,b]$ is disconnected (↯)

So $f([a,b])$ is cnt. #

Thm (Thm 3.5.2)

Every path-connected space P is connected

pf

case1: $P = \phi \Rightarrow P$ is cnt.

case2: $P \neq \phi$, i.e. $\exists x_0 \in P$

Since P is p-cnt, $\forall x \in P$, $\exists$ path $f_x : [a,b] \to P$ s.t. $f_x(0) = x_0$, $f_x(1) = x$.

By Lemma,

$$\Gamma_x := f_x([a,b]) \subseteq P \quad \text{is cnt.}$$

Since $x_0 \in \bigcap_{x \in P} \Gamma_x \neq \emptyset$,

$$P = \bigcup_{x \in P} \Gamma_x \subseteq P \quad \text{is connected}$$

$\#$

Continuous maps

$[a,b] \to M$

The idea of "continuous" paths can be generalized to continuous maps:

Def (Def 4.1.2, 4.1.3)

Let (M,d), (N,d') be metric spaces.

$S \subseteq M$, $f : S \to N$. $x_0 \in S$

(i) f is continuous at x_0 if $\forall \varepsilon > 0$

$\exists \delta > 0$ s.t.

Write $f \in C(S,N)$

$$d'(f(x), f(x_0)) < \varepsilon \quad \text{when} \quad d(x,x_0) < \delta, \ x \in S$$

(ii) If f is continuous at every point in S, we say f is continuous on S.

Thm

The following are equivalent:

(i) $f : S \to N$ is continuous at $x_0 \in S$

(ii) For any seq. $(x_n)_{n=1}^{\infty} \subseteq S$ with

$$\lim_{n\to\infty} x_n = x_0$$

we have

$$\lim_{n\to\infty} f(x_n) = f(x_0) \quad (= f(\lim_{n\to\infty} x_n))$$

pf: exer. (similar to the case of continuous paths)

Thm (Thm 4.1.4) $f : S \to N$, $S \subseteq (M, d)$, (N, d')

The following are equivalent:

(i) f is continuous on S

(ii) $\forall\, U \subseteq_{\text{open}} N$, $f^{-1}(U) \subseteq_{\text{open}} S$

(iii) $\forall\, F \subseteq_{\text{closed}} N$, $f^{-1}(F) \subseteq_{\text{closed}} S$

pf

(i) $\Rightarrow$ (ii) : exer (similar to Step 1 in the previous lemma)

(ii) $\Rightarrow$ (i) : Assume (ii) holds. $f : S \to N$

Given $x_0 \in S$, $\forall \varepsilon > 0$, since $D_N(f(x_0), \varepsilon) \subseteq_{\text{open}} N$

we have $f^{-1}(D_N(f(x_0), \varepsilon)) \subset_{\text{open}} S$

by (ii)

$\Rightarrow \exists \delta > 0$ s.t.

$$D_S(x_0, \delta) \subseteq f^{-1}(D_N(f(x_0), \varepsilon))$$

So $x \in D_S(x_0, \delta) \Leftrightarrow \begin{cases} x \in S \\ d(x, x_0) < \delta \end{cases}$

$\Rightarrow x \in f^{-1}(D_N(f(x_0), \varepsilon))$

$\Rightarrow f(x) \in D_N(f(x_0), \varepsilon)$, i.e.,

$d'(f(x), f(x_0)) < \varepsilon$

So f is continuous at x_0, $\forall x_0 \in S$.

$\Rightarrow$ (i) is true.

(i) $\Leftrightarrow$ (ii):

U open $N \Rightarrow f^{-1}(U)$ open S

$\Updownarrow$ $\Updownarrow$

$F = N \setminus U$ closed $N \Rightarrow S \setminus f^{-1}(U)$ closed S

$f^{-1}(N \setminus U)$

$= \{y \in S \mid f(y) \notin U\}$

$\Updownarrow$

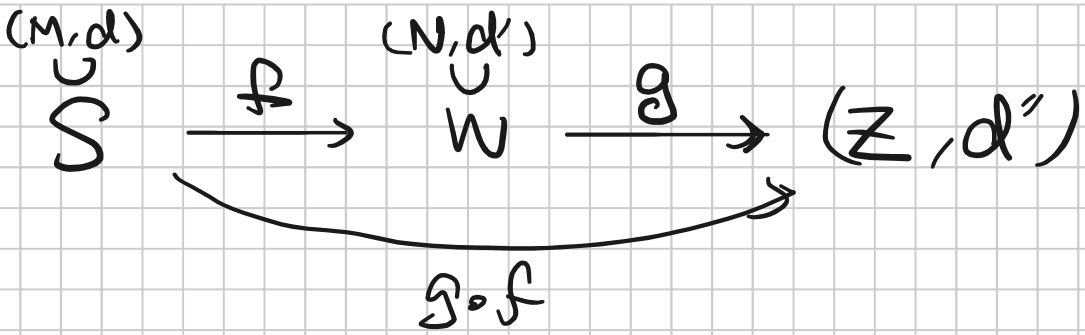
$f(y) \in N \setminus U$

$= f^{-1}(N \setminus U)$

#

Thm (Thm 4.3.1)

Assume



If f is continuous on S

g is " on W

then $g \circ f$ is continuous on S

Pf

$\forall\, U \underset{\text{open}}{\subset} Z$, $g^{-1}(U) \underset{\text{open}}{\subset} W$

$\Rightarrow f^{-1}(g^{-1}(U)) \underset{\text{open}}{\subset} S$

$\| $

$(g \circ f)^{-1}(U)$

$\Rightarrow g \circ f$ is continuous. $\#$

$\sqcup \quad \vee$

洪呈毅：

- 定義沒背熟。
- 寫考卷時請區分好 U 跟 V。

陳俊碩：

Problem 3
Let A be a subset of metric space (M, d). Determine whether each statement is true of false. Prove each true statement and give a counterexample to each false statement. All interiors, closures and boundaries are taken in M.
(a) $\text{int}(\text{cl}(A)) = \text{int}(A)$.

常錯：沒有指定 (M, d) 會出問題。例如這次答案最常見的反例是取 $A = \mathbb{Q}$。然而若考慮 $M = \mathbb{C}$，度量 d 取為 $d(z_1, z_2) = \sqrt{(z_1 - z_2)\overline{(z_1 - z_2)}}$，那麼

$$\text{cl}(\mathbb{Q}) = \mathbb{R} \subset \mathbb{C}. \qquad \text{int}(\mathbb{Q}) = \emptyset. \qquad \text{int}(\mathbb{R}) = \emptyset.$$

Problem 4
Let $S \subseteq \mathbb{R}$ be a nonemtpy bounded set containing at least two distinct points. Show that S has at least two boundary points.

常錯 1: 沒有說明 $\inf S$ 及 $\sup S$ 為何存在以及為何相異。
常錯 2: 對任意的 $\epsilon > 0$，我們不保證 $\inf S + \frac{\epsilon}{2}$ 和 $\sup S - \frac{\epsilon}{2}$ 屬於 S。

Problem 5
Prove that every compact metric space is complete.

常錯：證明一個 sequence 收斂必為 Cauchy sequence 無法說明所有 Cauchy sequence 收斂。

劉筱玟

Problem 6
常錯 1：取兩個數列的收斂子序列 x_{n_k}, y_{n_k} 時未說明為何兩者的指標 n_k 可以是相同的
常錯 2：未討論為何 d 是連續的
常錯 3：有人去討論 $\inf(K_1)$ ，但這在度量空間一般沒有定義

Problem 7
常錯 1：不熟悉 locally compact 的定義，誤以為所有開鄰域的 closure 都會是 compact
常錯 2：V 包含在 U 不會推得 $\text{cl}(V)$ 包含在 U
常錯 3：Closed 或 compact sets 的「有限」聯集也是 closed 或 compact ，有些人會忘記要是有限聯集
常錯 4：記得 closed and bounded $\Rightarrow$ compact 但不記得需要在 $\mathbb{R}^n$ 裡

F_n

$\Rightarrow \bigcup_{n\geq 1}^{\infty} F_n = (0,1]$

$\{ C_n \}$

$\shortparallel$

$(0, \dots, 0, \underset{n\text{-th}}{1}, 0, \dots)$

closed, bdd

NOT cpt

National Tsing Hua University

Advanced Calculus I — Midterm

Fall 2026

October 27, 2026 (Tuesday)

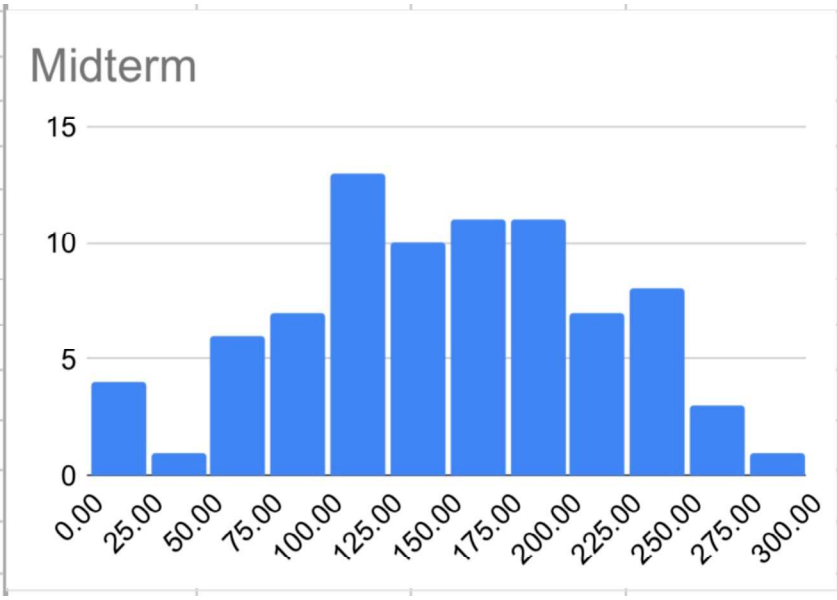
Instructor: Hsuan-Yi Liao

Name: Answer _____ **Student ID:** _____

- Time: **8:00–9:40 (100 minutes)**. Total: **300 points**.
- This document contains **7 questions** and **11 pages**, including this cover page.
- Give clear proofs or counterexamples. An unsupported answer will not receive full credit.
- You may use results proved in lectures unless the question asks you to prove that result.
- You may use an earlier part of a question in a later part, even if you have not proved the earlier part.
- Books, notes, calculators, cell phones, and other electronic devices are not permitted.
- Keep your student ID available for inspection.
- Write in the spaces provided. Clearly label any work continued elsewhere.

Distribution of Marks

Average	148.4756098
Average (except zero)	156.0897436
Quartile 0	0
Quartile 1	104.25
Quartile 2	150.5
Quartile 3	194.5
Quartile 4	294
標準差	66.74092635
非零數	78
不到50%人數	41
滿分人數	0



1. Let d and d' be equivalent metrics on a set M; that is, there exist constants $\alpha, \beta > 0$ such that

$$\circledast \qquad \alpha d(x,y) \le d'(x,y) \le \beta d(x,y) \qquad \text{for all } x, y \in M.$$

(a) (15 points) Suppose (x_n) is a sequence in M and $x_* \in M$. Prove that $x_n \to x_*$ in (M,d) if and only if $x_n \to x_*$ in (M,d').

"$\Rightarrow$" $x_n \to x_*$ in $(M,d) \iff \lim_{n\to\infty} d(x_n, x_*) = 0$

Since $\alpha d(x_n, x_*) \le d'(x_n, x_*) \le \beta d(x_n, x_*)$ $\forall n$, with $\beta d(x_n, x_*) \to 0$

we have $\lim_{n\to\infty} d'(x_n, x_*) = 0 \Rightarrow x_n \to x_*$ in (M,d')

"$\Leftarrow$" $x_n \to x_*$ in $(M,d') \iff \lim_{n\to\infty} d'(x_n, x_*) = 0$

By $\circledast$, (and $\alpha > 0$) $0 \le d(x_n, x_*) \le \frac{1}{\alpha} d'(x_n, x_*)$ $\forall n$, with $\frac{1}{\alpha} d'(x_n, x_*) \to 0$

$\Rightarrow \lim_{n\to\infty} d(x_n, x_*) = 0 \Rightarrow x_n \to x_*$ in (M,d) #

(b) (15 points) Prove that a subset $U \subseteq M$ is open in (M,d) if and only if it is open in (M,d').

"$\Rightarrow$" Assume U is open in (M,d).

$\Rightarrow \forall x \in U$, $\exists \delta > 0$ s.t.

$$D(x,\delta) = \{y \in M \mid d(x,y) < \delta\} \subseteq U$$

$\Rightarrow$ $D'(x, \alpha\delta) = \{y \in M \mid d'(x,y) < \alpha\delta\} \subseteq D(x,\delta) \subseteq U$ ($\alpha\delta > 0$)

($d'(x,y) \le \alpha d(x,y)$) $\Rightarrow d(x,y) < \delta$

$\Rightarrow U$ is open in (M,d')

"$\Leftarrow$" Similarly, if U is open in (M,d'), then

$\forall x \in U$, $\exists \varepsilon > 0$ s.t.

$$D'(x,\varepsilon) = \{y \in M \mid d'(x,y) < \varepsilon\} \subseteq U$$

$\Rightarrow$ $D(x, \frac{\varepsilon}{\beta}) = \{y \in M \mid d(x,y) < \frac{\varepsilon}{\beta}\} \subseteq D'(x,\varepsilon) \subseteq U$ ($\frac{\varepsilon}{\beta} > 0$)

($\frac{1}{\beta} d'(x,y) \le d(x,y)$) $\Rightarrow d'(x,y) < \varepsilon$

$\Rightarrow U$ is open in (M,d). #

2. Let (M, d) be a metric space and let $V \subseteq M$ be nonempty. Define $d_V : V \times V \to \mathbb{R}$ by $d_V(x, y) = d(x, y)$.

   (a) (15 points) Prove that (V, d_V) is a metric space by verifying the metric axioms.

For any $x, y, z \in V \subseteq M$, we have

(i) $d_V(x,y) = d(x,y) \geq 0$ and $d_V(x,y) = 0 \Leftrightarrow d(x,y) = 0 \Leftrightarrow x = y$

(ii) $d_V(x,y) = d(x,y) = d(y,x) = d_V(y,x)$

(iii) $d_V(x,z) = d(x,z) \leq d(x,y) + d(y,z)$

$= d_V(x,y) + d_V(y,z)$

$\Rightarrow (V, d_V)$ is a metric space. #

Question 2 (continued)

(b) (20 points) Let $A \subseteq V$. Prove that A is open in (V, d_V) if and only if there exists an open set U in (M, d) such that $A = V \cap U$.

"$\Rightarrow$" If A is open in (V, d_V), then $\forall x \in A, \exists \delta_x > 0$ s.t

$$D_V(x, \delta_x) = \{y \in V \mid d_V(x,y) < \delta_x\} \subseteq A$$

Note that since $d_V(x,y) = d(x,y) \quad \forall x, y \in V$,

$$D_V(x, \delta_x) = \{y \in M \mid d(x,y) < \delta_x,\ y \in V\} \quad - \circledast$$

$$= D(x, \delta_x) \cap V$$

where $D(x,\delta) = \{y \in M \mid d(x,y) < \delta_x\}$

$$\Rightarrow A = \bigcup_{x \in A} D_V(x, \delta_x) = \bigcup_{x \in A} (D(x, \delta_x) \cap V)$$

$$= \left(\bigcup_{x \in A} D(x, \delta_x)\right) \cap V = U \cap V$$

where $U = \bigcup_{x \in A} D(x, \delta_x)$ is open in M.

"$\Leftarrow$" If $A = U \cap V$ for some open set U in M, then

$\forall x \in U, \exists \delta_x > 0$ s.t. $D(x, \delta_x) \subseteq U$

$$\Rightarrow A = U \cap V = \left(\bigcup_{x \in U} D(x, \delta_x)\right) \cap V \overset{\circledast}{=} \bigcup_{x \in U} D_V(x, \delta_x) \text{ is open in } V. \#$$

(c) (10 points) Prove or disprove: a subset $U \subseteq V$ is open in (V, d_V) if and only if U is open in (M, d).

The statement is false:

Consider $(M, d) = (\mathbb{R}, d_e)$, $V = [-1, 1] \subseteq \mathbb{R}$

$U = (0, 1]$

Since $U = (0, 2) \cap V$, it is open in V by (b).

But $U = (0, 1]$ is NOT open in $M = \mathbb{R}$ because

for $1 \in U$, for any $\delta > 0$,

$$D(1, \delta) = (1-\delta, 1+\delta) \not\subseteq (0, 1] = U \quad \#$$

3. Let A be a subset of a metric space (M, d). Determine whether each statement is true or false. Prove each true statement and give a counterexample to each false statement. All interiors, closures, and boundaries are taken in M.

(a) (15 points) $\text{int}(\text{cl}(A)) = \text{int}(A)$.

Let $(M, d) = (\mathbb{R}, de)$, $A = \mathbb{Q}$

$\Rightarrow$ ① $\text{cl}(A) = \mathbb{R}$ $\Rightarrow$ $\text{int}(\text{cl}(A)) = \mathbb{R}$

② $\text{int}(A) = \text{int}(\mathbb{Q}) = \emptyset \neq \mathbb{R}$

$\forall x \in \mathbb{Q}$, $\forall \delta > 0$,

$D(x, \delta) = (x-\delta, x+\delta)$

contains irrational numbers

$\Rightarrow$ $D(x, \delta) \not\subseteq \mathbb{Q} = A$

$\Rightarrow$ $\text{int}(A) = \emptyset$

So $\text{int}(\text{cl}(A)) \neq \text{int}(A)$ #

(b) (15 points) If A is open, then $\partial(A) \subseteq M \setminus A$.

A is open $\Rightarrow$ $M \setminus A$ is closed $\Rightarrow$ $\text{cl}(M \setminus A) = M \setminus A$

$\Rightarrow \partial(A) = \text{cl}(A) \cap \text{cl}(M \setminus A) = \text{cl}(A) \cap (M \setminus A) \subseteq M \setminus A$

*

4. (35 points) Let $S \subseteq \mathbb{R}$ be a nonempty bounded set containing at least two distinct points. Show that S has at least two boundary points.

By assumption, $\exists\, u \neq v$ s.t. $u, v \in S$

Let $x = \inf(S)$, $y = \sup(S)$

S bdd $\Rightarrow$ $-\infty < x \leq u < v \leq y < \infty \Rightarrow x \neq y \in \mathbb{R}$

Claim: $x, y \in \partial(S)$

① $x = \inf(S) \Rightarrow \forall \varepsilon > 0$,

- $x - \frac{\varepsilon}{2} \notin S \Rightarrow x - \frac{\varepsilon}{2} \in D(x, \varepsilon) \cap (M - S) \Rightarrow D(x, \varepsilon) \cap (M - S) \neq \emptyset$
- $x + \frac{\varepsilon}{2}$ is NOT a lower bound $\Rightarrow \exists z \in S$ s.t. $z \leq x + \frac{\varepsilon}{2}$ ($x < z$)
  $\Rightarrow z \in D(x, \varepsilon) \cap S \Rightarrow D(x, \varepsilon) \cap S \neq \emptyset$

So $x \in \partial(S)$.

② $y = \sup(S) \Rightarrow \forall \varepsilon > 0$

- $y + \frac{\varepsilon}{2} \notin S \Rightarrow y + \frac{\varepsilon}{2} \in D(y, \varepsilon) \cap (M - S) \Rightarrow D(y, \varepsilon) \cap (M - S) \neq \emptyset$
- $\exists w \in S$ s.t. $y - \frac{\varepsilon}{2} \leq w \leq y \Rightarrow w \in D(y, \varepsilon) \cap S \Rightarrow D(y, \varepsilon) \cap S \neq \emptyset$

So $y \in \partial(S)$

So $\partial(S)$ has at least two points ($\inf S$ and $\sup S$) ✓

5. (35 points) Prove that every compact metric space is complete.

Assume (M, d) is a compact metric space.

Let $(x_n)_{n=1}^{\infty}$ be a Cauchy seq. in M.

Since M is cpt., $\exists$ a conv. subseq. $(x_{n_j})_{j=1}^{\infty}$

Let $\alpha = \lim_{j\to\infty} x_{n_j} \in M$.

$\Rightarrow \forall \varepsilon > 0$, $\exists N_1, N_2 \in \mathbb{N}$ s.t.

① $d(x_n, x_m) < \varepsilon/2 \quad \forall n, m \geq N_1$ ((x_n) is Cauchy)

② $d(x_{n_j}, \alpha) < \varepsilon/2 \quad \forall j \geq N_2$ ($\lim_{j\to\infty} x_{n_j} = \alpha$)

Let $N = \max\{N_1, N_2\}$ Then $n_N \geq n_{N_1} \geq N_1$

$\Rightarrow d(x_n, \alpha) \leq d(x_n, x_{n_N}) + d(x_{n_N}, \alpha)$

$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

$\forall n \geq N \ (\geq N_1)$

$\Rightarrow \lim_{n\to\infty} x_n = \alpha \in M$

So M is complete. #

6. Let (M, d) be a metric space.

   (a) (25 points) Let K_1 and K_2 be closed subsets of M. Suppose there are sequences of distinct points $(x_n) \subseteq K_1$ and $(y_n) \subseteq K_2$ such that

   $$d(x_n, y_n) \longrightarrow 0.$$

   Must $K_1 \cap K_2$ be nonempty? Justify your answer with a proof or a counterexample.

No.

Consider

$K_1 = \{(x,y) \in \mathbb{R}^2 \mid xy = 1\}$

$K_2 = \{(x,0) \in \mathbb{R}^2 \mid x \in \mathbb{R}\}$

$x_n = (n, \frac{1}{n}) \in K_1$

$y_n = (n, 0) \in K_2$

Check

① K_1 is closed: Assume $(z_n)_{n=1}^{\infty}$ is a convergent seq in K_1

$(z_n^1, z_n^2) \Rightarrow (z_n^1), (z_n^2)$ conv. in $\mathbb{R}$

Assume $\alpha = (\alpha_1, \alpha_2) = \lim_{n\to\infty} z_n$

$\Rightarrow \alpha_1 \cdot \alpha_2 = (\lim_{n\to\infty} z_n^1) \cdot (\lim_{n\to\infty} z_n^2) = \lim_{n\to\infty} (z_n^1 \cdot z_n^2) = \lim_{n\to\infty} 1 = 1$

$\Rightarrow \alpha \in K_1 \Rightarrow K_1$ closed.

② K_2 is closed: Assume $(z_n)_{n=1}^{\infty} \subseteq K_2$, conv. in $\mathbb{R}^2$

$(z_n^1, 0) \Rightarrow (z_n^1)$ conv. in $\mathbb{R}$

Assume $\alpha = (\alpha_1, \alpha_2) = \lim_{n\to\infty} z_n$

$\Rightarrow \alpha_2 = \lim_{n\to\infty} 0 = 0 \Rightarrow \alpha = (\alpha_1, 0) \in K_2 \Rightarrow K_2$ closed.

③ $d(x_n, y_n) = d((n, \frac{1}{n}), (n, 0))$

$= \frac{1}{n} \to 0$

④ $K_1 \cap K_2 = \emptyset$ because

$(x,y) \in K_1 \Rightarrow xy = 1 \Rightarrow y \neq 0 \Rightarrow (x,y) \notin K_2$

#

Question 6 (continued)

(b) (35 points) Let K_1 and K_2 be nonempty compact subsets of M. Define

$$d(K_1, K_2) := \inf\{d(x, y) \mid x \in K_1,\ y \in K_2\}.$$

Prove that there exist $a \in K_1$ and $b \in K_2$ such that $d(a, b) = d(K_1, K_2)$. Deduce that $d(K_1, K_2) > 0$ if $K_1 \cap K_2 = \varnothing$.

bdd below, nonempty

① Recall that for $S \subseteq \mathbb{R}$, $\exists z_n \in S$ s.t. $\lim_{n\to\infty} z_n = \inf(S)$

$\Rightarrow \exists\, x_n \in K_1,\ y_n \in K_2,\ n \in \mathbb{N}$, s.t.

$$d(K_1, K_2) = \lim_{n\to\infty} d(x_n, y_n)$$

② Since K_1 is cpt, $(x_n)_{n=1}^{\infty}$ has a conv. subseq. $(x_{n_j})_{j=1}^{\infty}$,

with $a = \lim_{j\to\infty} x_{n_j} \in K_1$

③ Since K_2 is cpt, $(y_{n_j})_{j=1}^{\infty}$ has a conv. subseq. $(y_{n_{j_k}})_{k=1}^{\infty}$

with $b = \lim_{k\to\infty} y_{n_{j_k}} \in K_2$

④ Since

$$d(K_1, K_2) \le d(a, b) \le d(a, x_{n_{j_k}}) + d(x_{n_{j_k}}, y_{n_{j_k}}) + d(y_{n_{j_k}}, b)$$

($d(a, x_{n_{j_k}}) \to 0$, $d(x_{n_{j_k}}, y_{n_{j_k}}) \to d(K_1, K_2)$, $d(y_{n_{j_k}}, b) \to 0$)

$$\| \inf\{d(x, y) \mid x \in K_1,\ y \in K_2\}$$

we have

$$d(a, b) = d(K_1, K_2)$$

by pinching thm.

⑤ If $K_1 \cap K_2 = \phi$, then $a \neq b$ ($a \in K_1,\ b \in K_2$)

$\Rightarrow$ $d(K_1, K_2) \overset{④}{=} d(a, b) > 0$

#

7. Recall that a metric space (M, d) is **locally compact** if every $x \in M$ has an open neighborhood O whose closure $\mathrm{cl}(O)$ is compact. Let (M, d) be locally compact and let $K \subseteq M$ be compact. All closures below are taken in M.

   (a) (50 points) Let $U \subseteq M$ be open with $K \subseteq U$. Prove that there exists an open set V such that

   $$K \subseteq V \subseteq \mathrm{cl}(V) \subseteq U$$

   and $\mathrm{cl}(V)$ is compact.

① $\forall x \in K \subseteq U$, open set O_x s.t. $x \in O_x$ and $\mathrm{cl}(O_x)$ is cpt.

② Since $O_x \cap U \ni x$ is open, $\exists \delta_x > 0$ s.t.
$D(x, 2\delta_x) \subseteq O_x \cap U$

③ Since $\{D(x, \delta_x) \mid x \in K\}$ is an open cover of K,
by cptness, $\exists x_1, \cdots, x_n$ s.t.
$K \subseteq \bigcup_{i=1}^{n} D(x_i, \delta_{x_i}) =: V$

④ Since $B(x_i, \delta_{x_i}) := \{y \in M \mid d(y, x_i) \le \delta_{x_i}\}$
is closed, so is
$F := \bigcup_{i=1}^{n} B(x_i, \delta_{x_i}) \supseteq K$ $\boxed{V \supseteq K}$

$\Rightarrow \mathrm{cl}(V) \subseteq F \subseteq \bigcup_{i=1}^{n} O_{x_i} \subseteq \bigcup_{i=1}^{n} \mathrm{cl}(O_{x_i})$

⑤ $\mathrm{cl}(O_{x_i})$ is cpt $\forall i = 1, \cdots, n$
$\Rightarrow$ so is $\bigcup_{i=1}^{n} \mathrm{cl}(O_{x_i})$

⑥ $\mathrm{cl}(V)$ is a closed subset of a cpt set $\bigcup_{i=1}^{n} \mathrm{cl}(O_{x_i})$
$\Rightarrow$ $\boxed{\mathrm{cl}(V) \text{ is cpt.}}$

⑦ $\mathrm{cl}(V) \subseteq F \subseteq \bigcup_{i=1}^{n} D(x_i, 2\delta_{x_i}) \subseteq \bigcup_{i=1}^{n} (O_{x_i} \cap U) \subseteq U$

By ④, ⑥, ⑦,
$K \subseteq V \subseteq \mathrm{cl}(V) \subseteq U$
with $\mathrm{cl}(V)$ cpt. #

Question 7 (continued)

(b) (15 points) Let $F \subseteq M$ be closed with $K \cap F = \varnothing$. Prove that there exist disjoint open sets $W_K, W_F \subseteq M$ such that

$$K \subseteq W_K, \qquad F \subseteq W_F,$$

and $\mathrm{cl}(W_K)$ is compact.

$K \cap F = \varnothing \Rightarrow K \subseteq (M \setminus F)$ open

(a) $\Rightarrow \exists\, V$ s.t.

$K \subseteq V \subseteq \mathrm{cl}(V) \subseteq (M \setminus F)$

with $\mathrm{cl}(V)$ cpt

Let $W_K = V$, $W_F = M \setminus \mathrm{cl}(V)$

$\Rightarrow$ W_K, W_F are open,

$V \subseteq \mathrm{cl}(V) \Rightarrow W_K \cap W_F \subseteq V \cap (M \setminus V) = \varnothing$

$K \subseteq W_K$, $F = M \setminus (M \setminus F) \subseteq M \setminus \mathrm{cl}(V) = W_F$

and $\mathrm{cl}(W_K) = \mathrm{cl}(V)$ is cpt. #