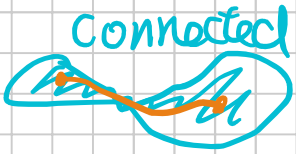
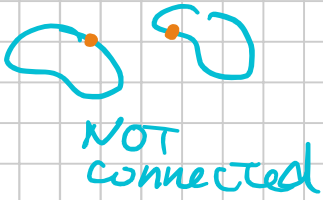
Advanced Calculus, week 8, Fall 2026

§ Connectedness





Path-connected

Recall

$f: [a,b] \longrightarrow \mathbb{R}$ is continuous at $c \in [a,b]$

($\mathbb{R}$: a metric sp.)

$\Leftrightarrow \lim_{x \to c} f(x) = f(c)$

$\Leftrightarrow$ For any seq. $(x_n) \subseteq [a,b]$ with $x_n \to c$,

$\rightarrow \lim_{n\to\infty} f(x_n) = f(c) \; (= f(\lim_{n\to\infty} x_n))$

$\Leftrightarrow \forall \varepsilon > 0 \; \exists \delta > 0$ s.t.

$\rightarrow |f(x) - f(c)| < \varepsilon$ when $(0<)\ |x-c| < \delta$, $x \in [a,b]$

Thm

Let (M,d) be a metric sp.,

$f: [a,b] \longrightarrow M$, $x_* \in [a,b]$.

Then the following are equivalent:

(i) $\forall \varepsilon > 0 \; \exists \delta > 0$ s.t.

$d(f(x), f(x_*)) < \varepsilon$

when $x \in [a,b]$ and $|x - x_*| < \delta$

(ii) If $(x_n)_{n=1}^{\infty}$ is a seq. in $[a,b]$ that converges to x_*, then

$$\lim_{n\to\infty} f(x_n) = f(x_*)$$

pf

(i) $\Rightarrow$ (ii):

Assume (i) is true: $\forall \varepsilon>0 \ \exists \delta>0$ s.t. $\forall x\in[a,b]$ $|x-x_*|<\delta \Rightarrow d(f(x),f(x_*))<\varepsilon$

If $(x_n) \subseteq [a,b]$ with $x_n \to x_*$, then

$\forall \varepsilon>0, \ \exists N_\varepsilon \in \mathbb{N}$ s.t.

$$|x_n - x_*| < \delta \qquad \forall n \geq N_\varepsilon$$

(i) $\Rightarrow$

$$d(f(x_n), f(x_*)) < \varepsilon \qquad \forall n \geq N_\varepsilon$$

$\Rightarrow \lim_{n\to\infty} f(x_n) = f(x_*) \Rightarrow$ (ii)

(ii) $\Rightarrow$ (i):

Assume (i) is NOT true:

$\exists \varepsilon_0>0$ s.t. $\forall \delta>0 \ \exists \ x_\delta \in [a,b]$ s.t.

$|x_\delta - x_*| < \delta$ and $d(f(x_\delta), f(x_*)) \geq \varepsilon_0$

For $\delta = \frac{1}{n} > 0$, $\exists \ x_n \in [a,b]$ s.t.

$|x_n - x_*| < \frac{1}{n} \Rightarrow \lim_{n\to\infty} x_n = x_*$

and $d(f(x_n), f(x_*)) \geq \varepsilon_0$

$\Rightarrow \exists\ (x_n) \subseteq [a,b]$ with $x_n \to x_*$, but

$$d(f(x_n), f(x_*)) \geq \varepsilon_0$$

$\Rightarrow \lim_{n\to\infty} f(x_n) \neq f(x_*) \Rightarrow$ (ii) is NOT true

So (ii) $\Rightarrow$ (i) ##

Def (Def 3.4.1)

A function $f: [a,b] \to M$ satisfying the conditions above at every point $x_* \in [a,b]$ is called a continuous function or (continuous) path

Let $C \subseteq (M,d)$.

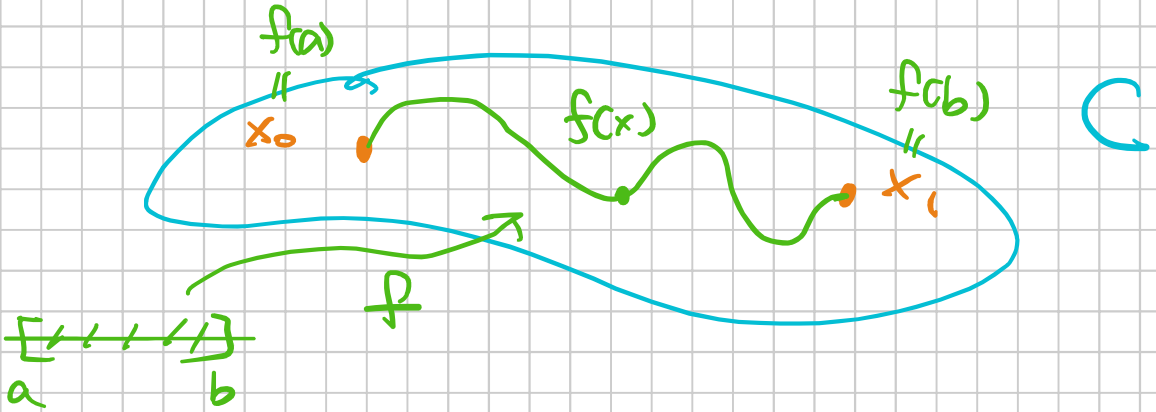
The set C is path-connected (路徑連通) if

$\forall\ x_0, x_1 \in C$, $\exists$ continuous path

$$f: [a,b] \to C$$

s.t.

$$f(a) = x_0 \quad , \quad f(b) = x_1$$



Example

① The empty set is path-cnt.

② Any interval in $\mathbb{R}$ is path-cnt

pf of ②

Let J be an interval ($J = (c,d)$ or $[c,d)$ or $(c,d]$ or $[c,d]$)

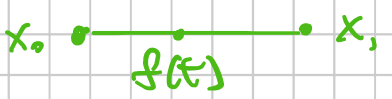
For $x_0, x_1 \in J$, define

$$f : [0,1] \to J, \qquad f(t) \in J \text{ between } x_0 \text{ and } x_1$$

$$f(t) = x_0 + t(x_1 - x_0) = (1-t)x_0 + tx_1$$

$\Rightarrow f$ is continuous, $f(0) = x_0$, $f(1) = x_1$

So J is path-cnt.



#

Remark

If $f : [a,b] \to M$ is Continuous, then

$$\tilde{f} : [0,1] \to M$$

$$\tilde{f}(t) = f(\underbrace{a + t(b-a)}_{\in [a,b]}) \in M$$

is continuous and

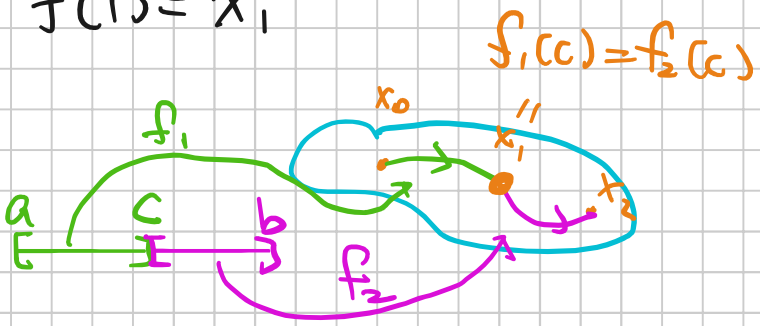
$$\tilde{f}(0) = f(a), \qquad \tilde{f}(1) = f(b)$$

So, for $x_0, x_1 \in C$, if $\exists$ continuous

$f : [a,b] \to C$ s.t. $f(a) = x_0$ $f(b) = x_1$

then $\exists$ continuous $\tilde{f}:[0,1]\to C$ s.t.

$\tilde{f}(0)=x_0$, $\tilde{f}(1)=x_1$



Lemma

Let $a<c<b$.

Suppose $f_1:[a,c]\longrightarrow M$ are continuous

$f_2:[c,b]\longrightarrow M$

s.t. $f_1(c)=f_2(c)$

Then $f:[a,b]\to M$,

$$f(t)=\begin{cases} f_1(t), & t\in[a,c] \\ f_2(t), & t\in[c,b] \end{cases}$$

is continuous

pf: exercise

e.g.



is path-connected

Prop

Let $C\subseteq(M,d)$. If $\exists\ x_*\in C$ s.t.

every point $x\in C$ can be joined to x_*

by a path in C (i.e. $\exists$ conti. $f:[a,b]\to C$ s.t. $f(a)=x_*$, $f(b)=x$)

then C is path-connected.

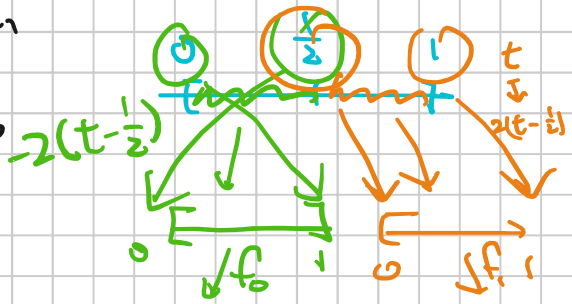
pf

$\forall\ x_0, x_1\in C$, by assumption $\exists$

$f_0, f_1 : [0,1] \to C$ s.t.

$f_0(0) = x_*, \quad f_0(1) = x_0$

$f_1(0) = x_*, \quad f_1(1) = x_1$



$\Rightarrow$ The function $f : [0,1] \longrightarrow C$

$$f(t) = \begin{cases} f_0(-2(t-\frac{1}{2})), & t \in [0, \frac{1}{2}] \\ f_1(2(t-\frac{1}{2})), & t \in [\frac{1}{2}, 1] \end{cases}$$

is continuous (Lemma + Remark)

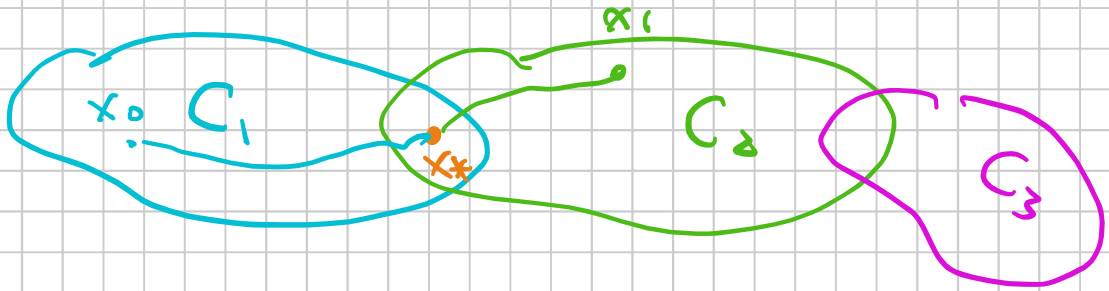
and $f(0) = f_0(1) = x_0$

$f(1) = f_0(1) = x_1$ #

Cor

If $C_1, C_2 \subseteq (M,d)$ are path-cnt. and $C_1 \cap C_2 \neq \emptyset$, then $\Rightarrow \exists\, x_* \in C_1 \cap C_2$

$C_1 \cup C_2$ is p-cnt.



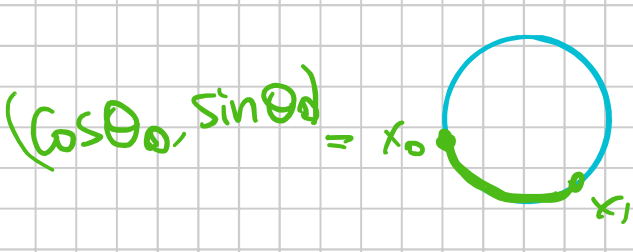
Cor

If $C_1, \cdots, C_n \subseteq (M,d)$ are p-cnt. and $C_k \cap C_{k+1} \neq \emptyset \quad \forall k = 1, \cdots, n-1$

then $\bigcup_{k=1}^{n} C_k$ is p-cnt.

Example

① $S^1 = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2=1\}$ is p-cnt,



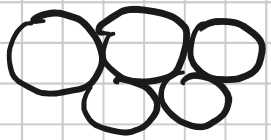
because $\forall\ x_0 = (\cos\theta_0, \sin\theta_0) \in S^1$
$x_1 = (\cos\theta_1, \sin\theta_1)$

$\exists\ f : [\theta_0, \theta_1] \longrightarrow S^1$,

$f(t) = (\cos t, \sin t)$

which is continuous (exer.) and

$f(\theta_0) = x_0$, $f(\theta_1) = x_1$.

②  is also p-cnt. by Cor.

Relatively open and closed sets

Def

Let $S \subseteq (M,d)$, $A \subseteq S$

(i) A is a relatively open subset of S if $\exists$ open set $U \subseteq M$ s.t. $A = U \cap S$

(ii) A is a relatively closed subset of S if $\exists$ closed set $F \subseteq M$ s.t. $A = F \cap S$

Remark (by Midterm)

$(S, d|_{S\times S})$ is a metric sp.

(i) A is relatively open "subspace topology"
$\Leftrightarrow$ A is open in $(S, d|_{S\times S})$

(ii) A is relatively close
$\Leftrightarrow$ A is closed in $(S, d|_{S\times S})$

Thm

A_α, $\alpha \in I$, are relatively open in S (relatively closed)

$\Rightarrow$ ① $\bigcup_{\alpha\in I} A_\alpha$ is relatively open in S

and, for $\alpha_1, \cdots, \alpha_n \in I$, $\bigcap_{\alpha\in I} A_\alpha$ is relatively closed in S

② $\bigcap_{i=1}^{n} A_{\alpha_i}$ is relatively open in S ($\bigcup_{i=1}^{n} A_{\alpha_i}$ is relatively closed in S)

③ $\emptyset, S$ are relatively closed in S.

Example

① $S = [0,1] \subseteq \mathbb{R}$ $\quad [0,1] \cap (0,2)$

Then $(0,1]$ is relatively open in S

"$\emptyset$, S are both relatively open and closed"

② $T = [0,1] \cup [2,3] \subseteq \mathbb{R}$

Then $[0,1] = T \cap (-1,2)$



$[0,1]$ is relatively open and relatively closed in T

$[0,1] = T \cap [0,1]$

$T \cap (1,4)$

"

$[2,3]$ is relatively open and

$[2,3] = T \cap [2,3]$

" closed in T