Advanced Calculus, week 7, Fall 2026

Recall that we were proving:

Thm (Heine-Borel. Thm 3.2.1)

A subset $K \subseteq \mathbb{R}^n$ is compact

$\Leftrightarrow$ K is closed and bounded

pf

"$\Rightarrow$" was done before

"$\Leftarrow$" Assume K is closed and bounded.

Let $(x_m)_{m=1}^{\infty} \subseteq K$ with

$$x_m = (x_{m,1}, x_{m,2}, \cdots, x_{m,n}) \in \mathbb{R}^n$$

Step 1 (Done last week)

K is bdd $\Rightarrow$ $(x_{m,i})_{m=1}^{\infty}$ is bdd $\forall i = 1, \cdots, n$.

Step 2 Construct a convergent subseq. of $(x_m)_{m=1}^{\infty}$

Recall from Calculus (1-dim'l Bolzano-Weierstrass theorem):

Every bounded seq. of real numbers has a convergent subseq.

We construct a convergent subseq. of $(x_m)_{m=1}^{\infty}$ as follows:

① $(x_{m,1})_{m=1}^{\infty}$ is a bdd seq. of real numbers

$\Rightarrow \exists$ subseq. $(x_{m_j,1})_{j=1}^{\infty}$ s.t.

$$\lim_{j\to\infty} x_{m_j,1} = \alpha_1$$

for some $\alpha_1 \in \mathbb{R}$

② $(x_{m_j,2})_{j=1}^{\infty}$ is subseq. of $(x_{m,2})_{m=1}^{\infty}$

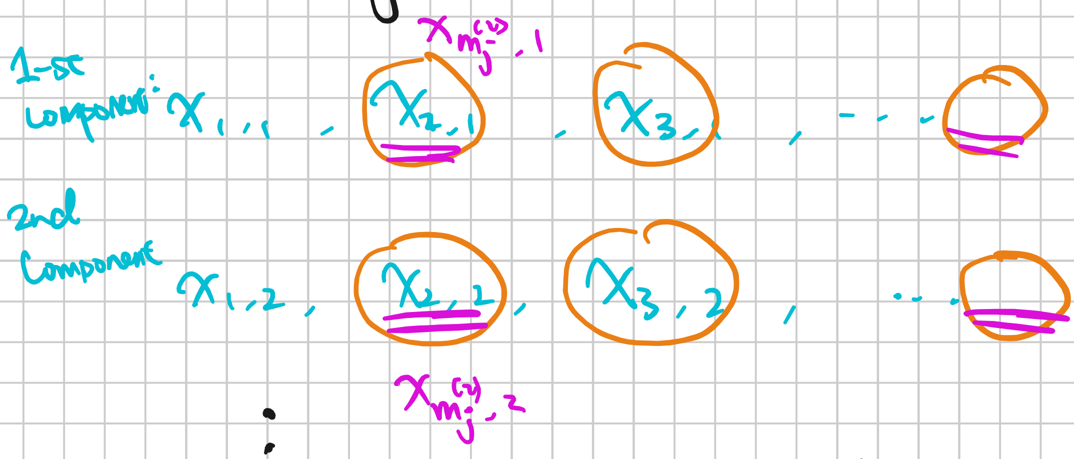
$\Rightarrow$ it is bdd

$\overset{B\text{-}W}{\Rightarrow}$ it has a convergent subseq.:

$$x_{m_j^{(2)},2} \longrightarrow \alpha_2 \in \mathbb{R}$$

NOTE:

$$x_{m_j^{(2)},1} \longrightarrow \alpha_1 \in \mathbb{R}$$



⋮

ⓝ For $(x_{m_j^{(n-1)},n})_{j=1}^{\infty}$ ← bdd, it has a conv. subseq.

$$x_{m_j^{(n)},n} \longrightarrow \alpha_n \in \mathbb{R}$$

NOTE:

$$x_{m_j^{(n)},\,n-1} \longrightarrow \alpha_{n-1}$$

$$\vdots$$

$$x_{m_j^{(n)},\,1} \longrightarrow \alpha_1$$

Then

$$x_{m_j^{(n)}} = (x_{m_j^{(n)},1}, x_{m_j^{(n)},2}, \dots, x_{m_j^{(n)},n})$$

$$\longrightarrow (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{R}^n$$

This shows that any seq. $(x_m)_{m=1}^{\infty} \subseteq K$ has a conv. subseq. $(x_{m_j^{(n)}})_{j=1}^{\infty}$.

Since K is closed, its limit is in K

$$(\alpha_1, \alpha_2, \dots, \alpha_n) \in K$$

$\Rightarrow K$ is Compact. #

Exercise:

Consider

$$\ell^2 = \left\{ (a_n)_{n=1}^{\infty} \subseteq \mathbb{R} \;\middle|\; \sum_{n=1}^{\infty} a_n^2 \text{ conv.} \right\}$$

$$d((a_n),(b_n)) = \sqrt{\sum_{n=1}^{\infty} (a_n - b_n)^2}$$

which is a metric sp. (HW1)

Show that

$$S = \left\{ (a_n) \in \ell^2 \;\middle|\; \sqrt{\sum_{n=1}^{\infty} a_n^2} = 1 \right\}$$

is closed and bdd, but NOT cpt.

Thm (Generalized Heine-Borel Thm, Thm 3.1.5)

A metric space (M, d) is compact

$\Leftrightarrow$ it is complete and totally bounded

Every Cauchy seq. conv.

$\forall \varepsilon > 0$, $\exists\, z_1, \dots, z_m$ s.t. $M \subseteq \bigcup_{j=1}^{m} D(z_j, \varepsilon)$

pf

"$\Rightarrow$"

By previous results, Compact $\Rightarrow$ totally bdd.

Need to show: cpt $\Rightarrow$ complete:

Given any Cauchy seq. (x_n) in M, we claim that it conv. in M.

Since M is cpt, (x_n) has a subseq. (x_{n_j}) s.t.

$$x_{n_j} \to \alpha$$

for some $\alpha \in M$.

So $\forall \varepsilon > 0$, $\exists N_1, N_2 \in \mathbb{N}$ s.t.

(i) $d(x_n, x_m) < \frac{\varepsilon}{2}$ $\quad \forall\, n, m \geq N_1$

(ii) $d(x_{n_j}, \alpha) < \frac{\varepsilon}{2}$ $\quad \forall\, j \geq N_2$

Let $N = \max\{N_1, N_2\}$. Then $\forall\, n \geq N$,

$$d(x_n, \alpha)$$

$n_N \geq n_{N_1} \geq N_1$

$$\leq \underbrace{d(x_n, x_{n_N})}_{< \frac{\varepsilon}{2}\ (i)} + \underbrace{d(x_{n_N}, \alpha)}_{< \frac{\varepsilon}{2}\ (ii)} \qquad N \geq N_2$$

$$= \varepsilon$$

$$\Rightarrow \lim_{n\to\infty} x_n = \alpha \in M$$

$\Rightarrow$ M is complete.

"$\Leftarrow$" Assume M is complete and totally bdd. Given a seq $(x_n)_{n=1}^{\infty}$ in M, we want to construct a Cauchy subseq.

Step 1: We inductively construct a nested seq. of subseq as follows:

① Set

$$(x_n^{(0)})_{n=1}^{\infty} = (x_n)_{n=1}^{\infty}$$

② For $m \geq 1$, since M is totally bdd, $\exists\ z_{m,1}, \cdots, z_{m,\ell_m} \in M$ s.t.

$$M \subseteq \bigcup_{j=1}^{\ell_m} D\left(z_{m,j}, \frac{1}{2^m}\right)$$

Since $(x_n^{(m-1)})_{n=1}^{\infty}$ has ∞ terms

and there are only finitely many disks : $D(z_{m,1}, \frac{1}{2^m}), \dots, D(z_{m,l_m}, \frac{1}{2^m})$

we know that at least one of the disks contain ∞ terms

Let's call this disk $D(z_m, \frac{1}{2^m})$ which contains a subseq. $(x_n^{(m)})_{n=1}^{\infty}$ of $(x_n^{(m-1)})_{n=1}^{\infty}$

Because $x_n^{(m)} \in D(z_m, \frac{1}{2^m})$ $\forall n$, we have

$$d(x_j^{(m)}, x_k^{(m)}) < \frac{1}{2^{m-1}} \qquad \forall j,k$$

Step2 : Extract the diagonal seq.

$$y_m := x_m^{(m)}$$

$x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, \dots$ $\quad d(x_j^{(1)}, x_k^{(1)}) < 1$

$x_1^{(2)}, x_2^{(2)}, x_3^{(2)}, \dots$ $\quad d(x_j^{(2)}, x_k^{(2)}) < \frac{1}{2}$

$x_1^{(3)}, x_2^{(3)}, x_3^{(3)}, \dots$ $\quad d(x_j^{(3)}, x_k^{(3)}) < \frac{1}{4}$

$\vdots$

$x_1^{(m)}, x_2^{(m)}, x_3^{(m)}, \dots, x_m^{(m)}, \dots, y_j$ $\quad d(x_j^{(m)}, x_k^{(m)}) < \frac{1}{2^{m-1}}$

$\vdots$

Then for $p > m \geq 1$, since

$$y_p = x_p^{(p)}$$

is contained in $(x_n^{(m)})_{n=1}^{\infty}$, we have

$$d(y_m, y_p) < \frac{1}{2^{m-1}}$$

$\forall \varepsilon > 0$, choose $N \in \mathbb{N}$ s.t. $\frac{1}{2^{N-1}} < \varepsilon$

$$\Rightarrow d(y_m, y_p) < \frac{1}{2^{m-1}} < \frac{1}{2^{N-1}} < \varepsilon$$

$\forall\ p > m \geq N$

$\Rightarrow (y_m)_{m=1}^{\infty}$ is a Cauchy seq.

$\Rightarrow$ it converges in M.

$\Rightarrow M$ is cpt. #

Thm (Finite Intersection Property, a.k.a. Nested Set Property, Thm 3.3.1)

Let (M, d) be a metric sp. and $F_n \subseteq M$, $n = 1, 2, 3, \cdots$, be nonempty compact subsets of M s.t.

$$F_1 \supseteq F_2 \supseteq F_3 \supseteq \cdots \supseteq F_n \supseteq F_{n+1} \supseteq \cdots$$

Then $\bigcap_{n=1}^{\infty} F_n \neq \phi$.

NOTE: The assumption "compact" is important: Consider

$[1,\infty) \supseteq [2,\infty) \supseteq [3,\infty) \supseteq \cdots$

$\Rightarrow \bigcap_{n=1}^{\infty} [n,\infty) = \emptyset$

pf

Suppose for contradiction $\bigcap_{n=1}^{\infty} F_n = \emptyset$.

$\Rightarrow M = M \setminus \emptyset = M \setminus \left(\bigcap_{n=1}^{\infty} F_n\right)$

$= \bigcup_{n=1}^{\infty} (M \setminus F_n)$ ($M \setminus F_n$ open)

So $\{ M \setminus F_n \mid n \in \mathbb{N} \}$ is an open cover of $(M2) F_1$.

Since F_1 is compact, $\exists N \in \mathbb{N}$ s.t.

$$\bigcup_{n=1}^{N} (M \setminus F_n) \supseteq F_1 \supseteq F_N$$

$$\bigcup_{n=1}^{N} (M \setminus F_n) = M \setminus F_N$$

$\Rightarrow M \setminus F_N \supseteq F_N$

$\Rightarrow F_N = F_N \cap (M \setminus F_N) = \emptyset$ ($\longrightarrow\!\!\!\!\!\!\times\!\!\!\longleftarrow$)

So $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$. ※

Example (Cantor Set)

Construction:

Let $F_0 = [0,1] \subseteq \mathbb{R}$



- Remove the middle-third from F_0:
$F_1 := F_0 - (\frac{1}{3}, \frac{2}{3}) = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$
- Remove the middle third of each subinterval of F_1:
$F_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$
$\vdots$
- Inductively, F_n is the union of 2^n disjoint closed intervals of length $\frac{1}{3^n}$

The Cantor set C is

$$C := \bigcap_{n=1}^{\infty} F_n$$

Properties of C:

① Nonempty and compact:

Since F_n is closed and bold, it is compact. (Heine-Borel)

Also, $F_n \neq \emptyset$, $\cdots \supseteq F_n \supseteq F_{n+1} \supseteq \cdots$.

by the Finite Intersection Property,

$$C = \bigcap_{n=1}^{\infty} F_n \neq \emptyset$$

Furthermore, since C is a closed subset of the compact set $[0,1]$, we have C is compact.

② Zero length:

The total length of F_n

$= 2^n \times \frac{1}{3^n} \longrightarrow 0$ as $n \to \infty$

"Total length of C is 0"

③ Perfect set:

The Cantor set C is a perfect set i.e. C is closed, and all points of C are accumulation points of C (the points of C = $f(C)$)

exer 2: perfect $\Rightarrow$ uncountable