Advanced Calculus, week 6, Fall 2026

Recall

① Let (M,d) be a metric sp., $K \subseteq M$.
K is sequentially compact (緊致) if every seq. in K admits a subsequence converging to a point in K

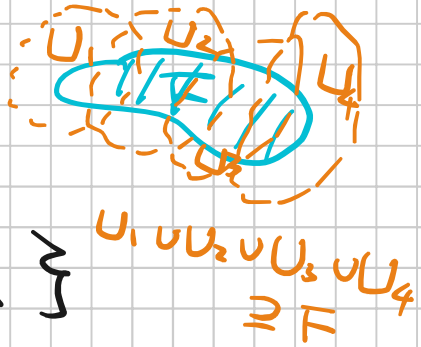
② K is seq. cpt. $\Rightarrow$ K is closed and bdd.

Compactness via open covers

Def (Def 3.1.2)
Let $F \subseteq (M,d)$ and

$$\mathcal{U} = \{ U_\alpha \mid \alpha \in A \}$$

be a family of some open sets in M.



(i) $\mathcal{U}$ is an open cover of F if

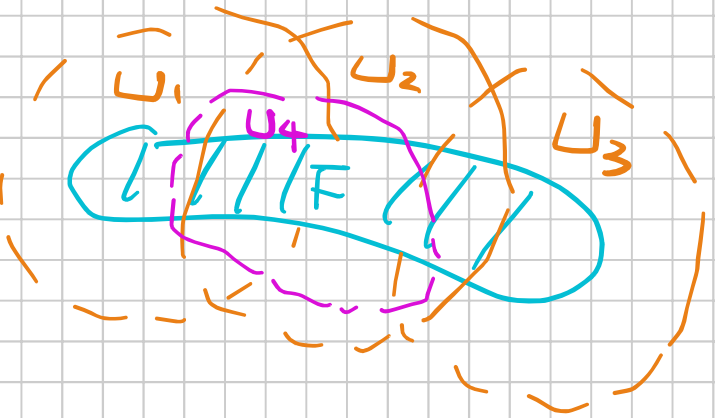
$$F \subseteq \bigcup_{\alpha \in A} U_\alpha$$

(ii) A subcollection

$$\mathcal{V} = \{ U_\alpha \mid \alpha \in B \} \quad (B \subseteq A)$$

covering F (i.e. $\bigcup_{\alpha \in B} U_\alpha \supseteq F$) is called an open subcover. If B is finite, $\mathcal{U}$ is called a finite subcover.

$\{U_1, U_2, U_3, U_4\}$ is an open cover of F,
$\{U_1, U_2, U_3\}$ is a subcover



$U_1 \cup U_2 \cup U_3 \cup U_4 \supseteq F$

$U_1 \cup U_2 \cup U_3 \supseteq F$

(iii) A subset $C \subseteq (M, d)$ is compact if every open cover of C admits a finite subcover

NOTE:

To prove a set $L \subseteq M$ is NOT cpt, it suffices to find one open cover of L that admits no finite subcover.

Example

① The real line $\mathbb{R}$ is NOT cpt

pf

Consider

$$\mathcal{U} = \{ \underbrace{(-n, n)}_{D(0,n)} \subseteq \mathbb{R} \mid n \in \mathbb{N} \}$$

$\Rightarrow$

- $(-n, n)$ is open in $\mathbb{R}$ $\forall n \in \mathbb{N}$
- $\bigcup_{n=1}^{\infty} (-n, n) \supseteq \mathbb{R}$

$\Rightarrow$ $\mathcal{U}$ is an open cover of $\mathbb{R}$.

But $\mathcal{U}$ admits NO finite subcover:

If $\exists$ finite $n_1, \cdots, n_N \in \mathbb{N}$ s.t.

$$(-n_1, n_1) \cup (-n_2, n_2) \cup \cdots \cup (-n_N, n_N) \supseteq \mathbb{R}$$

Consider

$$\tilde{N} = \max \{ n_1, \cdots, n_N \}$$

$$\Rightarrow (-\tilde{N}, \tilde{N}) \supseteq \bigcup_{j=1}^{N} (-n_j, n_j) \supseteq \mathbb{R}$$

But $\tilde{N}+1 \in \mathbb{R}$, $\tilde{N}+1 \notin (-\tilde{N}, \tilde{N})$ ($\longrightarrow\!\longleftarrow$)

So $\mathcal{U}$ has NO finite subcover

$\Rightarrow$ $\mathbb{R}$ is NOT cpt. #

② The interval $[0, 1) \subseteq \mathbb{R}$ is NOT cpt.

pf

Consider

$$\mathcal{U} = \{ (-1, 1 - \tfrac{1}{n}) \mid n \in \mathbb{N} \}$$

$$\Rightarrow \bigcup_{n=1}^{\infty} (-1, 1 - \tfrac{1}{n}) = (-1, 1) \supseteq [0, 1)$$

$\Rightarrow$ $\mathcal{U}$ is an open cover of $[0, 1)$.

Claim: $\mathcal{U}$ has NO finite subcover.

If $\mathcal{U}$ has a finite subcover, then

$\exists\, n_1, \cdots, n_N \in \mathbb{N}$ s.t.

$$\bigcup_{j=1}^{N} (-1, 1 - \tfrac{1}{n_j}) \supseteq [0, 1)$$

Let $\tilde{N} = \max \{ n_1, \cdots, n_N \}$

$$\Rightarrow (-1, 1 - \tfrac{1}{\tilde{N}}) \supseteq \bigcup_{j=1}^{N} (-1, 1 - \tfrac{1}{n_j}) \supseteq [0, 1)$$

But $1 > 1 - \frac{1}{\tilde{N}+1} > 1 - \frac{1}{\tilde{N}} \Rightarrow 1 - \frac{1}{\tilde{N}+1} \in [0, 1)$

$\not\in (-1, 1 - \frac{1}{\tilde{N}})$

$\Rightarrow \Leftarrow$

$\Rightarrow$ $\mathcal{U}$ has NO finite subcover

So $[0, 1)$ is NOT cpt. #

⑧ Let $F \subseteq (M, d)$ be a finite set.

$\Rightarrow$ F is cpt.

pf

Let $F = \{x_1, \cdots, x_n\}$

Given any open cover of F,

$$\mathcal{U} = \{U_\alpha \mid \alpha \in A\}$$

$\Rightarrow x_j \in F \subseteq \bigcup_{\alpha \in A} U_\alpha \ni x_j$

$\Rightarrow \forall j = 1, \cdots, n,$

$$x_j \in U_{\alpha_j} \quad \text{for some } \alpha_j \in A$$

$$\Rightarrow \bigcup_{j=1}^{n} U_{\alpha_j} \supseteq \{x_1, \cdots, x_n\} = F$$

i.e. $\{U_{\alpha_1}, \cdots, U_{\alpha_n}\}$ is a finite subcover

$\Rightarrow F$ is cpt. #

Thm

If $C \subseteq (M, d)$ is cpt, then C is closed and bounded.

pf

< $C = \phi \Rightarrow$ ok Assume $C \neq \phi$

Step1 (Bdd).

Fix $x_0 \in C$

$\{y \in M \mid d(y, x_0) < n\}$

$$\bigcup_{n=1}^{\infty} \underline{D(x_0, n)} = M \supseteq C$$

$\Rightarrow \{D(x_0, n)\}_{n=1}^{\infty}$ is an open cover of C

Since C is cpt, $\exists\ n_1, \cdots, n_N$ s.t.

$$\bigcup_{j=1}^{N} D(x_0, n_j) \supseteq C$$

Let $\tilde{N} = \max\{n_1, \cdots, n_N\}$

$$\Rightarrow D(x_0, \tilde{N}) \supseteq \bigcup_{j=1}^{N} D(x_0, n_j) \supseteq C$$

$\Rightarrow C$ is bounded.

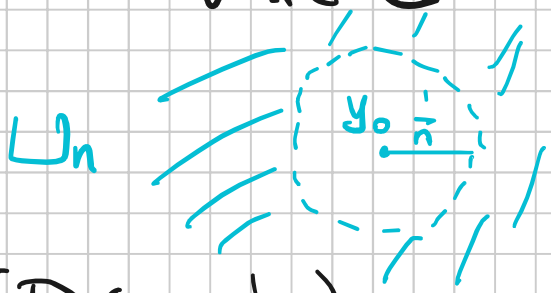
Step2: Before proving closedness, we first prove a lemma:

Lemma

Let $C \subseteq (M, d)$ be compact.

If $y_0 \in M \setminus C$, then $\exists\ N \in \mathbb{N}$ s.t.

$$d(x, y_0) \geq \frac{1}{N} \qquad \forall\ x \in C$$



pf

$\forall n \in \mathbb{N}$, let

$$U_n = M \setminus cl\left(D\left(y_0, \tfrac{1}{n}\right)\right)$$

which is open

Since $y_0 \notin C$, for $x \in C$

$\circ\!\!*$ $d(y_0, x) \geq \frac{1}{n_x}$ for some $n_x \in \mathbb{N}$

$\Leftrightarrow x \in U_{n_x}$

$\Rightarrow C \subseteq \bigcup_{x \in C} U_{n_x} \subseteq \bigcup_{n=1}^{\infty} U_n$

Since C is cpt, $\exists N$ s.t.

$U_N = \bigcup_{n=1}^{N} U_n \supseteq C$

$M - \overline{B}(y_0, \frac{1}{2}) = \{y \in M \mid d(y, y_0) \geq \frac{1}{2}\}$

That is, $\forall x \in C$,

$d(x, y_0) \geq \frac{1}{2}$ #

Step3 C is closed

Assume $y_0 = \lim_{n \to \infty} x_n$ for some $(x_n) \subseteq C$ (*)

Claim: $y_0 \in C$

If not, by Step2, $\exists N \in \mathbb{N}$ s.t.

$d(x, y_0) \geq \frac{1}{N}$ $\forall x \in C$

$\Rightarrow D(y_0, \frac{1}{N}) \cap C = \emptyset$

But (*) $\Rightarrow \exists \tilde{N}$ s.t. $x_n \in D(y_0, \frac{1}{N})$ $\forall n \geq \tilde{N}$

$\Rightarrow x_n \in C \cap D(y_0, \frac{1}{N})$ $\not\in$

So $y_0 \in C \Rightarrow C$ is closed. #

Example

① Since $\mathbb{R}$ is NOT bdd, it is NOT cpt.

② Since $[0,1) \subseteq \mathbb{R}$ is NOT closed, it is NOT cpt.

③ $\{\frac{1}{n} \in \mathbb{R} \mid n \in \mathbb{N}\}$ is NOT closed, it is NOT cpt.

④ $C = \{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{N}\}$ is cpt.

Pf

Let $\{U_\alpha\}_{\alpha \in A}$ be an open cover of C.

$\Rightarrow \exists\, \alpha_0 \in A$ s.t. $0 \in U_{\alpha_0}$

Step1 Since U_{α_0} is open, $\exists\, \delta > 0$ s.t.

$$D(0,\delta) = (-\delta, \delta) \in U_{\alpha_0}$$

Since $\lim_{n\to\infty} \frac{1}{n} = 0$, $\exists\, \tilde{N}$ s.t.

$$\left|\frac{1}{n} - 0\right| < \delta \quad \forall\, n \geq \tilde{N}$$

$\Rightarrow \frac{1}{n} \in (-\delta, \delta) \subseteq U_{\alpha_0} \quad \forall\, n \geq \tilde{N}$

Step2

Since $C \subseteq \bigcup_{\alpha \in A} U_\alpha$,

$\frac{1}{n} \in C \subseteq \bigcup_{\alpha \in A} U_\alpha$

$\exists\, \alpha_n \in A$ s.t.

$\frac{1}{n} \in U_{\alpha_n}$

$\Rightarrow \{1, \frac{1}{2}, \cdots, \frac{1}{\tilde{N}-1}\} \subseteq \bigcup_{j=1}^{\tilde{N}-1} U_{\alpha_j}$

So $C = \{0\} \cup \{\frac{1}{n} \mid n \geq \tilde{N}\} \cup \{\frac{1}{n} \mid 1 \leq n < \tilde{N}\}$

Step1 $\cap$ U_{α_0} ; $\bigcup_{j=1}^{\tilde{N}-1}$ $\cap$ Step2 U_{α_j}

$\subseteq \bigcup_{j=0}^{\tilde{N}-1} U_{\alpha_j}$

That is, $\{U_{\alpha_0}, \cdots, U_{\alpha_{\tilde{N}-1}}\}$ is a finite subcover.

$\Rightarrow C$ is cpt. #

Answer question:

Assume $A \subseteq (M, d)$ is closed. Show that

$\partial(\partial A) = \partial A$

Outline of pf: $\partial A = cl(A) - int(A)$

Step1: Show that $\partial A = A - int(A)$

Step2: Show that $int(\partial A) = \emptyset$

Step3: $\partial(\partial A) \overset{\text{Step1}}{=} \partial A - int(\partial A) \overset{\text{Step2}}{=} \partial A$

} Assume A is closed

Next goal: seq. cpt $\Leftrightarrow$ cpt (for metric sp.)

Def (Def 3.1.4)

A subset $F \subseteq (M,d)$ is totally bounded if $\forall \varepsilon > 0$ $\exists$ $x_1, \dots, x_n \in F$ s.t.

$$F \subseteq D(x_1,\varepsilon) \cup D(x_2,\varepsilon) \cup \cdots \cup D(x_n,\varepsilon)$$

easier: bdd $\Leftarrow$ F bdd

Prop

Every cpt set $C \subseteq (M,d)$ is totally bdd.

pf

Consider the open cover ($\forall \varepsilon > 0$)

$$C \subseteq \bigcup_{x \in C} D(x,\varepsilon)$$

$\overset{\text{cpt}}{\Rightarrow}$ $\exists$ $x_1, \dots, x_n \in C$ s.t. $C \subseteq \bigcup_{i=1}^{n} D(x_i,\varepsilon)$ $\Rightarrow$ C is tot. bdd. #

Prop

Every seq. cpt. set $K \subseteq (M, d)$ is totally bdd.

pf

Suppose K is NOT totally bdd., i.e. ($K \neq \phi$)

$\exists\, \varepsilon_0 > 0$ s.t. $\forall\, x_1, \cdots, x_n \in K$, $K \not\subseteq \bigcup_{i=1}^{n} D(x_i, \varepsilon_0)$

Construct $(y_m) \subseteq K$:

① $K \neq \phi \Rightarrow \exists\, y_1 \in K$

② $K \not\subseteq D(y_1, \varepsilon_0) \Rightarrow \exists\, y_2 \in K \setminus D(y_1, \varepsilon_0)$

③ $K \not\subseteq D(y_1, \varepsilon_0) \cup D(y_2, \varepsilon_0) \Rightarrow \exists\, y_3 \in K \setminus \left(\bigcup_{i=1}^{2} D(y_i, \varepsilon_0)\right)$

$\vdots$

(m) $K \not\subseteq \bigcup_{i=1}^{m-1} D(y_i, \varepsilon_0) \Rightarrow \exists\, y_m \in K \setminus \left(\bigcup_{i=1}^{m-1} D(y_i, \varepsilon_0)\right)$

$\vdots$

Then $(y_m) \subseteq K$ satisfies that

$$d(y_n, y_m) \geq \varepsilon_0 \quad \forall\, n \neq m$$

$\Rightarrow (y_n)$ has NO convergent subseq.

contradict to K is seq. cpt. #

Prop

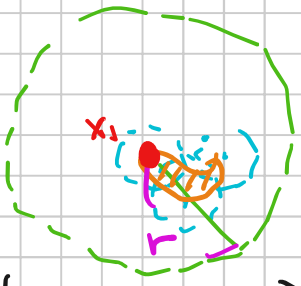
Every totally bdd set is bdd.

pf

If $F \subseteq (M,d)$ is totally bdd, then

for $\varepsilon = 1 > 0$, $\exists\ x_1, \cdots, x_n \in F$ s.t.

$$F \subseteq \overset{n}{\underset{i=1}{\cup}} D(x_i, 1)$$



Let $r = \max\{ d(x_1, x_i) + 100 \mid i = 1, \cdots, n \}$

Then $\forall\ y \in F$, we have

$y \in D(x_i, 1)$ for some i.

$$\Rightarrow\ d(x_1, y) \leq d(x_1, x_i) + d(x_i, y) \quad (d(x_i, y) < 1)$$

$$< d(x_1, x_i) + 100 \leq r$$

$$\Rightarrow\ y \in D(x_1, r)$$

$$\Rightarrow\ F \subseteq D(x_1, r) \Rightarrow F \text{ is bdd.}\ \#$$

Remark

Bdd. $\not\Rightarrow$ totally bdd.

e.g. $(M,d) = (\mathbb{N}, d_c)$ (d_c = discrete metric) $F = \mathbb{N}$

$\Rightarrow$ F is bdd because $F \subseteq D(1, 2) = \mathbb{N}$

F is NOT tot. bdd. $F \not\subseteq \overset{n}{\underset{i=1}{\cup}} D(x_i, \frac{1}{2}) = \{x_1, \cdots, x_n\}$

Lemma

If $K \subseteq (M,d)$ is seq. cpt, then

K is cpt.

Pf

Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ be an open cover of K.

Claim: $\exists\, \varepsilon > 0$ s.t. $\forall x \in K$,

$$D(x, \varepsilon) \subseteq U_{\alpha_x}$$

for some $\alpha_x \in A$.

pf of claim

If not, $\forall \varepsilon > 0 \ \exists x_\varepsilon \in K$ s.t. $D(x_\varepsilon, \varepsilon) \not\subseteq U_\alpha \ \forall \alpha \in A$

for $\varepsilon = \frac{1}{n} > 0$, $\exists x_n \in K$ s.t.

(*) $D(x_n, \frac{1}{n}) \not\subseteq U_\alpha \quad \forall \alpha \in A$

Since K is seq. cpt, $\exists$ subseq $(x_{n_j})_j$ of (x_n) s.t. $\lim_{j\to\infty} x_{n_j} = x_*$ for some $x_* \in K$.

$\Rightarrow \exists \alpha_* \in A$ s.t. $x_* \in U_{\alpha_*}$ ($K \subseteq \bigcup_{\alpha\in A} U_\alpha$)

Since U_{α_*} is open, $\exists \delta > 0$ s.t.

$$D(x_*, \delta) \subseteq U_{\alpha_*}$$

But for large j, ($n_j \geq j$)

$$d(x_{n_j}, x_*) < \frac{\delta}{2} \quad \text{and} \quad \frac{1}{n_j} < \frac{\delta}{2}$$

$$\Rightarrow D(x_{n_j}, \tfrac{1}{n_j}) \subseteq D(x_{n_j}, \tfrac{\delta}{2}) \subseteq D(x_*, \delta) \subseteq U_{\alpha_*}$$

contradict to (*)

So the claim is true

By Claim, $\exists\, \varepsilon > 0$, s.t. $\forall x \in K$,

$$D(x,\varepsilon) \subseteq U_{\alpha_x} \quad \text{for some } \alpha_x \in A.$$

Since K is seq cpt. $\Rightarrow$ K is tot. bdd.

$\Rightarrow \exists\, x_1, \cdots, x_n \in K$ s.t.

finite sub cover

$$K \subseteq \bigcup_{i=1}^{n} D(x_i, \varepsilon) \subseteq \bigcup_{i=1}^{n} U_{\alpha_{x_i}}$$

$\Rightarrow$ K is cpt. #

$C = [0,1]$ is cpt

$(\frac{1}{3} - \frac{1}{n}) \in Y = (\frac{1}{3}, \frac{2}{3})$ is NOT cpt

Lemma A closed subset Y of a cpt set C in (M,d) is cpt.

Pf

If $\{V_\alpha\}_{\alpha \in A}$ is an open cover of Y, then $\{V_\alpha\}_{\alpha \in A} \cup \{M \setminus Y\}$ is an open cover of C, because

$$\bigcup_{\alpha \in A} V_\alpha \cup (M \setminus Y) \supseteq Y \cup (M \setminus Y) = M \supseteq C$$

($\bigcup_{\alpha \in A} V_\alpha \supseteq Y$)

C is cpt

$\Rightarrow \exists\, V_{\alpha_1}, \cdots, V_{\alpha_n}$ s.t.

$$V_{\alpha_1} \cup \cdots \cup V_{\alpha_n} \cup (M \setminus Y) \supseteq C$$

$\Rightarrow$

$$V_{\alpha_1} \cup \cdots \cup V_{\alpha_n} \supseteq Y$$

$\forall y \in Y \subseteq C$, $y \in V_{\alpha_i}$ or $y \in (M \setminus Y)$

$\Rightarrow y \in V_{\alpha_i}$ for some i.

$\Rightarrow$ Y is cpt. #

Lemma

If $C \subseteq (M,d)$ is cpt, then C is seq. cpt.

pf

Assume $(y_n) \subseteq C$ has no subseq converging in C.

Consider

$$Y = \{y_n \mid n \in \mathbb{N}\}$$

$$\Rightarrow \underbrace{(\underbrace{Y \cup \partial(Y)}_{\text{"cl}(Y)} \cap C}_{\text{"}B = Y \cap C} \subseteq C$$

B is a closed subset in M

Lemma $\Rightarrow$ B is cpt.

NOT an acc pt

$\partial(Y) \cap C = \emptyset \Rightarrow \forall y_n \exists\ \varepsilon_n > 0$ s.t.

$$D(y_n, \varepsilon_n) \cap \underbrace{Y \cap C}_{B = Y} = \{y_n\}$$

$\Rightarrow$ $\{D(y_n, \varepsilon_n)\}_{n=1}^{\infty}$ is an open cover of $B = Y$

and has no finite subcover ($\not\rightarrow$ to Y cpt.)

So C is seq cpt. #

Thm (Bolzano–Weierstrass Thm, Thm 3.1.3)
A set $S \subseteq (M, d)$ is cpt $\Leftrightarrow$ S is seq. cpt.

Thm (Heine–Borel Thm, Thm 3.2.1)
A subset $K \subseteq (\mathbb{R}^n, d_e)$ is compact $\Leftrightarrow$ K is closed and bounded.

pf

"$\Rightarrow$" was done before.

"$\Leftarrow$" Assume $K \subseteq \mathbb{R}^n$ is closed, bdd.

Want to show: K is seq. cpt.

Let $(x_m)_{m=1}^{\infty} \subseteq K \subseteq \mathbb{R}^n$

$$x_m = (x_{m,1}, x_{m,2}, \cdots, x_{m,n})$$

where $x_{m,1}, \cdots, x_{m,n} \in \mathbb{R}$

Step1: K is bdd $\Rightarrow$ $(x_{m,i})_{m=1}^{\infty}$ is bdd $\forall i = 1, \cdots, n$

K is bdd $\Rightarrow \exists M > 0$ s.t. $\forall y \in K$,

$|y_i| \leq \sqrt{y_1^2 + \cdots + y_n^2} = d_2(y, 0) = \|y\|_2 < M$

$(x_m)_{m=1}^{\infty} \subseteq K \Rightarrow |x_{m,i}| < M \quad \forall m$

$\Rightarrow (x_{m,i})_{m=1}^{\infty}$ is bdd $\forall i = 1, \cdots, n$.

1. Let (M,d) be a metric space, and V be a nonempty subset of M. We know that $(V,d|_{V\times V})$ is a metric space.
   (a) (15 points) Prove that A is open in $(V,d|_{V\times V})$ if and only if there exists an open set U in (M,d) such that $A = V \cap U$.
   (b) (15 points) Prove or disprove that a subset $U \subseteq V$ is open in $(V,d|_{V\times V})$ if and only if U is open in (M,d).

第一題 (a)：

$D_M(x,r) = \{y \in M \mid d(x,y) < r\}$

- 把在 V 上的鄰域和在 M 上的鄰域混用，只使用 $D(x,r)$ 的符號(或是沒注意到 open in V 是不同於 open in M 的)，導致後面的推論會有問題，例如：A open in V => $\forall x \in A,\ \exists r(x) > 0$ such that $D(x,r(x)) \subseteq A$, 然後說 $D(x,r(x))$ open in M。

$D_V(x,r) = \{y \in V \mid d(x,y) < r\}$

- 沒有把 open in M 還是 in V 寫出來，只寫 open。
- 大多數人在寫存在 $r(x)$ 時不會把 x 寫出來，只寫r，但因為這題會討論 $\bigcup_{x\in A} D(x,r(x))$，所以建議把 r 跟 x 有關寫清楚。
- 少部分人不記得 $d|_{V\times V}$ 的意思，而去討論 (x,y) 在 $V\times V$ 上的鄰域。
- 在右推到左時，直接寫 "let $U = \bigcup_{x\in A} D(x,r(x))$"，但題目中的 U 是給定的，不能直接假設它是此聯集。

反例 ① $M=\mathbb{R}^2$, $V=$ 

② $M=\mathbb{R}$, $V=[0,1]$, $U=(\frac{1}{2},1]$ open in V, not open in $\mathbb{R}$

open in V, not open in $\mathbb{R}^2$

第一題 (b)：

- 沒有把 M 的空間寫出來，雖然多數人應該是直接默認 $(M,d) = (\mathbb{R}, d_e)$。

第二題： 2. (30 points) Show that for any subset A of a metric space M, $\text{int}(A) = A \setminus \partial(A)$.

- 在使用$x \in cl(A)$ 的等價敘述 $\forall r > 0, D(x,r) \cap A \neq \emptyset$ 時會把 $\forall$ 和 $\exists$ 弄混。
- 少部分人會寫 $D(x,r) \subseteq A \Rightarrow D(x,r) \cap (A\backslash\{x\}) \neq \emptyset$，但這不一定是對的，例如在 discrete topology。

$A = \{x\}$, $D(x,r) = \{y \in M \mid d(x,y) < r\}$, $\Rightarrow A \cap D(x,r) = \{x\}$

By 上課, $\text{int}(A) = cl(A) \setminus \partial A$

$A \setminus \partial A \subseteq cl(A) \setminus \partial A = \text{int}(A)$

上課 $\Rightarrow A \setminus \partial A \subseteq \text{int}(A)$

Average	31.55294118
Average (except zero)	36.24324324
Quartile 0	0
Quartile 1	10
Quartile 2	35
Quartile 3	50
Quartile 4	60
標準差	21.42067642
非零數	74
不到50%人數	37
滿分人數	9

