

Advanced Calculus, week 5, Fall 2026

Dense set

Def

Let $D \subseteq S \subseteq (M, d)$.

We say D is dense in S if

$$\text{cl}(D) \supseteq S$$

Example

① $\mathbb{Q}$ is dense in $\mathbb{R}$. ($\text{cl}(\mathbb{Q}) = \mathbb{R}$)

$\mathbb{Q}^n$ is dense in $\mathbb{R}^n \leftarrow \text{exer.}$

② $(0, 1) \subseteq (\mathbb{R}, d_e)$ is dense in $[0, 1]$ $\text{cl}([0, 1]) = [0, 1]$

③ Every set $D \subseteq (M, d)$ is dense in $\text{cl}(D)$.

Remark

D is dense in $S \Leftrightarrow \text{cl}(D) \supseteq S$

$\Leftrightarrow \forall y \in S \exists (x_n) \in D$ s.t.

$$\lim_{n \rightarrow \infty} x_n = y$$

Complete

Def (Def 2.8.1)

A seq. $(x_n) \subseteq (M, d)$ is Cauchy if

$$\forall \varepsilon > 0 \exists N_\varepsilon \text{ s.t.}$$

$$\uparrow \quad d(x_n, x_m) < \varepsilon \quad \forall n, m \geq N_\varepsilon$$

Good for determining if a seq. is convergent

exer: Any convergent seq. is Cauchy

A metric space (M, d) is complete if every Cauchy seq. in M converges in M .

Example

① $(\mathbb{R}, d_e)$ is complete (from Calculus)

② $(\mathbb{Q}, d_e)$ is NOT complete:

For example, $x_1 = 1, x_2 = 1.4, x_3 = 1.41$
convergent in $\mathbb{R} \Rightarrow$ Cauchy seq. $x_4 = 1.414, \dots$

$\Rightarrow (x_n) \subseteq \mathbb{Q}$, but $\lim_{n \rightarrow \infty} x_n = \sqrt{2} \notin \mathbb{Q}$

That is, (x_n) does NOT converge in $\mathbb{Q}$.

Thm (Thm 2.8.5)

For each $N \in \mathbb{N}$, $(\mathbb{R}^N, d_e)$ is complete.
pf

Assume $(x_m) \subseteq \mathbb{R}^N$ is Cauchy.

Denote $x_m = (x_{1,m}, x_{2,m}, \dots, x_{N,m}) \in \mathbb{R}^N$
Step 1 Then $\forall \varepsilon > 0, \exists N_\varepsilon$ s.t.

$$d_e(x_m, x_n) < \varepsilon \quad \forall m, n \geq N_\varepsilon$$

$$\sqrt{(x_{1,m} - x_{1,n})^2 + \dots + (x_{N,m} - x_{N,n})^2}$$

$$\sqrt{0 + \dots + 0 + (x_{j,m} - x_{j,n})^2 + 0 + \dots + 0}$$

$$|x_{j,m} - x_{j,n}| = d_e(x_{j,m}, x_{j,n})$$

$\Rightarrow (x_{j,m})_{m=1}^\infty$ is Cauchy in $\mathbb{R}$ for $j=1, \dots, N$.

Step 2 By completeness of $\mathbb{R}$, $\exists \alpha_j \in \mathbb{R}$ s.t.

$$\lim_{m \rightarrow \infty} x_{j,m} = \alpha_j \text{ in } \mathbb{R}$$

$\Rightarrow \lim_{m \rightarrow \infty} x_m = (\alpha_1, \dots, \alpha_N)$ in $\mathbb{R}^N$

$\forall \varepsilon > 0, \exists N_{j,\varepsilon}$ s.t. $|x_{j,m} - \alpha_j| < \frac{\varepsilon}{2N} \quad \forall m \geq N_{j,\varepsilon}$

$\Rightarrow d_e(x_m, \alpha) \leq \sum_{j=1}^N |x_{j,m} - \alpha_j| < \frac{\varepsilon}{2N} \cdot N < \varepsilon \quad \forall m \geq N_\varepsilon$

$N_\varepsilon = \max\{N_{1,\varepsilon}, N_{2,\varepsilon}, \dots, N_{N,\varepsilon}\}$

§ Topological properties and continuous maps

We will study "intrinsic topological properties", unlike open and closed
such as compact and connected
 $(0, 1)$ is NOT closed in $\mathbb{R}$ but it is closed in $(0, 1)$

We start with "Compact"

In metric spaces, $\exists$ 2 equivalent definitions of "Compact" — by sequences or by open cover.

Sequential compactness

Def (Def 3.1.1)

Let (M, d) be a metric sp, $K \subseteq M$.

We say K is sequentially compact if every seq. $(x_n) \subseteq K$ admits a subseq. (x_{n_j}) converging to a point in K

Remark

① Every closed interval $[a, b] \subseteq \mathbb{R}$ is sequentially compact

pf For $(x_n) \subseteq [a, b]$, it is bounded by Calculus: every bdd seq. in $\mathbb{R}$ has a conv. subseq.
 $\Rightarrow \exists$ subseq. (x_{n_j}) , $\exists \alpha \in \mathbb{R}$ s.t.
Bolzano-Weierstrass $\lim_{j \rightarrow \infty} x_{n_j} = \alpha$

By $\circledast$, we have $a \leq \alpha \leq b$, i.e. $\alpha \in [a, b]$

So $[a, b]$ is sequentially compact #

② To show $L \subseteq M$ is NOT $\overset{\text{seq}}{\text{cpt}}$, it suffices to construct $(x_n) \subseteq L$ that has NO subseq. converging in L

For example,

(a) $(0, 1)$ is NOT seq. cpt.

pf Consider $(x_n = \frac{1}{n+1})_{n=1}^{\infty} \Rightarrow \frac{1}{n+1} \in (0, 1)$

Every subseq. $x_{n_j} = \frac{1}{n_j+1}$ converges to 0 $\notin (0, 1)$

So $(0, 1)$ is NOT seq. cpt #

(b) $\mathbb{R}$ is NOT seq. cpt.

pf

Consider $(x_n = n)_{n=1}^{\infty} \Rightarrow n \in \mathbb{R}$

Every subseq.

$x_{n_j} = n_j$ is unbd.

$\Rightarrow (x_{n_j})_{j=1}^{\infty}$ does NOT converge in $\mathbb{R}$

So $\mathbb{R}$ is NOT seq. cpt. #

③ Every finite subset of a metric sp M is seq. cpt.

pf

Let $S = \{\alpha_1, \dots, \alpha_N\} \subseteq M$, and let

$(x_n)_{n=1}^{\infty}$ be any seq in S

$\Rightarrow x_n = \alpha_k$ for some $k \in \{1, \dots, N\}$

$\Rightarrow \exists k_0$ s.t. there are ∞ terms $(x_{n_j})_{j=1}^{\infty}$
with $x_{n_j} = \alpha_{k_0}$

$\Rightarrow \lim_{j \rightarrow \infty} x_{n_j} = \alpha_{k_0} \in S$

$\Rightarrow S$ is seq. cpt. #

Def

Let $B \subseteq (M, d)$. We say B is bounded

if $B \subseteq \underline{D}(x_0, R)$ for some $x_0 \in M$,
some $R > 0$.



Thm

If $K \subseteq (M, d)$ is seq. cpt., then
 K is closed and bounded in M .

pf

Step 1 (Closed):

Let $\alpha \in \text{cl}(K)$. Choose $(x_n) \subseteq K$ s.t.

$$\lim_{n \rightarrow \infty} x_n = \alpha.$$

Since K is seq. cpt., $\exists$ ^{subseq} $(x_{n_j})_{j=1}^{\infty}$ s.t.

$$\lim_{j \rightarrow \infty} x_{n_j} \in K$$

$$\Rightarrow \lim_{j \rightarrow \infty} x_{n_j} = \alpha \Rightarrow \alpha \in K$$

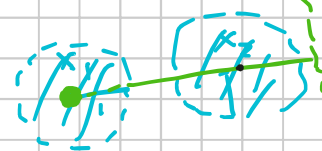
So K is closed

Step 2 (Bdd)

Assume K is NOT bdd. $(\Rightarrow K \neq \emptyset)$

① Fix $x_1 \in K$.

② Since K is NOT bdd, we have $K \not\subseteq D(x_1, 1)$
 $\Rightarrow \exists x_2 \in K \setminus D(x_1, 1)$



③ Note that

$$D(x_1, 1) \cup D(x_2, 1) \subseteq D(x_1, R)$$

where $R = d(x_1, x_2) + 2$

$$d(y, x_1) \leq d(y, x_2) + d(x_2, x_1) < 1 + d(x_2, x_1) < R$$

Since K is NOT bdd,

$$K \not\subseteq D(x_1, R) \Rightarrow K \not\subseteq D(x_1, 1) \cup D(x_2, 1)$$

$$\Rightarrow \exists x_3 \in K \setminus (D(x_1, 1) \cup D(x_2, 1))$$

⋮

K unbdd

$$\textcircled{1} \text{ Inductively, } \exists x_n \in K \setminus \left(\bigcup_{i=1}^{n-1} D(x_i, 1) \right)$$

So we find $(x_n) \subseteq K$ s.t.

$$d(x_n, x_m) \geq 1 \quad \forall n \neq m$$

$\Rightarrow$ Every subseq. (x_{n_j}) has the same property

$\Rightarrow (x_{n_j})$ is NOT Cauchy

$\Rightarrow (x_{n_j})$ is NOT convergent (in K)

$\Rightarrow K$ is NOT seq. cpt ($\rightarrow$)

So K is bdd.

$\nexists$

Quiz1

Jun-Shuo Chen

0.1 Problem 1

Let d_e denote the Euclidean metric on $\mathbb{R}^n$. For $x, y \in \mathbb{R}^n$, define

$$d(x, y) = \frac{d_e(x, y)}{1 + d_e(x, y)}$$

(a) Prove that $(\mathbb{R}^n, d)$ is a metric space.

常錯部分：

- 關於正定性的公理中，有一條要證明 $d(x, y) = 0 \iff x = y$ 。這裡有兩個方向，同學容易漏證明 $d(x, y) = 0 \implies x = y$ 。
- 此外， $d(x, x) = 0 \implies x = 0$ 是不正確的。

(b) Prove or disprove that d_e is equivalent to d .

提醒

$$x_n = (1, 0, 0, \dots, 0)$$

$$d_e(x_n, 0) = n \text{ is NOT bounded}$$

- 雖然 $d_e : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ 函數是無界的很顯然。但因為不一定所有的度量都是無界的（本題的 d 就不是），因此希望同學說明為何 d_e 是無界的。

0.2 Problem 2

For nonempty set $A, B \subset \mathbb{R}$, let $A + B = \{x + y | x \in A \text{ and } y \in B\}$. Show that $\sup(A + B) = \sup(A) + \sup(B)$.

常錯部分：

$$\sup(A) + B \quad \sup(A+B)$$

- 對 $\sup$ 的定義不了解或不清楚，不知道如何論述 $\sup(A) + \sup(B) \leq \sup(A + B)$ 。
- 不等式的方向容易顛倒，錯誤範例：For all $z \in A + B$, it can be written as $z = x + y$ for some $x \in A$ and $y \in B$. Since $x \leq \sup(A)$ and $y \leq \sup(B)$, one has

$$z \leq \sup(A) + \sup(B), \Rightarrow \sup(A+B) \leq \sup(A) + \sup(B)$$

Because $\sup(A + B)$ is upper bound of $A + B$, hence $\sup(A + B) \not\leq \sup(A) + \sup(B)$.

- for all 或 there exists 沒有宣告。
- 遺漏 A 跟 B 有無界的可能。

1 Problem 3

Construct two bounded sequence $(a_n)_{n=1}^\infty$ and $(b_n)_{n=1}^\infty$ such that

$$\overline{\lim}_{n \rightarrow \infty} (a_n + b_n) \leq \overline{\lim}_{n \rightarrow \infty} a_n + \overline{\lim}_{n \rightarrow \infty} b_n.$$

提醒

- 寫下 (a_n) 跟 (b_n) 請說明為何他們是 bounded sequence。此次由於同學舉例都過於顯然，沒有額外扣分。

96	Average		37.69767442
97	Average (except zero)		40.02469136
98	Quartile 0		0
99	Quartile 1		29
100	Quartile 2		41.5
101	Quartile 3		48
102	Quartile 4		57
103	標準差		15.04743071
104	非零數		81
105	不到50%人數		23
106	滿分人數		0

Quiz1

Score	Frequency
0.00	5
6.67	3
13.33	3
20.00	4
26.67	15
33.33	5
40.00	23
46.67	15
53.33	13
60.00	0