

Advanced Calculus, week 4, Fall 2026

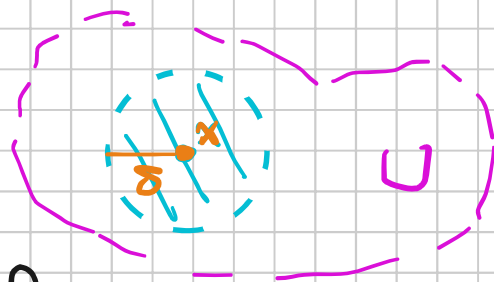
Recall

Let (M, d) be a metric space, $U, F \subseteq M$.

① U is open (in M) if $\forall x \in U \exists \delta = \delta_x > 0$ s.t.

$$D(x, \delta) \subseteq U$$

$$\{y \in M \mid d(x, y) < \delta\}$$



② F is closed (in M) if

$$F^c = M - F = \{x \in M \mid x \notin F\}$$

is open in M .

③ Basic properties:

$$\bigcap_{n=1}^{\infty} \left(-\frac{1}{n}, \frac{1}{n}\right) = \{0\}$$

is NOT open

- $\emptyset$ and M are open

- $U_\alpha, \alpha \in I$, are open $\Rightarrow \bigcup_{\alpha \in I} U_\alpha$ is open

- $U_1, \dots, U_k$ are open $\Rightarrow U_1 \cap \dots \cap U_k$ is open

Dually,

- $\emptyset$ and M are closed

- $F_\alpha, \alpha \in I$, are closed $\Rightarrow \bigcap_{\alpha \in I} F_\alpha$ is closed

- $F_1, \dots, F_k$ are closed $\Rightarrow F_1 \cup \dots \cup F_k$ is closed

Example

Let $x_0 \in (M, d)$, $r > 0$. Then

$$D(x_0, r) = \{y \in M \mid d(y, x_0) < r\}$$



is an open set

pf

Given $y_0 \in D(x_0, r)$, we want to find $\delta > 0$

$$d(x_0, y_0) < r$$

s.t. $D(y_0, \delta) \subseteq D(x_0, r)$.

Take

$$\delta = r - d(y_0, x_0) > 0$$

$\forall y \in D(y_0, \delta)$, we have

$$\begin{aligned} d(y, x_0) &\leq d(y, y_0) + d(y_0, x_0) \\ &< (r - d(y_0, x_0)) + d(y_0, x_0) \\ &= r \end{aligned}$$

$$\Rightarrow y \in D(x_0, r)$$

So $D(y_0, \delta) \subseteq D(x_0, r) \Rightarrow D(x_0, r)$ is open $\#$

Remark

① Let U be open in (M, d)

$$\Rightarrow \forall x \in U \exists \delta_x > 0 \text{ s.t. } D(x, \delta_x) \subseteq U$$

$$\Rightarrow \bigcup_{x \in U} D(x, \delta_x) \subseteq U$$

$$\Rightarrow U = \bigcup_{x \in U} D(x, \delta_x)$$



② Let d_0 be the discrete metric on $M = \mathbb{R}$

$$d_0(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

$\Rightarrow \forall x \in \mathbb{R},$

$D(x, \frac{1}{2}) = \{x\}$
is open in $(\mathbb{R}, d_0)$

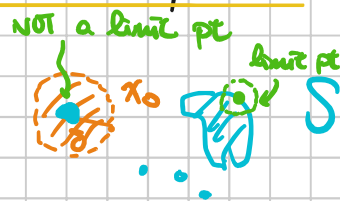
It's possible that
an open set is
finite.

Def (Def 2.4.1)

Let (M, d) be a metric sp, and $S \subseteq M$.

(i) A point $x_0 \in S$ is an isolated point of S if $\exists \delta > 0$ s.t.

$$D(x_0, \delta) \cap S = \{x_0\}$$



(ii) A point $x_0 \in M$ is an accumulation point (or a limit point) of S if $\forall \delta > 0,$

$$D(x_0, \delta) \cap (S \setminus \{x_0\}) \neq \emptyset$$

The set of accumulation points of S is denoted by $L(S)$

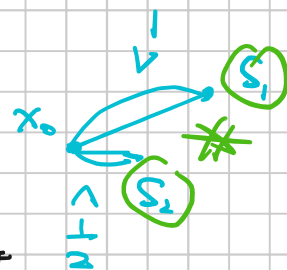
Prop

A point $x_0 \in M$ is an accumulation point of $S \iff \exists (s_n)_{n=1}^{\infty} \subseteq S$ s.t. $s_n \neq s_m$ $\forall n \neq m$

and $\lim_{n \rightarrow \infty} S_n = x_0$

pf

" $\Rightarrow$ " Assume $x_0 \in L(S)$.



For $\delta_1 = 1 > 0$, $\exists s_1 \in S$ s.t.

$$s_1 \in D(x_0, \delta_1) \cap (S - \{x_0\})$$

For $\delta_2 = \min\left\{\frac{1}{2}, \underline{d(s_1, x_0)}\right\} > 0$, $\exists s_2$ st

$$s_2 \in D(x_0, \delta_2) \cap (S - \{x_0\})$$

$$\Rightarrow d(s_2, x_0) < \frac{1}{2}$$

$$d(s_2, x_0) \neq d(s_1, x_0) \Rightarrow s_1 \neq s_2$$

$$\Rightarrow \underline{d(s_1, s_2)} \geq d(s_1, x_0) - d(s_2, x_0) > 0$$

$$\Rightarrow s_1 \neq s_2$$

For $\delta_3 = \min\left\{\frac{1}{3}, d(s_1, x_0), d(s_2, x_0)\right\} > 0$

$$\exists s_3 \in D(x_0, \delta_3) \cap (S - \{x_0\})$$

$$\Rightarrow d(x_0, s_3) < \frac{1}{3},$$

$$s_3 \neq s_1, \neq s_2$$

...

For $\delta_n = \min\left\{\frac{1}{n}, d(s_{n-1}, x_0), \dots, d(s_1, x_0)\right\} > 0$

$$\exists s_n \dots$$

$$\text{So } \exists (S_n)_{n=1}^{\infty} \subseteq S \quad \text{s.t.} \quad S_n \neq S_m \quad \forall n \neq m$$

$$\text{and} \quad d(S_n, x_0) < \frac{1}{n}$$

$$\Rightarrow \lim_{n \rightarrow \infty} S_n = x_0$$

$$\Leftarrow \text{Assume } \exists (S_n)_{n=1}^{\infty} \subseteq S, \quad S_n \neq S_m \quad \forall n \neq m$$

$$\lim_{n \rightarrow \infty} S_n = x_0$$

$$\text{Then } \forall \delta > 0, \exists N \quad \text{s.t.}$$

$$d(x_0, S_n) < \delta \quad \forall n \geq N$$

$$\Rightarrow S_n \in D(x_0, \delta) \cap S \quad \forall n \geq N$$

$$\Rightarrow D(x_0, \delta) \cap S \text{ is infinite}$$

$$\Rightarrow D(x_0, \delta) \cap (S - \{x_0\})$$

$$\text{is still infinite}$$

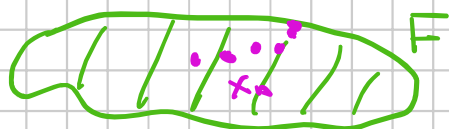
$$\Rightarrow D(x_0, \delta) \cap (S - \{x_0\}) \neq \emptyset$$

$$\Rightarrow x_0 \in L(S) \quad \#$$

Thm (Thm 2.4.2)

A set $F \subseteq (M, d)$ is closed

$$\Leftrightarrow L(F) \subseteq F \quad (\Leftrightarrow \underbrace{\forall (x_n)_{n=1}^{\infty} \subseteq F}_{\text{convergent}}, \underbrace{\lim_{n \rightarrow \infty} x_n}_{\in F})$$



pf

" $\Rightarrow$ " Assume F is closed.

Given $x_0 \in L(F)$, $\exists \underbrace{(S_n)_{n=1}^{\infty}} \subseteq F$ s.t.
 $S_n \neq S_m \ \forall n \neq m$ and $\lim_{n \rightarrow \infty} S_n = x_0$.

Claim: $x_0 \in F$

If not, $x_0 \in \underbrace{M-F}_{\text{open}} \Rightarrow \exists \delta > 0$ s.t.
 $D(x_0, \delta) \subseteq M-F$

Since $\lim_{n \rightarrow \infty} S_n = x_0$, $\exists N$ s.t. $M-F$

$d(S_n, x_0) < \delta \ \forall n \geq N \Rightarrow S_n \in \underbrace{D(x_0, \delta)}_{\cup} \ \forall n \geq N$

$\Rightarrow \underbrace{S_n \notin F}_{\text{orange}} \ \forall n \geq N \rightarrow \leftarrow$

So $x_0 \in F \Rightarrow L(F) \subseteq F$.

" $\Leftarrow$ " Assume $L(F) \subseteq F$.

Given any $x_0 \in M-F$, we have

$x_0 \notin L(F)$

Def $\Rightarrow \exists \delta > 0$ s.t.

$$D(x_0, \delta) \cap (F \setminus \{x_0\}) = \emptyset$$

$x_0 \in F$

$$\Rightarrow D(x_0, \delta) \cap F = \emptyset$$

$$\Rightarrow D(x_0, \delta) \subseteq M-F$$

$$\Rightarrow M-F \text{ is open} \Rightarrow F \text{ is closed} \quad \#$$

Example

① Any finite subset of (M, d) is closed

~~pf~~

If $F = \{f_1, \dots, f_m\}$, then

$$\mathcal{L}(F) = \emptyset \subseteq F$$

$\Rightarrow F$ is closed.

#

② Let $F = \{f_n \mid n \in \mathbb{N}\} \subseteq (M, d)$ satisfying

$$\underline{d(f_n, f_m) \geq 1} \quad \forall n \neq m$$

Then F is closed.

~~pf~~

If $(f_n)_{n=1}^{\infty}$ is convergent, then

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ s.t. } d(f_n, f_m) < \varepsilon \quad \forall n, m \geq N$$

$$\Rightarrow \mathcal{L}(F) = \emptyset \subseteq F$$

$\Rightarrow F$ is closed.

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Interior and closure

Recall that for $A \subseteq (M, d)$

$$\text{int}(A) = \{x \in A \mid \exists \delta > 0 \text{ s.t. } D(x, \delta) \subseteq A\}$$

Thm

The interior $\text{int}(A)$ is the largest open set contained in A . That is,

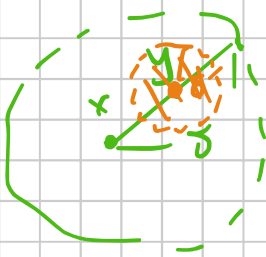
$$\text{int}(A) = \underbrace{\bigcup}_{B :=} \left\{ U \subseteq M \mid \begin{array}{l} U \subseteq A, \\ U \text{ is open in } M \end{array} \right\}$$

$\text{int}(A)$

Pf

To show $\text{int}(A) = B$, we verify
" $\text{int}(A) \subseteq B$ " and " $\text{int}(A) \supseteq B$ "

" $\subseteq$ " Since $\text{int}(A)$ is open, and $\text{int}(A) \subseteq A$,



$\forall x \in \text{int}(A), \exists \delta > 0$ s.t. $D(x, \delta) \subseteq A$

Note that $\forall y \in D(x, \delta)$,

$$\exists \delta' = \delta - d(y, x) > 0$$

s.t. $\underline{D(y, \delta')} \subseteq D(x, \delta) \subseteq A$

$$\begin{aligned} \stackrel{\Psi}{z} &\Rightarrow d(y, z) < \delta' = \delta - d(y, x) \\ &\Rightarrow d(z, x) \leq d(z, y) + d(y, x) \\ &\quad < \delta' + d(y, x) = \delta \\ &\Rightarrow z \in D(x, \delta) \end{aligned}$$

$$\Rightarrow y \in \text{int}(A) \Rightarrow D(x, \delta) \subseteq \text{int}(A)$$

$\Rightarrow \text{int}(A)$ is open.

we have $\text{int}(A) \subseteq B$.

" $\supseteq$ " $\forall x \in B, \exists U$ s.t.

$$(i) \quad x \in U$$

$$(ii) \quad U \subseteq A \quad (\Leftrightarrow A^c \subseteq U^c)$$

$$(iii) \quad U \text{ is open in } M$$

Recall

$$z \in \bigcup_{x \in I} A_x$$

$$\Rightarrow z \in A_x \text{ for some } x \in I$$

By (i) and (iii), $\exists \delta > 0$ s.t.,

$$D(x, \delta) \subseteq U$$

$$\text{By (ii), } D(x, \delta) \subseteq U \subseteq A$$

$$\Rightarrow x \in \text{int}(A)$$

$$\Rightarrow B \subseteq \text{int}(A)$$

Conclusion

Since $\text{int}(A) \subseteq B$ and $B \subseteq \text{int}(A)$,

we have

$$\text{int}(A) = B$$

#

Def (Def 2.5.1)

Let $F \subseteq (M, d)$. The closure of F , denoted $\overline{F}$ or $\text{cl}(F)$, is the smallest closed set containing F .

Remark

$$(i) \quad \text{cl}(F) = \bigcap \left\{ G \subseteq M \mid \begin{array}{l} G \supseteq F \\ G \text{ is closed in } M \end{array} \right\}$$

$$(ii) \quad \text{cl}(F) \text{ is closed in } (M, d).$$

$$(iii) \quad F \text{ is closed} \Leftrightarrow F = \text{cl}(F)$$

Thm (Prop 2.5.2)

$$\text{cl}(F) = F \cup L(F)$$

pf

" $\supseteq$ " Given any closed set $G \supseteq F$, we have

- $L(F) \subseteq L(G) \quad \because F \subseteq G$
- $L(G) \subseteq G \quad \because G \text{ is closed}$

$$\Rightarrow L(F) \subseteq G \quad \text{for all such } G$$

$$\Rightarrow L(F) \subseteq \bigcap G = \text{cl}(F)$$

Since we also have $F \subseteq \text{cl}(F)$,

$$F \cup L(F) \subseteq \text{cl}(F).$$

" $\subseteq$ " To show $\text{cl}(F) \subseteq F \cup L(F)$, it suffices to verify that $F \cup L(F)$ is closed, i.e.

$$L(F \cup L(F)) \subseteq F \cup L(F)$$

$\forall x_0 \in L(F \cup L(F))$, $\exists$ distinct seq.

$(g_n)_{n=1}^{\infty} \subseteq F \cup L(F)$ s.t. $\lim_{n \rightarrow \infty} g_n = x_0$
call $g_{n_j} \in F$

case 1: If infinitely many $g_n \in F$, then
 $x_0 = \lim_{j \rightarrow \infty} g_{n_j} \in L(F) \subseteq F \cup L(F)$

case 2: If only finite $g_n \in F$, then as many
 $g_n \in L(F) \Rightarrow \exists$ subseq $g_{n_j} \in L(F)$

Since $g_{n_j} \in L(F)$, for $\delta = \frac{1}{j} > 0$,
 $\exists f_j \in D(g_{n_j}, \frac{1}{j}) \cap F \neq \emptyset$

$$\Rightarrow d(f_j, x_0) \leq d(f_j, g_{n_j}) + d(g_{n_j}, x_0) \\ < \frac{1}{j} + d(g_{n_j}, x_0) \rightarrow 0 \text{ as } j \rightarrow \infty$$

$$\Rightarrow \lim_{j \rightarrow \infty} f_j = x_0 \in L(F) \subseteq F \cup L(F)$$

$$\text{So } L(F \cup L(F)) \subseteq F \cup L(F)$$

$$\Rightarrow F \cup L(F) \text{ is closed and } \supseteq \bar{F}$$

$$\Rightarrow F \cup L(F) \supseteq cl(F)$$

Conclusion

$$F \cup L(F) = cl(F) \quad \#$$

Example

In $(\mathbb{R}, d_e)$, if $F = \{\frac{1}{n} \mid n \in \mathbb{N}\}$, then
 $L(F) = \{0\}$

$$\Rightarrow cl(F) = F \cup L(F) = \{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{N}\}$$

Prop

Let $F \subseteq (M, d)$. The following are equivalent:

$$i) \quad x_0 \in cl(F)$$

$$(ii) D(x_0, \delta) \cap F \neq \emptyset \quad \forall \delta > 0$$

$$(iii) \exists x_n \in F, n \in \mathbb{N}, \text{ s.t. } \lim_{n \rightarrow \infty} x_n = x_0$$

pf

$(ii) \Leftrightarrow (iii)$: exercise.

$(i) \Rightarrow (ii)$: If $x_0 \in \text{cl}(F) = F \cup L(F)$, then

$$\begin{aligned} \text{case 1: } x_0 \in F &\Rightarrow x_0 \in D(x_0, \delta) \cap F \quad \forall \delta > 0 \\ &\Rightarrow D(x_0, \delta) \cap F \neq \emptyset \quad \forall \delta > 0 \end{aligned}$$

$$\text{case 2: } x_0 \in L(F)$$

$$\Rightarrow D(x_0, \delta) \cap (F \setminus \{x_0\}) \neq \emptyset$$

$$\Rightarrow D(x_0, \delta) \cap F \neq \emptyset \quad \forall \delta > 0.$$

$\Rightarrow (ii)$

$(ii) \Rightarrow (i)$: Assume $D(x_0, \delta) \cap F \neq \emptyset \quad \forall \delta > 0$.

$$\text{case 1: } x_0 \in F \Rightarrow x_0 \in \text{cl}(F) \Rightarrow (i)$$

$$\text{case 2: } x_0 \notin F$$

$$\Rightarrow D(x_0, \delta) \cap (F \setminus \{x_0\}) = D(x_0, \delta) \cap F \neq \emptyset \quad \forall \delta > 0$$

$$\Rightarrow x_0 \in L(F) \subseteq \text{cl}(F)$$

$$\Rightarrow x_0 \in \text{cl}(F) \Rightarrow (i)$$

#

Example

True or false:

$$\textcircled{1} \text{int}(\text{cl}(A)) = \text{int}(A).$$

$$\text{False: } A = \mathbb{Q} \Rightarrow \text{int}(A) = \emptyset, \text{cl}(A) = \mathbb{R} \\ \subseteq (\mathbb{R}, d_e) \text{int}(\text{cl}(A)) = \mathbb{R}$$

② $\text{cl}(A) \cap A = A$

True: $A \subseteq \text{cl}(A) \Rightarrow \text{cl}(A) \cap A = A.$

③ $\text{cl}(\text{int}(A)) = A$

False: $A = \mathbb{Q} \subseteq \mathbb{R} \Rightarrow \text{int}(A) = \emptyset, \text{cl}(\text{int}(A)) = \emptyset \neq \mathbb{Q} = A$

④ $\text{cl}(A \cap B) = \text{cl}(A) \cap \text{cl}(B)$

False: $A = (0, 1), B = (1, 2)$

$\Rightarrow \text{cl}(A) = [0, 1], \text{cl}(B) = [1, 2]$

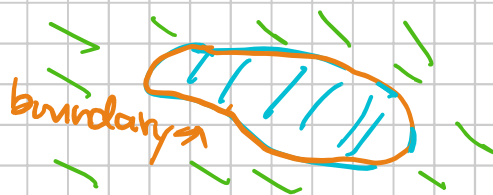
$\Rightarrow \text{cl}(A) \cap \text{cl}(B) = \{1\} \neq$

True $\text{cl}(A \cap B) = \text{cl}(\emptyset) = \emptyset$

⑤ $\text{int}(A \cap B) = \text{int}(A) \cap \text{int}(B)$

True: exercise.

Boundary



Def (Def 2.6.1)

Let $S \subseteq (M, d)$. The boundary of S is

$$\partial S (= \text{bd}(S)) = \text{cl}(S) \cap \text{cl}(M - S)$$

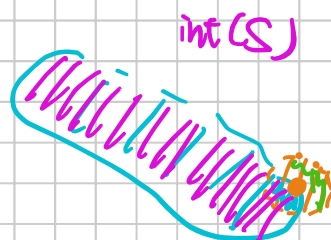
Example

① For $S = \mathbb{Q} \cap [0, 1] \subseteq \mathbb{R}$,

$$\partial S = \text{cl}(\mathbb{Q} \cap [0, 1]) \cap \text{cl}(\mathbb{R} - (\mathbb{Q} \cap [0, 1]))$$

$$= [0, 1] \cap \mathbb{R} = [0, 1]$$

$$\begin{aligned} \textcircled{2} \quad \partial([0, 1]) &= \text{cl}([0, 1]) \cap \text{cl}((-\infty, 0] \cup (1, \infty)) \\ &= [0, 1] \cap ((-\infty, 0] \cup [1, \infty)) \\ &= \{0, 1\} \end{aligned}$$



Remark

$$\text{iv) } x_0 \in \partial S \iff \forall \varepsilon > 0,$$

$$D(x_0, \varepsilon) \cap S \neq \emptyset$$

$$D(x_0, \varepsilon) \cap (M - S) \neq \emptyset$$

$$\text{iii) } \text{cl}(S) = \text{int}(S) \cup \partial S$$

pf of iii)

$$\text{"}\supseteq\text{"} \quad \text{int}(S) \subseteq S \subseteq \text{cl}(S)$$

$$\partial(S) = \text{cl}(S) \cap \text{cl}(M - S) \subseteq \text{cl}(S)$$

$$\Rightarrow \text{int}(S) \cup \partial S \subseteq \text{cl}(S)$$

$$\text{"}\subseteq\text{"} \quad \forall x \in \text{cl}(S).$$

$$\text{case 1: } x \in \text{int}(S) \Rightarrow x \in \text{RHS}$$

$$\text{case 2: } x \notin \text{int}(S) \Rightarrow x \in \text{cl}(S) \setminus \text{int}(S)$$

$$\bullet \quad x \in \text{cl}(S) \Rightarrow \forall \varepsilon > 0, D(x, \varepsilon) \cap S \neq \emptyset$$

$$\bullet \quad x \notin \text{int}(S) \Rightarrow \forall \varepsilon > 0 \quad D(x, \varepsilon) \not\subseteq S \Rightarrow$$

$$x \in \text{int}(S) \iff \exists \delta > 0 \text{ st. } D(x, \delta) \subseteq S \quad D(x, \varepsilon) \cap (M - S) \neq \emptyset$$

$$\Rightarrow x \in \partial S \Rightarrow x \in \text{RHS}$$

$$\text{So } \text{cl}(S) \subseteq \text{int}(S) \cup \partial S$$

#

Prop

$$\text{(i)} \quad \partial S = \text{cl}(S) \setminus \text{int}(S)$$

$$\text{(ii)} \quad \text{int}(S) = S \setminus \partial S$$

$$\text{(iii)} \quad \text{cl}(S) = S \cup \partial S = \text{int}(S) \cup \partial S$$

pf: exercise