Advanced Calculus, week 3, Fall 2026

Recall

① Let (x_n) be a seq. in a metric sp M.

A point $\alpha \in M$ is a cluster point of (x_n) accumulation pt

$\Leftrightarrow$ $\exists$ subseq $(x_{n_j})_{j=1}^{\infty}$ of $(x_n)_{n=1}^{\infty}$ s.t.

$$\lim_{j\to\infty} x_{n_j} = \alpha$$

② Let $S \subseteq \mathbb{R}$ (or $S \subseteq \mathbb{R} \cup \{-\infty\} \cup \{\infty\}$)

$$\sup S = \begin{cases} -\infty & \text{if } S = \emptyset \\ +\infty & \text{if } S \text{ is NOT bdd above} \\ \alpha & \text{if } S \neq \emptyset \text{, bdd above and } \alpha \text{ satisfies:} \end{cases}$$

(a) α is an upper bound

(b) $\forall \varepsilon > 0 \ \exists \ s_\varepsilon \in S$ s.t. $\alpha - \varepsilon < s_\varepsilon (\leq \alpha)$

③ Let $(x_n)_{n=1}^{\infty}$ be a seq. in $\mathbb{R}$.

$$\overline{\lim_{n\to\infty}} x_n = \inf_{n\geq 1} \sup\{x_k \mid k \geq n\}$$

$$\underline{\lim_{n\to\infty}} x_n = \sup_{n\geq 1} \inf\{x_k \mid k \geq n\}$$

Remark

If $\sup\{x_k \mid k \geq n\} = +\infty \ \forall n$

then $\inf_{n\geq 1} \sup\{x_k \mid k \geq n\} = +\infty$

" $-\infty <$ real numbers $< +\infty$ "

e.g. $x_k = k$

Remark

Let C be the set of cluster points of a bdd seq. $(x_n)_{n=1}^{\infty}$. Then

① $\overline{\lim_{n\to\infty}} x_n = \sup C$

$$\underline{\lim}_{n\to\infty} x_n = \inf C$$

② $\overline{\lim}_{n\to\infty} x_n = A$

$\Leftrightarrow$ $\forall \varepsilon > 0$, $\exists N_\varepsilon$ and $\exists x_{n_j}$ s.t.

(i) $x_n < A+\varepsilon \quad \forall n \geq N_\varepsilon$

(ii) $A-\varepsilon < x_{n_j} \quad \forall j \geq N_\varepsilon$

Example

Let $S \subseteq \mathbb{R}$ be a nonempty bdd set.

$\alpha = \sup S$. Assume $\alpha \notin S$.

Show that

① S contains ∞ elements

② $\exists\ s_n \in S$, $n = 1, 2, \cdots$, s.t.

$s_n \neq s_m \quad \forall n \neq m$

and $\lim_{n\to\infty} s_n = \alpha$

pf

① If $S = \{s_1, s_2, \cdots, s_N\}$. then

$\sup S = \max S \in S$ ($\rightarrow\leftarrow$)

$\Rightarrow$ S contains ∞ elements. #

② (i) $\alpha = \sup S \Rightarrow \alpha - 1$ is NOT an upper bound

$\Rightarrow \exists\ s_1 \in S$ s.t.

$$\alpha \underset{\neq}{\geq} s_1 > \alpha - 1$$

<u>NOTE</u>: since $\alpha \notin S$, $s_1 \in S \Rightarrow s_1 \neq \alpha$

(ii) Set $\varepsilon_2 = \min\{\frac{1}{2}, \underbrace{\alpha - s_1}_{>0}\} > 0$

$\alpha - \varepsilon_2$ is NOT an upper bound

$\Rightarrow \exists\ s_2 \in S$ s.t. $\quad \varepsilon_2 \leq \frac{1}{2}, \leq \alpha - s_1$

$$\alpha > s_2 > \alpha - \varepsilon_2 \geq \alpha - \frac{1}{2}$$

$$\geq \alpha - (\alpha - s_1)$$

$$= s_1$$

$$\Rightarrow \quad \alpha > s_2 > \alpha - \frac{1}{2}$$

$$\alpha > s_2 \underset{\neq}{\geq} s_1$$

$\vdots$

(n) Inductively, we get $s_n \in S$ s.t.

$(*)\quad \alpha > s_n > \alpha - \frac{1}{n}$

$(**)\quad \alpha > s_n > s_{n-1} > \cdots > s_2 > s_1$

$(*) \Rightarrow \lim_{n \to \infty} s_n = \alpha$ by 夾擠

$(**) \Rightarrow (s_n)_{n=1}^{\infty}$ is strictly increasing

$\Rightarrow s_n \neq s_m \quad \forall n \neq m$ #

§ Open sets and closed sets

Recall

$\forall$ open set $U \ni x_*$ $\exists N_U$ s.t. $x_n \in U$ $\forall n \geq N_U$ $\Longleftrightarrow$

$$\lim_{n\to\infty} x_n = x_* \quad \text{in} \quad (M, d)$$

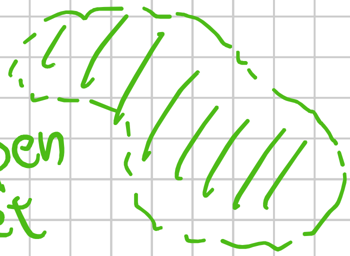
$\Leftrightarrow$ $\forall \varepsilon > 0$ $\exists N_\varepsilon$ s.t.

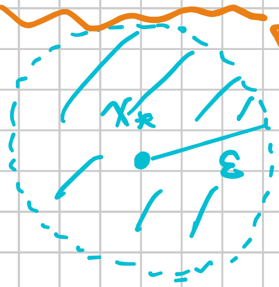
$$\boxed{d(x_n, x_*) < \varepsilon} \qquad \forall n \geq N_\varepsilon$$

$\Leftrightarrow$ $x_n \in D(x_*, \varepsilon) = \{y \in M \mid d(y, x_*) < \varepsilon\}$

(ε-) neighborhood of x_* 鄰域

open set





In general, "$D(x_*, \varepsilon)$" can be replaced by "open set". That is,

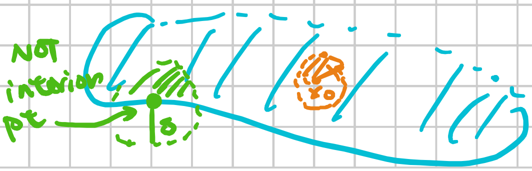
neighborhood of x_* can be understood as an "open set" U s.t. $x_* \in U$.

Def (Def 2.1.1 and 2.2.1)

Let (M, d) be a metric space.

Let $U \subseteq M$.

$x \in U \Rightarrow x \in M$ $\quad$ C



(i) A point $x_0 \in U$ is an interior point 內點 of U if $\exists\, \delta > 0$ s.t.

$$D(x_0, \delta) = \{y \in M \mid d(y, x_0) < \delta\} \subseteq U$$

(ii) The interior of U

$= \{$ all interior points of $U\}$

notation

$= \mathrm{int}(U)$ or $\mathring{U}$

(iii) U is an open set 開集合 in M if

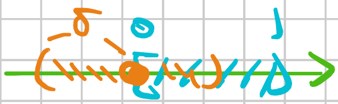
$$U = \mathrm{int}(U)$$

i.e. $\forall\, x \in U\ \exists\, \delta > 0$ s.t.

$$D(x, \delta) \subseteq U$$

e.g. • $U = (0,1)$ or (a,b) is open in $(\mathbb{R}, d_0)$ (open interval)

• $[0,1]$ and $[0,1)$ are NOT open in $\mathbb{R}$



$D(0, \delta) = (-\delta, \delta)$

Thm (Prop. 2.1.3)

The collection of open sets in a metric space (M, d) satisfies:

(i) ϕ and M are open

empty set — whole space

some index set. NOT necessity finite, NOT necessity countable

(ii) If U_α, $\alpha \in A$, are open, then

$$\bigcup_{\alpha\in A} U_\alpha \text{ is also open}$$

(iii) If $U_1, \cdots, U_n$ are open, then

$$\bigcap_{k=1}^{n} U_k \text{ is open}$$

pf

No point

(i) $\forall\, x_0 \in M$ (ϕ), take any positive $\delta > 0$

$$D(x_0, \delta) = \{ y \in M \mid d(y, x_0) < \delta \} \subseteq M.$$

$\Rightarrow$ M is open

(ii) $\forall\, x_0 \in \bigcup_{\alpha\in A} U_\alpha$ $\exists\, \alpha_0 \in A$ s.t. $x_0 \in U_{\alpha_0}$

Since U_{α_0} is open, $\exists\, \delta > 0$ s.t.

$$D(x_0, \delta) \subseteq U_{\alpha_0} \subseteq \bigcup_{\alpha\in A} U_\alpha$$

$\Rightarrow$ $\bigcup_{\alpha\in A} U_\alpha$ is open.

(iii) $\forall\, x_0 \in \bigcap_{k=1}^{n} U_k$, we have

U_k is open $x_0 \in U_k \quad \forall k = 1, \dots, n.$

$\Rightarrow$ $\delta_k > 0$ s.t.

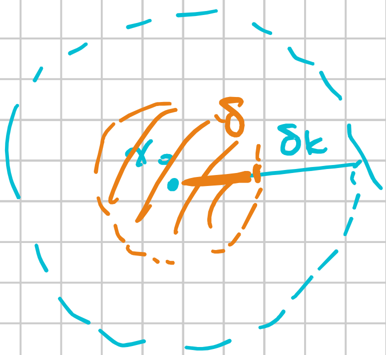
$$D(x_0, \delta_k) \subseteq U_k \qquad \forall k = 1, \dots, n$$

Take $\delta = \min\{\delta_1, \cdots, \delta_n\} > 0$

$\delta \leq \delta_k \quad \forall k = 1, \cdots, n$

$\Rightarrow \quad D(x_0, \delta) \subseteq D(x_0, \delta_k) \subseteq U_k$

$\forall k = 1, \cdots, n$



$\Rightarrow \quad D(x_0, \delta) \subseteq \bigcap_{k=1}^{n} U_k$

$\Rightarrow \quad \bigcap_{k=1}^{n} U_k$ is open #

<u>Example</u>

① Let $(M, d) = (\mathbb{R}, d_e)$. $a < b \in \mathbb{R}$

Then $(a, b) = \{x \in \mathbb{R} \mid a < x < b\}$ is open in $\mathbb{R}$

pf

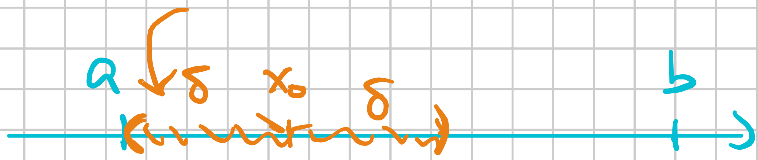
NOTE: $D(x_0, \delta) = \{y \in \mathbb{R} \mid \underbrace{\underbrace{d(y, x_0)}_{|y - x_0|} < \delta}_{x_0 - \delta < y < x_0 + \delta}\}$

$= (x_0 - \delta, x_0 + \delta)$

To show (a, b) is open, given $x_0 \in (a, b)$,

take $\delta = \min\{x_0 - a, b - x_0\} > 0$

$\Rightarrow \quad \underline{D(x_0, \delta)} = (x_0 - \delta, x_0 + \delta) \subseteq (a, b)$



$\Rightarrow \quad (a, b)$ is open #

② By ①, $(-\frac{1}{n}, \frac{1}{n})$ is open in $\mathbb{R}$ $\forall n\in\mathbb{N}$.

Show that

$$\bigcap_{n=1}^{\infty}\left(-\frac{1}{n}, \frac{1}{n}\right) = \{0\}$$

exercise

is NOT open

pf

For $x_0 = 0 \in \{0\}$, $\forall \delta > 0$,

$$D(x_0, \delta) = (0-\delta, 0+\delta) \not\subseteq \{0\}$$

$\in \frac{\delta}{2} \neq 0$ $\qquad$ $\not\ni \frac{\delta}{2}$

$\Rightarrow \{0\}$ is NOT open in $\mathbb{R}$ #

Def (Def 2.3.1)

metric sp.

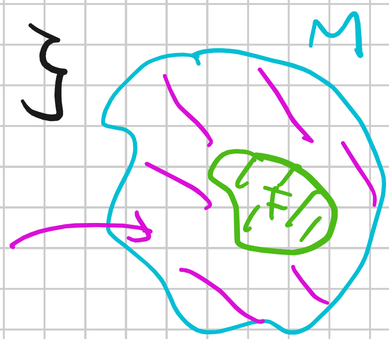
A subset $F \subseteq (M, d)$ is a closed set iff

閉集合

$$M \setminus F = \{y \in M \mid y \notin F\}$$

is open

$F^c = M - F$



Example (NOTE: $[a,b)$ is NOT closed, NOT open)

$[a,b]$ are closed in $\mathbb{R}$ $\forall a \le b$.

pf

That is, we want to show

$$\mathbb{R} \setminus [a,b] = (-\infty, a) \cup (b, +\infty)$$

is open.

Step 1: $(-\infty, a)$ is open

$\forall x_0 \in (-\infty, a)$, $\exists\ \delta = a - x_0 > 0$

$$D(x_0, \delta) = (x_0 - \delta, \underbrace{x_0 + \delta}_{=a}) \subseteq (-\infty, a)$$

$\Rightarrow (-\infty, a)$ is open

Step 2: (b, ∞) is open (similar to Step 1)

Step 3: By Thm, Step 1, Step 2,

$$(-\infty, a) \cup (b, \infty)$$

is open $\Rightarrow [a,b]$ is closed. #

Thm (Prop 2.3.2)

The collection of closed sets in (M, d) satisfies:

(i) ϕ and M are closed $\left(\begin{matrix} \phi^c = M \\ M^c = \phi \end{matrix} \right)$

(ii) If F_α, $\alpha \in A$, are closed, then

$\bigcap_{\alpha \in A} F_\alpha$ is closed $\left(\left(\bigcap_{\alpha \in A} F_\alpha \right)^c = \bigcup_{\alpha \in A} F_\alpha^c \right)$

(iii) If $F_1, \cdots, F_n$ are closed, then

$\bigcup_{k=1}^{n} F_k$ is closed $\left(\left(\bigcup_{k=1}^{n} F_k\right)^c = \bigcap_{k=1}^{n} F_k^c \right)$

__Remark__

F is closed $\Leftrightarrow F^c = M - F$ is open

$\Leftrightarrow \underline{\forall x_0 \in F^c}\ \exists \delta > 0$ s.t. $\underline{D(x_0, \delta) \subseteq F^c}$

$\Updownarrow$

$\forall x_0 \notin F$

$\Leftrightarrow D(x_0, \delta) \cap F = \emptyset$