Advanced Calculus, week 2, Fall 2026

NOT an equivalence relation

<u>Recall</u> e.g. $S=\mathbb{R}$, $\sim\, := \{(a,b)\in\mathbb{R}\times\mathbb{R} \mid a\leq b\} \subseteq \mathbb{R}\times\mathbb{R}$ $1\leq 2$ but $2\not\leq 1$

$\sim\, = \,\leq$

A relation $\sim$ on a set S is a subset of $S\times S$. Write $x\sim y$ if $(x,y)\in\,\sim$.

An equivalence relation is a relation satisfying: e.g. "$=$" is an equivalence relation

- $x\sim x \quad \forall x\in S$
- $x\sim y \Rightarrow y\sim x \quad \forall x,y\in S$
- $x\sim y,\ y\sim z \Rightarrow x\sim z \quad \forall x,y,z\in S$

Given an equivalence relation $\sim$ on S,

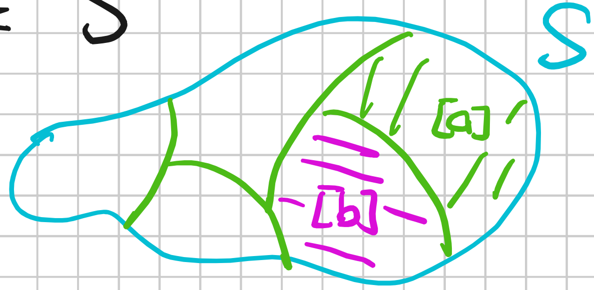
$$S/\!\sim \;:= \{[a] \mid a\in S\}$$

where

$$[a] = \{x\in S \mid x\sim a\} \subseteq S$$

NOTE: $S/\!\sim$ is a partion of S, i.e.

- $[a]\cap[b] = \emptyset$ if $a\nsim b$
- $\bigcup_{[a]\in S/\sim} [a] = S$



Prop
The equivalence of metrics on set M is an equivalence relation

$S = \{ \text{all metrics on } M \}$

Norm ver:

Let $\|\ \|$ and $\|\ \|'$ be two norms on a vector space V. They are <u>equivalent</u> if $\exists\ \alpha, \beta > 0$ s.t.

$$\alpha \|v\| \le \|v\|' \le \beta \|v\| \quad \forall v \in V$$

Notation: $\|\ \| \sim \|\ \|'$

Prop

- Equivalent norms induce equivalent metrics.
- Equivalence of norms is an equivalence relation.

pf: postponed

Thm

Any two norms on $\mathbb{R}^n$ are equivalent

($\Rightarrow \exists$ standard notion of limit on $\mathbb{R}^n$) which comes from Euclidean metric

Remark

The discrete metric d_0 on $\mathbb{R}^n$ is NOT equivalent to the Euclidean metric d_2.

If they're equivalent, then $\exists\ \alpha, \beta > 0$ s.t.

$$\alpha d_0(x,y) \le \underline{d_2(x,y)} \le \beta \underbrace{d_0(x,y)}_{\overset{\text{0, or 1}}{\shortparallel}} \quad \forall x,y \in \mathbb{R}^n$$

$d_2((n,0,\cdots,0),(0,\cdots,0)) \le \beta$ (→←) bdd = bounded

$= n$ is NOT bdd above

So $\nexists$ norm $\| \ \|$ on $\mathbb{R}^n$ s.t. $d_0(x,y) = \|x-y\|$

§ The real line

Def (§ Introduction)

Let S be a set.

(i) If $S = \phi$, then we say S is a finite set and has zero element.

countable one-to-one onto map

(ii) If $\exists$ $\overset{\shortparallel}{\text{bijection}}$ $f: \{1, 2, \cdots, n\} \to S$, then S is a finite set with n elements.

(iii) Suppose S is NOT finite. If $\exists$ bijection $f: \mathbb{N} = \{1, 2, 3, \cdots\} \longrightarrow S$, then

S is an infinitely countable set.

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Equivalently, S is infinitely countable
$\Leftrightarrow$ its elements can be arraged
in a sequence

$$f(1),\ f(2),\ f(3),\ \cdots$$

(iv) If S is NOT cases (i)–(iii), then
S is uncountable

Remark

the number of elements / cardinality of A

- If $\exists$ 1-1 map $f: A \to B$, then $\#(A) \le \#(B)$
- If $A \subseteq B$, then $\#(A) \le \#(B)$
- If $A \subseteq B$ and B is countable, then A is countable.

Prop (Prop 1.1.9)
The set of rational numbers $\mathbb{Q}$ is a countable set.

pf

Let $\mathbb{Q}_+ := \{ \frac{p}{q} \in \mathbb{Q} \mid \frac{p}{q} > 0 \}$

Step1: Prove $\mathbb{Q}_+$ is countable

Step 1-1 • $\forall x \in \mathbb{Q}_+$, $\exists\ p, q \in \mathbb{N}$ s.t. $x = \frac{p}{q}$

• If we require p and q are 互質 (relatively prime), then p and q are uniquely determined.

• So we have a 1-1 map

$$\mathbb{Q} \longrightarrow \mathbb{N} \times \mathbb{N}, \quad x = \frac{p}{q} \longmapsto (\underbrace{p, q}_{互質})$$

$\Rightarrow \#(\mathbb{N} \times \mathbb{N}) \geq \#(\mathbb{Q})$

It suffices to show: $\mathbb{N} \times \mathbb{N}$ is countable

Step 1-2: Prove $\mathbb{N} \times \mathbb{N}$ is countable. by arranging elements in $\mathbb{N} \times \mathbb{N}$ into a seq:

$$\begin{array}{cccc}
\overset{a_1}{\shortparallel}\ (1,1) & \longrightarrow\ \overset{a_2}{\shortparallel}\ (1,2) & a_4\ (1,3) & a_7\ \cdots \\
a_3 = (2,1) & (2,2)\ a_5 & (2,3) & \cdots \\
a_6\ (3,1) & (3,2) & (3,3) & \cdots \\
\vdots & \vdots & \vdots &
\end{array}$$

$\Rightarrow \mathbb{N} \times \mathbb{N}$ is countable $\Rightarrow \mathbb{Q}_+$ is countable

Step 2: Prove $\mathbb{Q}$ is countable

$\mathbb{Q} = \mathbb{Q}_+ \cup \{0\} \cup \mathbb{Q}_-$

$\mathbb{Q}_- := \{ -x^{\in \mathbb{Q}} \mid x \in \mathbb{Q}_+ \}$

By Step 1, we can express

$$Q_+ = \{ b_1, b_2, b_3, b_4, \cdots \}$$

$$\Rightarrow \quad 0$$

$$Q_- = \{ -b_1, -b_2, -b_3, \cdots \}$$

$$\Rightarrow \quad Q = \{ 0, b_1, -b_1, b_2, -b_2, b_3, -b_3, \cdots \}$$

$\Rightarrow$ Q is countable #

Lemma

e.g. $(0, 1, 0, 1, 0, 1, \cdots)$ $\quad (0, 1, 1, 1, 1, 0, 0, 1, \cdots)$

Let

$$S = \{ (a_n)_{n=1}^{\infty} \mid a_n = 0 \text{ or } 1 \}$$

Then S is uncountable.

pf

Assume S is countable. Then

$$S = \{ s_1, s_2, s_3, s_4, \cdots \}$$

where

$$s_n = (s_{n,k})_{k=1}^{\infty} = s_{n,1}, s_{n,2}, s_{n,3}, \cdots$$

and

$$s_{n,k} \in \{ 0, 1 \}$$

$s_1 = s_{1,1}, s_{1,2}, s_{1,3} \cdots$

$s_2 = s_{2,1}, s_{2,2}, s_{2,3} \cdots$

$s_3 = s_{3,1}, s_{3,2}, s_{3,3} \cdots$

$\vdots$

Define $t = (t_n)_{n=1}^{\infty}$ by

$$t_n := \begin{cases} 0 & \text{if } s_{n,n} = 1 \\ 1 & \text{if } s_{n,n} = 0 \end{cases}$$

Then $t \neq S_n \quad \forall n$

($t = S_n$ means $t_k = S_{n,k} \quad \forall k \in \mathbb{N}$

$\overset{k=n}{\Rightarrow} \quad t_n = S_{n,n} \quad \rightarrow\leftarrow$)

$t \notin \{S_1, S_2, S_3, \dots \}$

But $t \in S \qquad (\rightarrow\leftarrow)$

So S is uncountable #

Thm (Thm 1.2.18)

The set $\mathbb{R}$ of real numbers is uncountable.

pf.

Assume $\mathbb{R}$ is Countable

$\Rightarrow [0,1) \subseteq \mathbb{R}$ is also Countable

correction

$$\boxed{\begin{array}{l}\text{But we have a 1-1 map} \\ S \longrightarrow [0,1) \subseteq \mathbb{R} \\ (S_n)_{n=1}^{\infty} \longmapsto 0.S_1S_2S_3\cdots = \sum_{n=1}^{\infty} \frac{S_n}{10^n}\end{array}}$$

By Lemma, S is uncountable

$\Rightarrow [0,1)$ is also uncountable $(\rightarrow\leftarrow)$

So $\mathbb{R}$ is uncountable #

Limit sup/inf and cluster points

Def

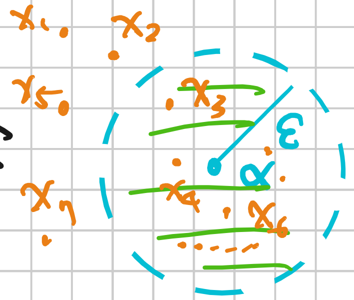
Let $(x_n)_{n=1}^{\infty}$ be a seq. in a metric sp (M,d)

A point $\alpha \in M$ is called a cluster point (or accumulation point) of (x_n) 聚點

if $\forall \varepsilon > 0$, the open disk

$$D(\alpha, \varepsilon) := \{ x \in M \mid d(x, \alpha) < \varepsilon \}$$

contains infinitely many terms of (x_n).



That is, α is a cluster pt.

$\Leftrightarrow$ $\forall \varepsilon > 0$, $\exists$ subseq. (x_{n_j}) of (x_n) s.t. $d(x_{n_j}, \alpha) < \varepsilon$

Remark

$\alpha \in M$ is a cluster pt of (x_n)

$\Leftrightarrow$ $\exists$ $(x_{n_j})_j$, a subseq. of $(x_n)_n$, s.t.

$$\lim_{j \to \infty} x_{n_j} = \alpha$$

pf

"$\Rightarrow$" Assume α is a cluster pt of (x_n)

① For $\varepsilon = \frac{1}{1} > 0$, $\exists$ $x_{n_1} \in D(\alpha, \frac{1}{1})$

② For $\varepsilon = \frac{1}{2} > 0$, since there are ∞ terms in $D(\alpha, \frac{1}{2})$, we can choose n_2 s.t.

- $n_2 > n_1$
- $x_{n_2} \in D(\alpha, \frac{1}{2})$

③ Inductively, for $\varepsilon = \frac{1}{j} > 0$, $\exists\ n_j$ s.t.

- $n_j > n_{j-1}$
- $x_{n_j} \in D(\alpha, \frac{1}{j})$ $\Leftrightarrow$ $d(x_{n_j}, \alpha) < \frac{1}{j}$

So we have a subseq

$(x_{n_j})_{j=1}^{\infty} = x_{n_1}, x_{n_2}, x_{n_3}, \cdots$

of (x_n) s.t.

$\lim_{j\to\infty} x_{n_j} = \alpha$

$\forall \varepsilon > 0$, $\exists N_\varepsilon = [\frac{1}{\varepsilon}] + 1$ s.t. $\frac{1}{j} < \varepsilon \quad \forall j \geq N_\varepsilon$

$\Rightarrow d(x_{n_j}, \alpha) < \frac{1}{j} < \varepsilon \quad \forall j \geq N_\varepsilon$

"$\Leftarrow$" Assume $\exists$ subseq (x_{n_j}) of (x_n) s.t.

$\lim_{j\to\infty} x_{n_j} = \alpha$

$x_{n_j} \in D(\alpha, \varepsilon)$

$\Updownarrow$

Then $\forall \varepsilon > 0$, $\exists N_\varepsilon$ s.t. $d(x_{n_j}, \alpha) < \varepsilon \quad \forall j \geq N_\varepsilon$

$\Rightarrow x_{n_j} \in D(\alpha, \varepsilon) \quad \forall j \geq N_\varepsilon$ $\quad \infty$ terms

$x_{n_{N_\varepsilon}}, x_{n_{N_\varepsilon+1}}, \cdots$

$\Rightarrow \alpha$ is a cluster pt #

Example

Define

$$x_n = \begin{cases} \frac{1}{n} \in \mathbb{R} & \text{if } n \text{ is odd} \\ n \in \mathbb{R} & \text{if } n \text{ is even} \end{cases}$$

Show that 0 is its unique cluster pts.

pf

Step1 0 is a cluster pt of (x_n)

Since the subseq $(x_{2j+1})_{j=1}^{\infty} = \left(\frac{1}{2j+1}\right)_{j=1}^{\infty}$

converges to 0, we conclude that

0 is a cluster pt. of (x_n) (by Remark)

Step2 $\nexists$ other cluster pts

Assume α is a cluster pt of (x_n).

Then $\exists$ subseq $(x_{n_j})_{j=1}^{\infty}$ s.t. $\lim_{j\to\infty} x_{n_j} = \alpha$.

Claim

n_j is odd for all but finite j.

($\Leftrightarrow$ $\exists N$ s.t. n_j is odd $\forall j \geq N$)

If not, $\exists$ $n_{j_1}, n_{j_2}, n_{j_3}, \cdots$ which are all even

$\Rightarrow$ $x_{n_{j_k}} = n_{j_k}$ which is unbdd

$\Rightarrow$ x_{n_j} is unbdd

$\Rightarrow$ $\lim_{j\to\infty} x_{n_j}$ does NOT exist,

contradiction to $\lim_{j\to\infty} x_{n_j} = a$

So $\exists N$ s.t. n_j is odd $\forall j \geq N$

This proves the claim.

$\leftarrow$ NOTE: $n_j \in \mathbb{N}$, $n_{j+1} > n_j$ $\forall j$

exercise

$n_j \geq j$

So $\quad x_{n_j} = \dfrac{1}{n_j} \qquad \forall j \geq N$

$\longrightarrow 0 \quad$ as $\quad j \to \infty$

$$\Rightarrow \quad \alpha = \lim_{j \to \infty} x_{n_j} = 0$$

So 0 is the only cluster pt. #

Recall the following thm from Calculus:

<u>Thm</u> (Bolzano–Weierstrass, Thm 1.4.3)

Every bounded seq. $(x_n)_{n=1}^{\infty}$ in $\mathbb{R}$ has a convergent subseq.

Equivalently, every bdd seq. has at least one cluster point.

<u>Def</u> (§1.3)

Let $S \subseteq \mathbb{R}$. We define the least upper bound (supremum) $\sup S = \operatorname{lub} S$ of S as follows:

(i) If $S = \emptyset$, then $\sup S = -\infty$

(ii) If $S \neq \emptyset$ and not bounded above (i.e. $\forall n \in \mathbb{N}\ \exists\, s_n \in S$ s.t. $s_n \geq n$)

then $\sup S = +\infty$

(iii) If $S \neq \phi$ and bounded above (i.e. $\exists M$ s.t. $s \leq M\ \forall s \in S$)

then $\sup S$ is the real number α satisfying:

(a) α is an upper bound of S

i.e. $s \leq \alpha \quad \forall s \in S$

(b) $\forall \varepsilon > 0$, $\alpha - \varepsilon$ is NOT an upper bound of S, i.e. $\exists\, s_\varepsilon \in S$ s.t. $\alpha - \varepsilon < s_\varepsilon \ (\leq \alpha)$

Similarly, one can define the greatest lower bound (infimum), $\inf S = \operatorname{glb} S$, of S

Remark

$-S := \{-s \mid s \in S\}$ " $\mathbb{R}$ "

For $\underset{\neq \phi}{S} \subseteq \mathbb{R}$, we have

(i) $\inf(S) = -\sup(-S)$

(ii) $\sup(S) = -\inf(-S)$

Def (Def 1.5.3)

Let $(x_n)_{n=1}^{\infty}$ be a real seq. Define

(i) $\limsup_{n\to\infty} x_n = \overline{\lim_{n\to\infty}} x_n$

$$:= \inf_{n\geq 1} \sup\{x_k \mid k\geq n\}$$

if the limit exists

$$\left(= \lim_{n\to\infty} \sup\{x_k \mid k\geq n\}\right)$$

(ii) $\liminf_{n\to\infty} x_n = \underline{\lim}_{n\to\infty} x_n$

$$:= \sup_{n\geq 1} \inf\{x_k \mid k\geq n\}$$

$$= \sup\left\{ \inf\{x_k \mid k\geq n\} \mid n\geq 1 \right\}$$

Example

Let $x_n = (-1)^n$. Then

① $\sup\{\underbrace{x_k}_{=(-1)^k} \mid k\geq n\} = 1 \quad \forall n\in\mathbb{N}$

$\Rightarrow \overline{\lim_{n\to\infty}} x_n = \inf_{n\geq 1} \sup\{x_k \mid k\geq n\} = \inf_{n\geq 1}\{1\} = 1$

② $\inf\{\underbrace{x_k}_{=(-1)^k} \mid k\geq n\} = -1 \quad \forall n\in\mathbb{N}$

$\Rightarrow \underline{\lim}_{n\to\infty} x_n = \sup_{n\geq 1} \inf\{x_k \mid k\geq n\} = \sup_{n\geq 1}\{-1\}$

$= -1$ #

Example

① $a_n = -n \Rightarrow \overline{\lim_{n\to\infty}} a_n = \inf_{n\geq 1} \underbrace{\sup\{\overset{-k}{a_k} \mid k\geq n\}}_{=-n}$

$-n, -n-1, -n-2, \cdots$

$= -\infty$

② Similarly, $b_n = n \Rightarrow \lim\limits_{n\to\infty} b_n = +\infty$.

<u>Thm</u>

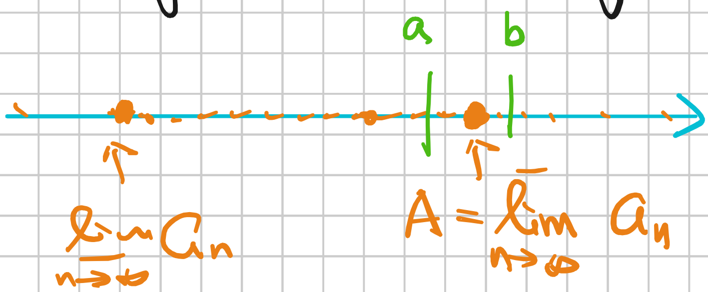
Let $(a_n)_{n=1}^{\infty}$ be a bdd seq. $A\in\mathbb{R}$.

Then $A = \overline{\lim\limits_{n\to\infty}}\, a_n \iff A$ satisfies

(i) $\forall\, b > A\ \exists\, N_b \in \mathbb{N}$ s.t. $a_n < b\ \ \forall n \geq N_b$

(ii) $\forall\, a < A\ \exists$ subseq $(a_{n_j})_{j=1}^{\infty}$ s.t.

$a_{n_j} > a \qquad \forall j \in \mathbb{N}$



<u>pf</u>

Let $M_n = \sup\{a_k \mid k \geq n\} \Rightarrow \overline{\lim\limits_{n\to\infty}}\, a_n = \inf\limits_{n\geq 1} M_n$

NOTE: $(M_n)_{n=1}^{\infty}$ is a decreasing seq s.t.

$$\lim_{n\to\infty} M_n = \inf_{n\geq 1} M_n = \overline{\lim_{n\to\infty}}\, a_n$$

"$\Rightarrow$" Suppose $A = \overline{\lim\limits_{n\to\infty}}\, a_n = \inf\limits_{n\geq 1} M_n$

Want to show (i) and (ii):

(i) $\forall\, b > A$. Since

最大下界 $\rightarrow \inf\limits_{n\geq 1} M_n < b \leftarrow$ 不再是下界

$\exists\, N_b \in \mathbb{N}$ s.t. $M_{N_b} < b$

$$\Rightarrow \quad a_k \le \underset{\sup\{a_k \mid k \ge N_b\}}{\underset{\shortparallel}{M_{N_b}}} < b \quad \forall k \ge N_b$$

(ii) $\forall\, a < A = \inf\limits_{n\ge 1} M_n \le M_n \quad \forall n \in \mathbb{N}$

$$\underset{\text{NOT upper bound}}{a} < \underset{\text{least upper bound}}{\sup\{a_k \mid k \ge n\}}$$

For $n=1$, $\exists\, n_1 \ge 1$ s.t. $a_{n_1} > a$

For $n = n_1+1$, $\exists\, n_2 \ge n_1+1$ s.t. $a_{n_2} > a$

$\vdots$

For $n=(n_j)+1$, $\exists\, n_{j+1} \ge (n_j)+1$ s.t. $a_{n_{j+1}} > a$

$\vdots$

Inductively, we get $(a_{n_j})_{j=1}^{\infty}$ s.t. $a_{n_j} > a \quad \forall j \quad \Rightarrow$ (ii)

"$\Leftarrow$" Suppose $A \in \mathbb{R}$ satisfies (i) and (ii).

We want to show $\overline{\lim\limits_{n\to\infty}}\, a_n = A$ by showing $\overline{\lim\limits_{n\to\infty}}\, a_n \le A$ and $\overline{\lim\limits_{n\to\infty}}\, a_n \ge A$

"$\le$" By (i), $\forall\, b > A$ $\exists\, N_b$ s.t.

an upper bound $\quad a_n < b \quad \forall n \ge N_b$

$$\Rightarrow \quad M_{N_b} = \sup\{a_k \mid k \ge N_b\} \le b$$

$$\Big(\underset{M_n \le b}{\overset{\vee\!/}{}} \quad \forall n \ge N_b \Big)$$

$$\Rightarrow \quad \overline{\lim_{n\to\infty}} a_n = \inf_{n\geq 1} M_n \leq M_{N_b} \leq b$$

$$\forall\, b > A$$

$$\Rightarrow \quad \overline{\lim_{n\to\infty}} a_n \leq A$$

If not, $\overline{\lim_{n\to\infty}} a_n > A$, take

$$b = \frac{\left(\overline{\lim_{n\to\infty}} a_n + A\right)}{2} > A$$

$$\overline{\lim_{n\to\infty}} a_n >$$

$\rightarrow \leftarrow$ to $\overline{\lim_{n\to\infty}} a_n \leq b$ $\forall\, b > A$

" $\geq$ " follows from (ii)

exercise.

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