Advanced Calculus, week 11, Fall 2026

Recall

① $f: M \to N$ is Continuous

$\Leftrightarrow$ $U \subseteq_{\text{open}} N \Rightarrow f^{-1}(U) \subseteq_{\text{open}} M$

$\Leftrightarrow$ $\forall x_0 \in M, \forall \varepsilon > 0$ $\exists \delta = \delta(x_0, \varepsilon) > 0$ s.t.

$d_M(f(x), f(x_0)) < \varepsilon$ when $d_N(x, x_0) < \delta$

Now assume $f: M \to N$ is continuous. Then

② M is connected $\Rightarrow$ $f(M)$ is connected

③ M is compact $\Rightarrow$ $f(M)$ is compact.

Example (Quiz 3. Problem 1)

Let (K, d) be a compact metric sp. and $f: K \to K$ satisfy

$\circledast$ $d(f(x), f(y)) = d(x, y)$ $\forall x, y \in K$.

Prove $f(K)$ is closed in K.

pf

Step 1 f is continuous:

$\forall x_0 \in K$ $\forall \varepsilon > 0$, $\exists \delta = \varepsilon > 0$ s.t.

when $d(x_0, x) < \delta = \varepsilon$, $d(f(x_0), f(x)) \overset{\circledast}{=} d(x_0, x) < \varepsilon$

Step2 Use ③ and properties of cpt sp..

K is cpt, f conti. ⇒ f(K) cpt.

(Recall: a cpt subset in a metric sp is closed)

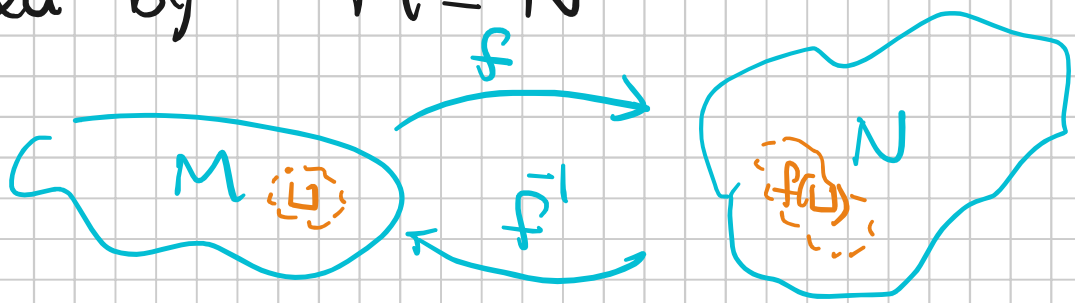
⇒ f(K) is closed in K. ∎

Topological invariants

Def

Let (M, d_M), (N, d_N) be metric sp.

A bijection (= 1-1 and onto map) $f: M \to N$ is called a homeomorphism if both f and $f^{-1}: N \to M$ are continuous. If such a bijection exists, then we say M and N are homeomorphic, denoted by $M \cong N$



∃ one-to-one correspondence between {open sets in M} and {open sets in N}

A property P of a space is called a topological property (or topogical

invariant) if it is preserved under homeomorphisms. That is, if $M \cong N$, then M has the property P $\Leftrightarrow$ N has the property P

Example

Assume $M \underset{f^{-1}}{\overset{f}{\rightleftarrows}} N$ are continuous.

① M is compact $\Leftrightarrow$ N is compact

③ ⇓ : $N = f(M)$ cpt

⇓ : $M = f^{-1}(N)$ cpt

So compactness is a topo. invariant.

② M is connected $\Leftrightarrow$ N is connected

② ⇓ : $N = f(M)$ cnt

⇓ : $M = f^{-1}(N)$ cnt

So connectedness is a topo invariant.

Remark

$(0,1) \cong \mathbb{R}$ (Hint: $\tan : (-\frac{\pi}{2}, \frac{\pi}{2}) \xrightarrow[\text{onto}]{(-1)} \mathbb{R}$)

$(0,1)$ is NOT closed in $\mathbb{R}$, but

$\mathbb{R}$ is closed in $\mathbb{R}$

So "closed in $\mathbb{R}$" is NOT a topo. invariant.

Uniform continuity

Def (Def 4.6.1)

Let $f: (M, d_M) \to (N, d_N)$, $S \subseteq M$.

We say f is **uniformly continuous** on S if $\forall \varepsilon > 0$ $\exists \delta = \delta(\varepsilon) > 0$ (indep. of x, y) s.t.

$$d_N(f(x), f(y)) < \varepsilon$$

when $x, y \in S$, $d_M(x, y) < \delta$

NOTE: f is uniformly continuous on S $\Rightarrow$ f is continuous on S

Thm (Thm 4.6.2)

If $K \subseteq (M, d_M)$ is cpt and $f: K \to N$ is continuous, then f is uniformly continuous on K.

pf

f is NOT conti. at $x_0 \Leftrightarrow$ $\exists \varepsilon_0 > 0$ $\forall \delta > 0$, $\exists y_\delta \in K$ s.t. $d(y_\delta, x_0) < \delta$, $d(f(y_\delta), f(x_0)) \geq \varepsilon_0$ (indep. of δ)

Suppose f is NOT unif. conti. on K.

Then $\exists \varepsilon_0 > 0$ s.t. $\forall \delta > 0$ $\exists x_\delta, y_\delta \in K$ (dep. on δ)

s.t. $d_M(x_\delta, y_\delta) < \delta$ but $d_N(f(x_\delta), f(y_\delta)) \geq \varepsilon_0$

take $\delta = \frac{1}{n}$ $\Rightarrow$ x_n, y_n with $d_M(x_n, y_n) < \frac{1}{n}$

$d_N(f(x_n), f(y_n)) \geq \varepsilon_0$ $\forall n \in \mathbb{N}$

Since K is cpt, $\exists$ subseq. $(x_{n_i})_i$ s.t.

$x_{n_i} \to p \in K$

$\because d_M(x_{n_i}, y_{n_i}) < \frac{1}{n_i} \to 0$

$\therefore y_{n_i} \to p \in K$

Since f is conti. at p,

$0 < \varepsilon_0 \le d_N(f(x_{n_i}), f(y_{n_i}))$

$\le d_N(f(x_{n_i}), f(p)) + d_N(f(p), f(y_{n_i}))$

$\to 0$

$(\to\leftarrow)$

So f is unif. conti. #

§ Seq. of functions

$\lim_{m\to\infty} f(x_m) = f(\lim_{m\to\infty} x_m)$

Q1: If $f_n(x)$ is continuous / differentiable / bounded $\forall n$,

is $\lim_{n\to\infty} f_n(x)$ continuous / differentiable / bounded ?

$\lim_{m\to\infty}(\lim_{n\to\infty} f_n(x_m)) = \lim_{n\to\infty} f_n(\lim_{m\to\infty} x_m)$

Q2: Do we have

$\left(\lim_{n\to\infty} f_n(x)\right)' \overset{?}{=} \lim_{n\to\infty} f_n'(x)$

$$\int_a^b \left(\lim_{n\to\infty} f_n(x)\right) dx \overset{?}{=} \lim_{n\to\infty} \int_a^b f_n(x)\,dx$$

A: NO. We need extra condition.

Pointwise Convergence and Uniform Convergence

Def (Def 5.1.1, 5.1.2)

Let $g_n : S \subseteq (M,d) \longrightarrow (N,d')$ be a sequence of functions.

(i) $(g_n)_{n=1}^{\infty}$ is **pointwise convergent on S** if $\forall x \in S$, $(g_n(x))_{n=1}^{\infty}$ converges in N, i.e. $\forall x \in S\ \forall \varepsilon > 0$, $\exists N_{\varepsilon,x} \in \mathbb{N}$ s.t. $\forall n \geq N_{\varepsilon,x}$,

$$d'(g_n(x), g(x)) < \varepsilon$$

dep. on x

$\exists g : S \to N$.

(ii) $(g_n)_{n=1}^{\infty}$ is **uniformly convergent on S** if $\forall \varepsilon > 0\ \exists N_\varepsilon \in \mathbb{N}$ s.t. indep. of x

$\forall n \geq N_\varepsilon, \forall x \in S$,

$$d'(g_n(x), g(x)) < \varepsilon,$$

for some function $g : S \to N$.

"$g_n \to g$ unif. on S"

g_n converges to g uniformly on S

Notations:

$$C(M) = C(M, \mathbb{R}) = C^0(M, \mathbb{R})$$
$$= \{ f : M \to \mathbb{R} \mid f \text{ is conti.} \}$$

Thm (Prop. 5.1.4)

If $f_n \in C(M, N)$ $\forall n$, and

$f_n \to f$ unif. on M,

then $f \in C(M, N)$.

pf

Let $x_0 \in M$. Given $\varepsilon > 0$ $\exists N_\varepsilon \in \mathbb{N}$ s.t.

$$d'(f_n(x), f(x)) < \varepsilon/3$$

$\forall x \in M$, $\forall n \geq N_\varepsilon$ (i.e. $= N_\varepsilon$)

Since $f_{N_\varepsilon} \in C(M, N)$, $\exists \delta > 0$ s.t.

$$d'(f_{N_\varepsilon}(x), f_{N_\varepsilon}(x_0)) < \varepsilon/3$$

$\forall x \in M$ with $d(x, x_0) < \delta$

So, when $x \in M$, $d(x, x_0) < \delta$,

$$d'(f(x), f(x_0)) \leq d'(f(x), f_{N_\varepsilon}(x)) + d'(f_{N_\varepsilon}(x), f_{N_\varepsilon}(x_0)) + d'(f_{N_\varepsilon}(x_0), f(x_0))$$

($< \varepsilon/3$, $< \varepsilon/3$, $< \varepsilon/3$)

$$< \varepsilon$$

So $f \in C(M,N)$. #

Example

① Consider

$$f_n(x) = x^n \quad \text{on} \quad [0,1]$$

Then $f_n \in C[0,1]$ $\forall n \in \mathbb{N}$, and

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} x^n = \begin{cases} 0, & x \in [0,1) \\ 1, & x = 1 \end{cases}$$

which is NOT continuous on $[0,1]$

So $(f_n)_{n=1}^{\infty}$ is pointwise convergent but NOT uniformly convergent

② Consider

$$g_n : [0, \tfrac{1}{2}] \to \mathbb{R}, \quad g_n(x) = x^n$$

Then $(g_n)_{n=1}^{\infty}$ is uniformly convergent.

because $\forall \varepsilon > 0$ $\exists N_\varepsilon \in \mathbb{N}$ s.t.

$(\frac{1}{2})^{N_\varepsilon} < \varepsilon$ $\overset{\text{e.g.}}{=}$ $\max\{\lceil -\log_2 \varepsilon \rceil + 100, 1\}$

$$\Rightarrow |g_n(x) - 0| = x^n \leq (\tfrac{1}{2})^n < \varepsilon$$

$\forall x \in [0, \frac{1}{2}]$ $\forall n \geq N_\varepsilon$ #

③ Consider

$$h_n(x) = n x (1-x)^n$$

on $[0,1]$

(a) $\lim_{n\to\infty} h_n(x) = 0 \qquad \forall x \in [0,1]$

"pointwise conv."

(In fact, $(h_n)_{n=1}^{\infty}$ is NOT unif. conv.)

(b) $\max_{x\in[0,1]} h_n(x) = h_n\left(\frac{1}{n+1}\right)$

Solve $h_n'(x) = 0$, $h_n''\left(\frac{1}{n+1}\right)$

(c) $\lim_{n\to\infty} \max_{x\in[0,1]} h_n(x) = \lim_{n\to\infty} n \left(\frac{n}{n+1}\right)^{n+1}$ (exponent $1-\frac{1}{n+1}$)

$$= e^{-1} \neq 0$$

$$\max_{x\in[0,1]} \left(\lim_{n\to\infty} h_n(x) \right)$$

Prop

Let $S \neq \phi \subseteq (M,d)$, $f_n : S \to \mathbb{R}$ converges uniformly on S to $f : S \to \mathbb{R}$

Then

(ii) If f_n is bdd above $\forall n$,
then f is bdd above, and

$$\lim_{n\to\infty}\left(\sup_{x\in S} f_n(x)\right) = \sup_{x\in S} f(x)$$

(iii) If f_n is bdd below $\forall n$,
then f is bdd below, and

$$\lim_{n\to\infty}\left(\inf_{x\in S} f_n(x)\right) = \inf_{x\in S} f(x)$$

pf

We first prove (ii):

Step1: f is bdd above

Since $f_n \to f$ unif. on S, for $\varepsilon = 1 > 0$

$\exists\, N_1 \in \mathbb{N}$ s.t. $\forall n \geq N_1$ $\forall x \in S$

$$|f_n(x) - f(x)| < 1$$

In particular, for $n = N_1$, we have

$$f_{N_1}(x) - 1 < f(x) < f_{N_1}(x) + 1 \qquad \forall x \in S$$

($f(x)$: ← bdd above; $f_{N_1}(x)+1$: ← bdd above)

$$\leq \sup_{x\in S} f_{N_1}(x) + 1 < \infty$$

($\sup_{x\in S} f_{N_1}(x) + 1$: $\tilde{M}$ indep. of x)

$\Rightarrow f$ is bdd above

Step2: $\lim$ commutes with $\sup$

Fix an arbitrary $\varepsilon > 0$.

Since $f_n \to f$ unif, $\exists\ N_\varepsilon \in \mathbb{N}$ s.t.

$\forall\ n \geq N_\varepsilon \quad \forall x \in S,$

$$f_n(x) - \varepsilon < f(x) < f_n(x) + \varepsilon$$

$$\Rightarrow \sup_{x\in S}\left(f_n(x)-\varepsilon\right) \leq \sup_{x\in S} f(x) \leq \sup_{x\in S}\left(f_n(x)+\varepsilon\right)$$

$$= \left(\sup_{x\in S} f_n(x)\right) - \varepsilon$$

$$\Rightarrow 0 \leq \left| \sup_{x\in S} f_n(x) - \sup_{x\in S} f(x) \right| \leq \varepsilon \quad \forall n \geq N_\varepsilon$$

$$\Rightarrow 0 \leq \lim_{n\to\infty} \left| \sup_{x\in S} f_n(x) - \sup_{x\in S} f(x) \right|$$

$$\leq \lim_{n\to\infty} \left| \sup_{x\in S} f_n(x) - \sup_{x\in S} f(x) \right| \leq \varepsilon$$

$\forall \varepsilon > 0$!!

$$\Rightarrow 0$$

$$\Rightarrow \lim_{n\to\infty} \left| \sup_{x\in S} f_n(x) - \sup_{x\in S} f(x) \right| = 0$$

$$\Rightarrow \lim_{n\to\infty} \left(\sup_{x\in S} f_n(x) \right) = \sup_{x\in S} f(x) \Rightarrow (i)$$

For (ii), applying (i) to $(-f_n)$:

Since $-f_n \to -f$ unif. , by (i)

$-f$ is bdd above $\Leftrightarrow$ f is bdd below

and

$$\lim_{n\to\infty}\left(\sup_{x\in S}(-f_n(x))\right) = \sup_{x\in S}(-f(x))$$

$$= -\inf_{x\in S} f_n(x) \qquad \to -\inf_{x\in S} f(x)$$

$\Rightarrow$ (ii) #

Criteria for unif. Conv.

Thm (Uniform Cauchy Criterion, Thm 5.2.1)

Let $f_n : S \subseteq (M, d_M) \longrightarrow (N, d_N)$, where (N, d_N) is complete. Then

$(f_n)_{n=1}^{\infty}$ converges uniformly on S

$\Leftrightarrow$ it is uniformly Cauchy, i.e.

$\forall \varepsilon > 0$, $\exists N_\varepsilon \in \mathbb{N}$ (indep. of x) s.t.

$$d_N(f_n(x), f_m(x)) < \varepsilon$$

$\forall n, m \geq N_\varepsilon$, $\forall x \in S$

pf

"$\Rightarrow$" Suppose $f_n \to f$ unif. on S. Then

$\forall \varepsilon > 0 \ \exists N_\varepsilon \in \mathbb{N}$ s.t.

$$d_N(f_n(x), f(x)) < \frac{\varepsilon}{2} \quad \forall n \geq N_\varepsilon, \forall x \in S$$

$\Rightarrow \forall n, m \geq N_\varepsilon$

$$d_N(f_n(x), f_m(x)) \leq d_N(f_n(x), f(x)) + d_N(f(x), f_m(x))$$

where $d_N(f_n(x), f(x)) < \frac{\varepsilon}{2}$ and $d_N(f(x), f_m(x)) < \frac{\varepsilon}{2}$

$$< \varepsilon$$

$\Rightarrow (f_n)$ is unif. Cauchy.

"$\Leftarrow$" Suppose (f_n) is unif. Cauchy,

$\Rightarrow (f_n(x))_{n=1}^{\infty}$ is Cauchy in (N, d_N) $\forall x \in S$

N is complete

$\Rightarrow$ its limit exists:

$$f(x) := \lim_{n \to \infty} f_n(x)$$

Claim: $f_n \to f$ unif. on S

Since (f_n) is unif. Cauchy,

$\forall \varepsilon > 0 \ \exists N_\varepsilon \in \mathbb{N}$, s.t.

$$d_N(f_n(x), f_m(x)) < \frac{\varepsilon}{2} \quad \forall n, m \geq N_\varepsilon, \ \forall x \in S$$

Fix $x \in S$, $n \geq N_\varepsilon$.

(arbitrary)

$$\lim_{m\to\infty} \left(d_N(f_n(x), f_m(x)) \right) \leq \frac{\varepsilon}{2}$$

$\|$ d_N is conti. (HW9)

$$d_N\left(f_n(x), \lim_{m\to\infty} f_m(x)\right)$$

$\|$

$$d_N(f_n(x), f(x)) \leq \frac{\varepsilon}{2} < \varepsilon$$

$\Rightarrow$ $d_N(f_n(x), f(x)) < \varepsilon \quad \forall n \geq N_\varepsilon \ \forall x \in S$

$\Rightarrow$ $f_n \to f$ unif. on S #

Thm (Weierstrass M-test, Thm 5.2.2)

Let $f_n : S \xrightarrow{\subseteq (M, d_M)} N \overset{= (N, \|\cdot\|)}{\longrightarrow}$ where N is a complete normed vector space. (e.g. $N = \mathbb{R}^n$)

(a.k.a. Banach space), i.e. $(N, \|\cdot\|)$ is a normed vector space s.t. the induced metric space (N, d_N) is complete.

($d_N(x,y) = \|x-y\|$)

If $\exists K \in \mathbb{N}$ and $\exists$ convergent series

$\sum_{n=1}^{\infty} M_n$, with $M_n \geq 0$, s.t.

$$\|f_n(x)\| \leq M_n \quad \forall n \geq K \quad \forall x \in S,$$

then $\sum_{n=1}^{\infty} f_n$ converges uniformly on S

i.e. $\left(\sum_{n=1}^{m} f_n\right)_{m=1}^{\infty}$

pf Let $S_m := \sum_{n=1}^{m} f_n$

Since $\sum_{n=1}^{\infty} M_n$ Conv. $\Rightarrow \left(\sum_{n=1}^{m} M_n\right)_{m=1}^{\infty}$ is Cauchy

, $\forall \varepsilon > 0 \; \exists N_\varepsilon \geq K$ s.t.

$$\sum_{i=m+1}^{n} M_i < \varepsilon \quad \forall \; n > m \geq N_\varepsilon$$

For $n > m \geq N_\varepsilon$,

$$\|S_n(x) - S_m(x)\| \leq \sum_{i=m+1}^{n} \|f_i(x)\| \leq M_i$$

$$\leq \sum_{i=m+1}^{n} M_i < \varepsilon$$

$\Rightarrow (S_n)_{n=1}^{\infty}$ is unif. Cauchy on S.

$\Rightarrow$ it converges unif. on S. #

Example

Show that $\sum_{n=1}^{\infty} \frac{(\sin nx)^2}{n^2}$ Conv. unif. on $\mathbb{R}$

pf $(N, \|\cdot\|) = (\mathbb{R}, |\cdot|)$

Since $\left|\frac{(\sin nx)^2}{n^2}\right| \le \frac{1}{n^2} \quad \forall x \in \mathbb{R}$

and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ Conv.

by Weierstrass M-test,

$\sum_{n=1}^{\infty} \frac{(\sin nx)^2}{n^2}$ conv. unif. on $\mathbb{R}$ #

§ Interchange of limits

Q: For operation T (or property), such $T = \sup, \inf, \lim, \text{conti.}, \frac{d}{dx}, \int dx, \dots$, do we have

$$\lim_{n\to\infty} T(f_n) = T\left(\lim_{n\to\infty} f_n\right) ?$$

A: In general, wrong for pointwise Conv. seq.

but "true in some sense" for unif. Conv. seq.

Interchange of infinite summations

Consider $T = \lim_{n\to\infty}$

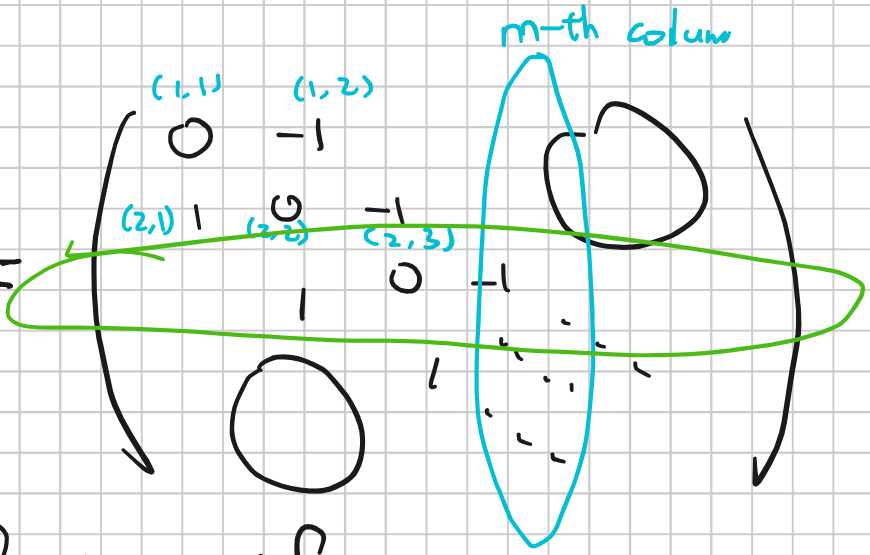
For symmetry, Consider double series

$$\sum_{n,m=1}^{\infty} a_{n,m}$$

Example

Define

$$(a_{n,m}) = \begin{pmatrix} 0 & -1 & & \\ 1 & 0 & -1 & \\ & 1 & 0 & -1 \\ & & 1 & \ddots \end{pmatrix}$$



i.e.

$$a_{n,m} = \begin{cases} -1 & \text{if } m = n+1 \\ 1 & \text{if } m = n-1 \\ 0 & \text{otherwise} \end{cases}$$

Q:

$$\sum_{m=1}^{\infty}\left(\sum_{n=1}^{\infty} a_{n,m}\right) \neq \sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty} a_{n,m}\right)$$

A: NO, because ...

$$\sum_{n=1}^{\infty} a_{n,m} = \begin{cases} 1, & \text{if } m = 1 \\ 0, & \text{if } m \geq 2 \end{cases}$$

$$\Rightarrow \sum_{m=1}^{\infty}\left(\sum_{n=1}^{\infty} a_{n,m}\right) = 1$$

$$\sum_{m=1}^{\infty} a_{n,m} = \begin{cases} -1 & \text{if } n=1 \\ 0 & \text{if } n \geq 2 \end{cases}$$

$$\Rightarrow \sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty} a_{n,m}\right) = -1 \neq 1 = \sum_{m=1}^{\infty}\left(\sum_{n=1}^{\infty} a_{n,m}\right) \quad \#$$

Q: Under what condition, we have

$$\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty} a_{n,m}\right) = \sum_{m=1}^{\infty}\left(\sum_{n=1}^{\infty} a_{n,m}\right) ?$$

Thm

Suppose

$$\sum_{m=1}^{\infty}\left(\sum_{n=1}^{\infty} |a_{n,m}|\right) < \infty$$

Then

$$\sum_{m=1}^{\infty}\left(\sum_{n=1}^{\infty} a_{n,m}\right) = \sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty} a_{n,m}\right)$$

Remark: This can be proved by

a method of uniform convergence.

---> next week.