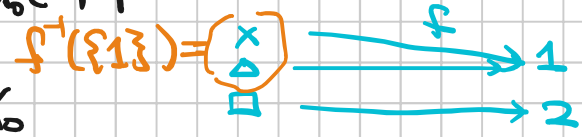
Advanced Calculus, week 10, Fall 2026

Recall

For $f: A \to B$, $T \subseteq B$,

$f^{-1}(T) := \{x \in A \mid f(x) \in T\}$



Let $f: (M,d) \to (N,d')$, $x_0 \in M$

- f is continuous at x_0
  $\Leftrightarrow \forall \varepsilon > 0 \ \exists \delta > 0$ s.t. $d(x,x_0) < \delta \Rightarrow d'(f(x), f(x_0)) < \varepsilon$
  $\Leftrightarrow \forall (x_n)_{n=1}^{\infty} \subseteq M$ with $\lim_{n\to\infty} x_n = x_0$, we have $\lim_{n\to\infty} f(x_n) = f(x_0)$
- f is continuous (on M)
  $\Leftrightarrow f$ is continuous at x_0 $\forall x_0 \in M$
  $\Leftrightarrow U \subseteq_{\text{open}} N \Rightarrow f^{-1}(U) \subseteq_{\text{open}} M$ ($f^{-1}(U) = \{x \in M \mid f(x) \in U\}$)
  $\Leftrightarrow F \subseteq_{\text{closed}} N \Rightarrow f^{-1}(F) \subseteq_{\text{closed}} M$
- For $S \subseteq M$, f is continuous on S
  $\Leftrightarrow f|_S : (S, d|_{S\times S}) \to (N, d')$ is continuous
- $M \xrightarrow{f} N \xrightarrow{g} Z$, $g \circ f$: f, g conti. $\Rightarrow g \circ f$ conti.

Thm

Let $f: (M,d) \to \mathbb{R}^n$ be written as

$f(x) = (f_1(x), f_2(x), \cdots, f_n(x))$

f is continuous at $x_0 \in M$
$\Leftrightarrow f_1, \cdots, f_n$ are continuous at x_0

Thm f is continuous (on M) (notation: $f \in C(M, \mathbb{R}^n)$)
$\Leftrightarrow f_1, \cdots, f_n \in C(M, \mathbb{R})$

pf: exer.

Hint: $|f_j(x) - f_j(x_0)| \leq \|f(x) - f(x_0)\|_2 \leq \sum_{j=1}^{n} |f_j(x) - f_j(x_0)|$

Example of conti. fn

① $\pi_j : \mathbb{R}^n \to \mathbb{R}$, $\pi_j(x_1, \cdots, x_n) = x_j$
is continuous. ("the j-th projection")

pf: $\forall \vec{x}_0 \in \mathbb{R}^n$
$\forall \varepsilon > 0$, $\exists\ \delta = \varepsilon > 0$ s.t.

$\|\vec{x} - \vec{x}_0\| < \delta = \varepsilon \Rightarrow |\pi_j(\vec{x}) - \pi_j(\vec{x}_0)|$

$\|\vec{x} - \vec{x}_0\| = \sqrt{\sum_{j=1}^{n} (x_j - x_{0j})^2} \geq |x_j - x_{0j}| = |\pi_j(\vec{x}) - \pi_j(\vec{x}_0)|$

$\leq \|\vec{x} - \vec{x}_0\| < \delta = \varepsilon$ #

② Constant functions are continuous.

③ $+ : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $(x, y) \mapsto x + y$
is continuous.

pf $\forall (x_0, y_0) \in \mathbb{R} \times \mathbb{R}$
$\forall \varepsilon > 0$, $\exists\ \delta = \frac{\varepsilon}{\sqrt{2}} > 0$ s.t.

$\|(x, y) - (x_0, y_0)\| < \delta = \frac{\varepsilon}{\sqrt{2}}$

$\Rightarrow |(x+y) - (x_0+y_0)| \leq |x - x_0| + |y - y_0|$

$\leq \sqrt{2} \cdot \|(x, y) - (x_0, y_0)\| < \sqrt{2} \cdot \frac{\varepsilon}{\sqrt{2}} = \varepsilon$ #

Cauchy-ineq: $|x - x_0| \cdot 1 + |y - y_0| \cdot 1 \leq \sqrt{|x - x_0|^2 + |y - y_0|^2} \cdot \sqrt{1^2 + 1^2}$

$\frac{|x - x_0| + |y - y_0|}{\sqrt{2}} \leq \|(x, y) - (x_0, y_0)\| = \sqrt{(x - x_0)^2 + (y - y_0)^2}$

④ $\cdot : \mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{R}$, $(x,y) \mapsto x\cdot y$ is continuous

pf

Recall: $(x_n, y_n) \to (x_0, y_0)$ in $\mathbb{R}^2$

(exer.) $\Leftrightarrow \lim_{n\to\infty} x_n = x_0$, $\lim_{n\to\infty} y_n = y_0$

$\forall (x_n, y_n) \in \mathbb{R}^2 \to (x_0, y_0)$, we have

- $(x_n, y_n) = x_n \cdot y_n \longrightarrow x_0 y_0$ (by Calculus)

$\Rightarrow$ "$\cdot$" is continuous #

⑤ $\mathbb{R}^n \to \mathbb{R}$, $\vec{x} = (x_1, \cdots, x_n) \mapsto c + \sum_{i=1}^{n} \underbrace{r_i x_i}_{g_i}$ is continuous, where $c, r_i \in \mathbb{R}$.

pf

Step 1:

$(\vec{x} \xrightarrow{g_i} r_i x_i) = \mathbb{R}^n \xrightarrow{f} \mathbb{R}^2 \xrightarrow{\cdot} \mathbb{R}$ (conti. by ④)

$\vec{x} \mapsto (r_i, x_i)$

$\vec{x} \mapsto (f_1(\vec{x}), f_2(\vec{x}))$

$f_1(\vec{x}) = r_i$ (Constant) is continuous

$f_2(\vec{x}) = x_i = \pi_i(\vec{x})$ is continuous (①)

Thm $\Rightarrow f$ is continuous

$\Rightarrow g_i = \cdot \circ f$ is continuous.

Remark

More generally, if $f, g : M \to \mathbb{R}$ are

Continuous, then $f.g : M \to \mathbb{R}$, $x \mapsto f(x)g(x)$ is also continuous.

$f+g$ $\quad$ $f(x)+g(x)$

pf

$$f.g = M \xrightarrow{x \mapsto (f(x), g(x))} \mathbb{R}\times\mathbb{R} \xrightarrow{\cdot} \mathbb{R}$$

$x \mapsto (f(x), g(x))$: Conti. by Thm

$\cdot$: Conti. by ④

$+$: Conti. by ③

$f(x)+g(x)$

is continuous. #

Step2 Use Step1 and Remark:

$(\vec{x} \mapsto r_i x_i)$ are continuous

Remark $\Rightarrow \sum_{i=1}^{n} r_i x_i$ is continuous

$(\vec{x} \mapsto c)$ is continuous

$\Rightarrow$ $c + \sum_{i=1}^{n} r_i x_i$ is continuous #

⑥ Every real polynomial $\mathbb{R}^n \to \mathbb{R}$ (of n variables) is the multiplication and addition of π_j and constant functions,

e.g. $2 x_1 x_2 x_3^2 + x_1^2 \cdot x_2^4 - x_1 + 5$

x_1: π_1, x_3: π_3, 5: constant

So it every (real) polynomial is continuous.

($\mathbb{R}^{n^2}$)

⑦ $\det : M_n(\mathbb{R}) \to \mathbb{R}$ is a polynomial

$A = (a_{ij})$

$\det(A) = \sum_{\sigma \in S_n} \operatorname{sign}(\sigma)\, a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}$

$d(A,B) = \sqrt{\sum_{i,j} (a_{ij} - b_{ij})^2}$

$\Rightarrow$ it is continuous

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

$$\det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$



$$GL_n(\mathbb{R}) = \{ A \in M_n(\mathbb{R}) \mid \det(A) \neq 0 \}$$

$$= \det^{-1}\underbrace{(\mathbb{R} - \{0\})}_{\cap \text{ open}} \subseteq_{\text{open}} M_n(\mathbb{R})$$

$$\mathbb{R}$$

□

(8) $$\{ \underbrace{x^2 + 2xy + 3y^2}_{f(x,y)} < 1 \}$$ $(x,y) \in \mathbb{R}^2$

$f: \mathbb{R}^2 \to \mathbb{R}$ is continuous

$$= f^{-1}\underbrace{(-\infty, 1)}_{\cap \text{ open}} \subseteq_{\text{open}} \mathbb{R}^2$$

$$\mathbb{R}$$

$$\{ (x,y) \in \mathbb{R}^2 \mid \underbrace{xy}_{g(x,y)} < 1 \}$$

$g: \mathbb{R}^2 \to \mathbb{R}$ is continuous

$$= g^{-1}\underbrace{(-\infty, 1)}_{\cap \text{ open}} \subseteq_{\text{open}} \mathbb{R}^2$$

$$\mathbb{R}$$

※

Applications to Connected sets

Thm

Let $C \subseteq (M, d)$. The following are equivalent

(i) C is connected

(ii') $\emptyset$ and C are the only open closed set in C (relatively)

(iii') $\nexists$ continuous map $f: C \to \mathbb{R}$ s.t. $f(C) = \{0, 1\}$

pf

(i) $\Leftrightarrow$ (ii') : Last week

(ii') $\Rightarrow$ (iii) : Assume $\exists$ such f

Since $\{0\} \subset \{0, 1\}$ (open closed) and f is conti.

we have

$f^{-1}(\{0\}) \subset C$ (open closed.)

$\{f(x) \mid x \in C\} = f(C) = \{0, 1\}$ $\neq$

$\Rightarrow \exists x \in C$ s.t. $f(x) = 0$ $\neq \emptyset$

$\Rightarrow x \in f^{-1}(\{0\})$

$\neq C$

$\exists y \in C$ s.t. $f(y) = 1 \neq 0$

$y \notin f^{-1}(\{0\})$

$\Rightarrow C$ is NOT connected ($\rightarrow\leftarrow$)

(iii) holds.

(iii) $\Rightarrow$ (ii') : Assume $\exists$ $G \subset C$ (open closed), $\neq C$, $\neq \emptyset$

((ii') is wrong)

Define

($\Rightarrow f(C) = \{0, 1\}$)

$f: C \to \mathbb{R}$, $f(x) = \begin{cases} 0 & x \in G \\ 1 & x \notin G \end{cases}$

Claim: f is continuous

$\forall\, U \underset{\text{open}}{\subseteq} \mathbb{R}$, $\quad f^{-1}(U) = \{x \in C \mid f(x) \in U\}$

4 possibilities

$$= \begin{cases} \phi & \underset{\text{open}}{\subseteq} C \quad 0,1 \notin U \\ G & \underset{\text{open}}{\subseteq} C \quad 0 \in U,\ 1 \notin U \\ C \setminus G & \underset{\text{open}}{\subseteq} C \quad 1 \in U,\ 0 \notin U \\ C & \underset{\text{open}}{\subseteq} C \quad 0,1 \in U \end{cases}$$

$\Rightarrow f$ is conti. ($\rightarrow\!\leftarrow$ to iii)

So (ii) is true $\quad \#$

Cor

Let $C \subseteq (M,d)$ be connected.

If $C \subseteq A \subseteq cl(C)$, then

A is cnt.

pf

If A is NOT cnt, then $\exists$ conti.

$f: A \to \mathbb{R}$, $\quad f(A) = \{0,1\}$

NOTE: $f|_C : C \to \mathbb{R}$ is continuous

Thm

$\Rightarrow f(C) \neq \{0,1\}$

Assume $f(C) = \{0\}$ ($f(C) = \{1\}$ is similar)

Since $A \subseteq cl(C)$, $\forall x \in A\ \exists x_n \in C$ s.t.

$\lim_{n\to\infty} x_n = x \Rightarrow f(x) = \lim_{n\to\infty} \underbrace{f(x_n)}_{=0} = 0$

$\Rightarrow f(A) = \{0,1\}$ $\rightarrow\leftarrow$

So A is cnt. #

Recall

- $f: M \to N$ is continuous $\Leftrightarrow \forall U \underset{\text{open}}{\subseteq} N,\ f^{-1}(U) \underset{\text{open}}{\subseteq} M$
- The following are equivalent:
  - C is connected
  - ϕ and C are the only open closed sets in C
  - $\nexists$ continuous map $f: C \to \mathbb{R}$ s.t. $f(C) = \{0,1\}$

Thm (Thm 4.2.1)

Assume $f: M \to N$ is continuous, and $C \subseteq M$ is connected. Then

$f(C)$ is Connected.

$= \{f(x) \in N \mid x \in C\}$

pf

If $f(C)$ is NOT connected, then $\exists$ continuous $g: f(C) \longrightarrow \mathbb{R}$ s.t. $g(f(C)) = \{0,1\}$.

Then $f \circ g : C \to \mathbb{R}$ is conti. s.t.

$$(f \circ g)(C) = g(f(C)) = \{0, 1\}$$

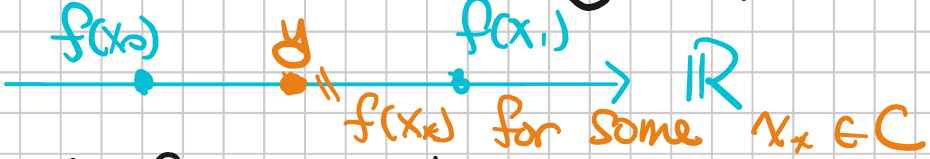
$\Rightarrow$ C is NOT connected $(\rightarrow\!\leftarrow)$

So $f(C)$ is connected. #

<u>Thm</u> (Intermediate Value Thm, Thm 4.5.1)

Let $C \subseteq (M, d)$ be connected, and $f : C \to \mathbb{R}$ is continuous.

If $x_0, x_1 \in C$, and $y \in \mathbb{R}$ is between



$f(x_0)$ and $f(x_1)$, then $\exists\, x_* \in C$ s.t. $f(x_*) = y$

<u>pf</u>

By Thm, $f(C) \subseteq \mathbb{R}$ is connected. ($\neq \phi$)

$\Rightarrow$ $f(C)$ is a point or an interval.

$\Rightarrow$ $\forall\, y_0 \leq y_1 \in f(C)$, $y_0 \leq y \leq y_1 \Rightarrow y \in f(C)$

Since $f(x_0), f(x_1) \in f(C)$, any y between $f(x_0)$ and $f(x_1)$ is in $f(C)$, i.e.

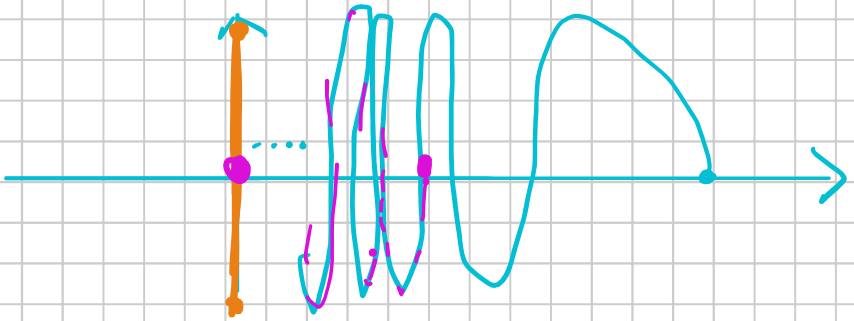
$\exists\, x_* \in C$ s.t. $f(x_*) = y$. #

Recall:

path-connected $\Rightarrow$ connected.

Example (Topologist's Sine Curve)

Let $\Gamma = \left\{ (x, \sin\frac{1}{x}) \in \mathbb{R}^2 \mid 0 < x \leq \frac{1}{\pi} \right\}$



$\Rightarrow$ Γ is path-connected (HW)

$\Rightarrow$ Γ is connected

$\Rightarrow$ $cl(\Gamma)$ is connected (By Cor. on Tue)

but NOT path-connected (HW)

↑ can be shown by IVT.

Q: Under what condition, we have

Connected $\Rightarrow$ path-connected.

Thm

If $U \subseteq \mathbb{R}^n$ is open and connected,

then U is path-connected.

pf

case1: $U = \emptyset \Rightarrow$ it is path-connected

case2: $U \neq \emptyset$. Fix $x_0 \in U$.

Consider

$$P = \{ x \in U \mid \exists \text{ conti. path } \gamma : [0,1] \to U \text{ s.t. } \gamma(0) = x_0,\ \gamma(1) = x \}$$

Since $x_0 \in P$, we have $P \neq \emptyset$.

Claim:

P is both open and closed in U.

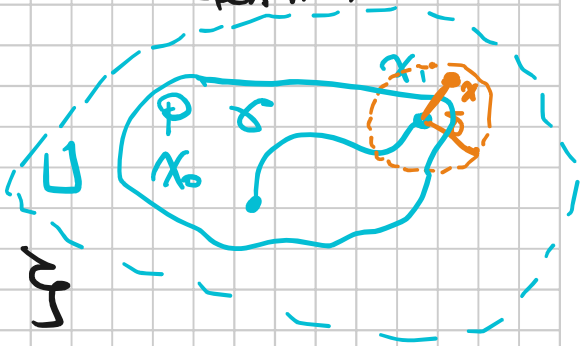
Step1: $P \subseteq_{\text{open}} U$ $\Longrightarrow$ $\exists$ conti. $\gamma : [0,1] \to U$ s.t. $\gamma(0) = x_0$, $\gamma(1) = x_1$

Let $x_1 \in P \subseteq U$. Since $U \subseteq_{\text{open}} \mathbb{R}^n$, $\exists \delta > 0$ s.t.

$$D_{\mathbb{R}^n}(x_1, \delta) \subseteq U$$

$$\| \qquad \|$$

$$\{ x \in \mathbb{R}^n \mid d(x_1, x) < \delta \}$$



Then for any $x \in D(x_1, \delta)$, define

$$\gamma_x : [0,1] \longrightarrow \mathbb{R}^n$$

$$\gamma_x(t) = \begin{cases} \gamma(2t) & 0 \le t \le \frac{1}{2} \\ x_1 + (2t-1)(x - x_1) & \frac{1}{2} \le t \le 1 \end{cases}$$

$\beta(t)$!!

- γ_x is continuous because γ is continuous, β is continuous

each component is a poly

and $\gamma(2\cdot\frac{1}{2}) = \beta(\frac{1}{2}) = x_1 + \underbrace{(2\cdot\cancel{\tfrac{1}{2}}-1)}_{=0}(x-x_1) = x_1$

$\gamma(2\cdot\frac{1}{2}) = \gamma(1) = x_1$

- $\forall\, t\in[0,1]$, $\gamma_x(t) \in U$, because

$0 \le t \le \frac{1}{2} \Rightarrow \gamma_x(t) = \gamma(2t) \in U$

$\frac{1}{2} \le t \le 1 \Rightarrow \gamma_x(t) = \beta(t) \in D(x_1,\delta)$

$d(\beta(t), x_1) = \|\beta(t) - x_1\|$

$= \|(2t-1)(x-x_1)\|$

$= \underbrace{|2t-1|}_{\le 1,\ 0\le} \cdot \underbrace{\|x-x_1\|}_{<\delta}$

$\le \|x-x_1\| < \delta$

So $\gamma_x(1) = \beta(1) = x \in P$

$\Rightarrow D(x_1,\delta) \subseteq P \Rightarrow P$ is open

Step 2: $P \underset{\text{closed}}{\subseteq} U \Leftrightarrow \exists\, F \underset{\text{closed}}{\subseteq} \mathbb{R}^n$ s.t. $P = F\cap U$

$\Leftrightarrow U\setminus P = U\setminus(F\cap U) = U\setminus F \underset{\text{open}}{\subseteq} \mathbb{R}^n$

Given $x_1 \in U\setminus P$, $\exists\, \delta>0$ s.t.

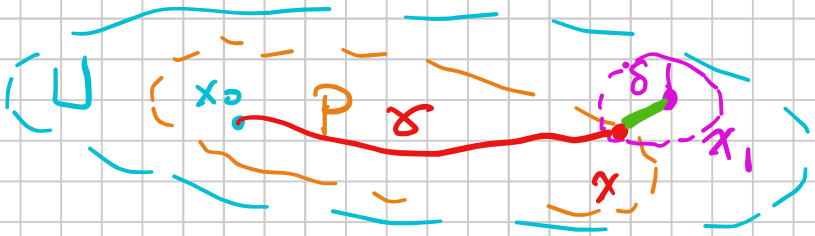
$D(x_1,\delta) \subseteq U$ $\quad (\because U \underset{\text{open}}{\subseteq} \mathbb{R}^n)$

Note that if

$D(x_1,\delta)\cap P \neq \phi$,

then $\exists\, x \in D(x_1,\delta)\cap P$.

$\Rightarrow \exists\ \gamma: [0,1] \to U$ s.t. $\gamma(0)=x_0$, $\gamma(1)=x$



$\Rightarrow \alpha: [0,1] \to \mathbb{R}^n$

$$\alpha(t) = \begin{cases} \gamma(2t), & 0 \le t \le \frac{1}{2} \\ x + (2t-1)(x_1 - x), & \frac{1}{2} \le t \le 1 \end{cases}$$

(similar to the arguement for open)

is Continuous, $\alpha(t) \in U$

$\Rightarrow \alpha(1) = x_1 \in P$ ($\to\leftarrow$)

So $D(x_1, \delta) \subseteq U \setminus P$

$\Rightarrow U \setminus P$ is open $\Rightarrow P$ closed U

Conclusion:

P open U, P closed U, $P \neq \phi$

U is connect $\Rightarrow P = U$

$\Rightarrow U$ is path-connected. #

Def

Let (M, d) be a metric sp., $x \in M$.

The connected component of x is

$$C_x = \bigcup \{A \subseteq M \mid A \text{ is cnt.}, x \in A\}$$

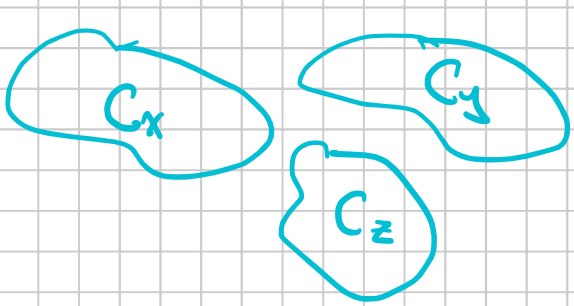
Thm

Let $x \in (M, d)$. Then

(i) C_x is cnt. and closed in M

(ii) If $A \subseteq M$ is cnt. and $C_x \subseteq A$, then $A = C_x$

(iii) $\forall\, x, y \in M$, either $C_x = C_y$ or $C_x \cap C_y = \phi$



pf

(i) [Recall: If C_α ($\alpha \in A$) are connected, $\bigcap C_\alpha \neq \phi$, then $\bigcup C_\alpha$ is cnt.]

$\Rightarrow$ C_x is cnt.

$\Rightarrow$ $cl(C_x)$ is cnt. and contains x

$\Rightarrow$ $cl(C_x) \subseteq C_x$ $\Rightarrow$ C_x is closed.

(ii) $A \subseteq M$ is cnt. and $C_x \subseteq A$. $\Rightarrow x \in A$ $\Rightarrow A \subseteq C_x$ $\Rightarrow A = C_x$

(iii) $C_x \cap C_y \neq \phi \overset{(i)}{\Rightarrow} C_x \cup C_y$ is cnt. ($\supseteq C_x$, $\supseteq C_y$, $\ni x$)

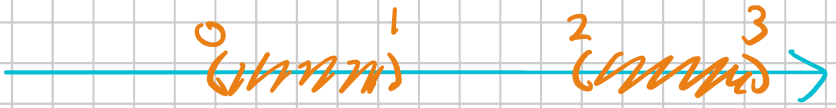
$\overset{(3)}{\Rightarrow}$ $C_x = C_x \cup C_y = C_y$ #

Example

① $S = (0,1) \cup (2,3) \subseteq \mathbb{R}$.

The connected components of S are

$(0,1)$ and $(2,3)$



② $\mathbb{Q} \subseteq \mathbb{R}$ is "totally disconnected"

i.e. $\forall x \in \mathbb{Q}$, its connected component is $\{x\}$.

If $x \neq y \in \mathbb{Q}$, $\exists\, c < y$, $c \notin \mathbb{Q}$

$\Rightarrow (-\infty, c)$, (c, ∞) separate $\mathbb{Q}$

Applications to Compact sets

Thm (Thm 4.2.2)

Let $K \subseteq (M,d)$ be compact.

If $f: K \to N$ is continuous, then $f(K)$ is compact.

pf

Given any open cover $\{U_\alpha\}_{\alpha \in I}$ of $f(K)$,

$$f^{-1}(U_\alpha) \underset{\text{open}}{\subseteq} K \quad \text{and}$$

$$\bigcup_{\alpha \in I} f^{-1}(U_\alpha) = K$$

i.e. $\{f^{-1}(U_\alpha)\}$ is an open cover of K

$\Rightarrow \exists\, \alpha_1, \cdots, \alpha_m \in I$ s.t.

$$\bigcup_{i=1}^{m} f^{-1}(U_{\alpha_i}) = K$$

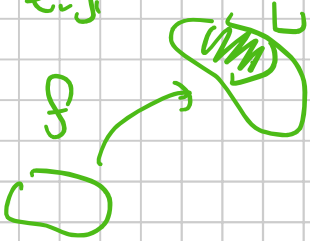
$f(x) \in U_{\alpha_i}$

$\Updownarrow$

x

$$\Rightarrow \bigcup_{i=1}^{m} U_{\alpha_i} \supseteq \bigcup_{i=1}^{m} f(\underbrace{f^{-1}(U_{\alpha_i})}_{\subseteq U_\alpha})$$

e.g.



$$= f\left(\bigcup_{i=1}^{m} f^{-1}(U_{\alpha_i})\right)$$

$$= f(K)$$

$\Rightarrow U_{\alpha_1}, \cdots, U_{\alpha_m}$ is a finite subcover of $f(K)$

#

Thm (Thm 4.4.1)

If K is cpt, $f: K \to \mathbb{R}$, then $\exists\, x_0, x_1 \in K$ s.t. $f(x_0) = \inf f(K)$, $f(x_1) = \sup f(K)$

pf

$f(K)$ is cpt in $\mathbb{R} \Rightarrow$ closed, bdd

$\Rightarrow \alpha = \inf f(K), \quad \beta = \sup f(K) \in \mathbb{R}$

$$cl(f(K)) = f(K)$$

$\Rightarrow \exists\, x_0, x_1 \in K$ s.t. $f(x_0) = \alpha, \quad f(x_1) = \beta.$

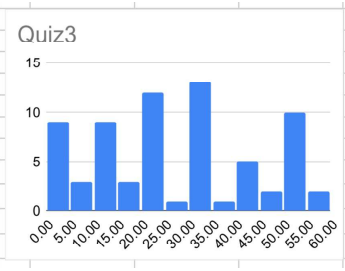
$\square$

Cor (Extreme Value Thm).

If $f: [a,b] \to \mathbb{R}$ is continuous, then

f takes a max and min. in $[a,b]$.

Average	27.68421053
Average (except zero)	31.40298507
Quartile 0	0
Quartile 1	10
Quartile 2	30
Quartile 3	45.25
Quartile 4	60
標準差	18.84760676
非零數	67
不到50%人數	37
滿分人數	6



Quiz 3

Common Mistakes and C

Problem 8. Let (K, d) be a compact metric space and let $f : K \to K$ satisfy

$$d(f(x), f(y)) = d(x, y) \qquad \text{for all } x, y \in K.$$

Prove that $f(K)$ is a closed subset of K.

Common Mistakes

1. **Using a theorem not yet covered.**

   A student writes:

   > Since f is continuous and K is compact, $f(K)$ is compact and hence closed.

   This argument is mathematically correct, but it uses the theorem that continuous images of compact spaces are compact, which is not yet allowed.

   The student should instead give a direct argument using sequential compactness.

2. **Failing to establish closedness in the correct ambient space.**

   The goal is to prove that $f(K)$ is closed *in* K.

   In a sequential proof, one should begin with a sequence (y_n) in $f(K)$ converging to some $y \in K$, and then prove that $y \in f(K)$.

3. **Not justifying convergence under f.**

   A student writes

   $$x_{n_j} \to x \quad \Longrightarrow \quad f(x_{n_j}) \to f(x)$$

   without explanation.

   This implication is correct, but students should justify it using the given assumption:

   $$d(f(x_{n_j}), f(x)) = d(x_{n_j}, x) \to 0.$$

Problem 6. Let $r > 0$, and let $D(0, r)$ be the open ball of radius r centered at 0 in $\mathbb{R}^n$. Define

$$S(0, r) := \{x \in \mathbb{R}^n \mid \|x\|_2 = r\}.$$

Let B be a nonempty subset of $S(0, r)$, and set

$$C := B \cup D(0, r).$$

Prove that C is path-connected.

Common Mistakes

1. **Confusing the open and closed balls.**

   A student constructs

   $$\gamma(t) = (1 - t)x + ty$$

   and proves only

   $$\|\gamma(t)\|_2 \le r.$$

This shows that $\gamma(t)$ belongs to the closed ball

$$\overline{D(0,r)} = D(0,r) \cup S(0,r),$$

but not necessarily to

$$C = D(0,r) \cup B.$$

Since B may be a proper subset of the sphere, additional justification is needed when $\|\gamma(t)\|_2 = r$.

2. Using a strict triangle inequality without justification.

A student writes

$$\|(1-t)x + ty\|_2 < (1-t)\|x\|_2 + t\|y\|_2$$

without explaining why the inequality is strict.

The ordinary triangle inequality gives only $\leq$.

When $x \neq y$, $\|x\|_2 = \|y\|_2 = r$, and $0 < t < 1$, strict inequality follows from the equality condition in the Euclidean triangle inequality.

This is an important step and should be justified.

3. Not verifying continuity at the joining point.

A student defines a piecewise path

$$\gamma(t) = \begin{cases} (1-2t)x, & 0 \leq t \leq \frac{1}{2}, \\ (2t-1)y, & \frac{1}{2} \leq t \leq 1, \end{cases}$$

but only states that both pieces are continuous.

The student should also verify that both pieces agree at $t = \frac{1}{2}$.

Then the entire path is continuous by the pasting lemma (if available), or a proposition stated in class, or by a direct continuity argument.

4. Proving paths to the origin without completing the argument.

A student proves that every $x \in C$ can be joined to 0 by a path, but does not explain why arbitrary $x, y \in C$ can be joined.

The missing step is to reverse the path from y to 0 and concatenate it with the path from x to 0.

This is usually a minor omission if the construction is otherwise clear.

This is the same as proving C is star-shaped, and then apply a proposition in class, but students should still mention the usage of it.

5. Failing to verify that the path stays inside C.

A student verifies that

$$\gamma(0) = x, \qquad \gamma(1) = y,$$

and that γ is continuous, but does not prove

$$\gamma([0,1]) \subseteq C.$$

Continuity and correct endpoints alone are insufficient.

The requirement that the entire path lies in C is essential to the definition of path-connectedness.