Advanced Calculus, week 1, Fall 2026

Topology of metric spaces

The theory of calculus is based on limits:

① limit of a seq: $\lim_{n\to\infty} a_n = a$

$\forall \varepsilon>0 \ \exists N_\varepsilon$ s.t. $|a_n-a|<\varepsilon \quad \forall n\geq N_\varepsilon$

② $\lim_{x\to x_0} f(x)=L$: $\forall \varepsilon>0 \ \exists \delta>0$ s.t.

"distance" $|f(x)-L|<\varepsilon \quad \forall\, 0<|x-x_0|<\delta$

③ $\lim_{t\to t_0} \vec{f}(t)=\vec{L}$: $\forall \varepsilon>0 \ \exists \delta>0$ s.t.

$\|\vec{f}(t)-\vec{L}\|<\varepsilon \quad \forall\, 0<|t-t_0|<\delta$

④ $\lim_{(x,y)\to(x_0,y_0)} f(x,y)=L$: $\forall \varepsilon>0 \ \exists \delta>0$ s.t.

"near L" "near (x_0,y_0)"

$|f(x,y)-L|<\varepsilon \quad \forall\, 0<\sqrt{(x-x_0)^2+(y-y_0)^2}<\delta$

In all of these limits, "distance" and "neighborhood" play fundamental role.

鄰域

This observation leads the study of "distance" and "topology"

$\Leftarrow$ "metric"

Metric spaces

In $\mathbb{R}^n = \{(x_1, x_2, \cdots, x_n) \mid x_1, \cdots, x_n \in \mathbb{R}\}$, where $x = (x_1, x_2, \cdots, x_n)$, $y = (y_1, \cdots, y_n)$

$\exists$ natural distance:

$$d_e(x,y) = \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2 + \cdots + (x_n-y_n)^2}$$

Euclidean

NOTE: d_e is a map

$$d_e : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$$

s.t.

(i) $d_e(x,y) \geq 0 \quad \forall x,y \in \mathbb{R}^n$

(ii) $d_e(x,y) = 0 \iff x = y$

(iii) $d_e(x,y) = d_e(y,x) \quad \forall x,y \in \mathbb{R}^n$

(iv) $d_e(x,z) \leq d_e(x,y) + d_e(y,z) \quad \forall x,y,z \in \mathbb{R}^n$

triangle inequality

Now we consider a nonempty set S

Def (Chapter 2)

If a map $d: S\times S \to \mathbb{R}$, $(a,b)\mapsto d(a,b)$ satisfies

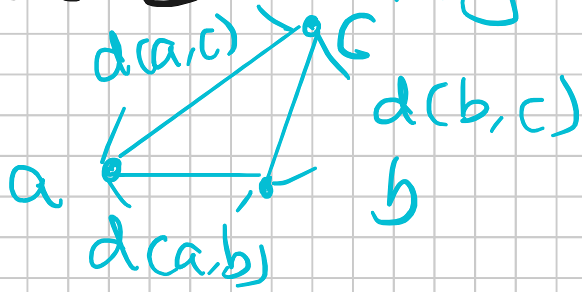
(i) $d(a,b) \geq 0 \quad \forall a,b\in S$

(ii) $d(a,b) = 0 \iff a=b$

(iii) $d(a,b) = d(b,a) \quad \forall a,b\in S$

(iv) $d(a,c) \leq d(a,b) + d(b,c)$

$\forall a,b,c\in S$ (triangle inequality)



then d is called a metric on S and the pair (S,d) is called a metric space.

度量空間/賦距空間

Example

① Let $S=\mathbb{R}$ and $d(x,y) = |x-y|$

Then $(\mathbb{R}, d)$ is a metric space.

② Let $S=\mathbb{R}^n$ and $d_e(x,y)=\sqrt{(x_1-y_1)^2+\cdots+(x_n-y_n)^2}$

Then $(\mathbb{R}^n, d_e)$ is a metric space.

<u>Def</u> (Prop 2.7.2)

Let (M,d) be a metric space, and $(x_n)_{n=1}^{\infty}$ a sequence in M

(i.e. $x_1, x_2, x_3, \cdots \in M$)

(x_n) <u>converges</u> to $x_* \in M$ if

$\forall \varepsilon > 0 \ \exists N_\varepsilon$ s.t.

$$d(x_n, x_*) < \varepsilon \quad \forall \ n \geq N_\varepsilon$$

$$\Leftrightarrow \lim_{n\to\infty} d(x_n, x_*) = 0$$

<u>Example</u>

Let $S \neq \phi$. Define, for $s_1, s_2 \in S$,

$$d_0(s_1,s_2) := \begin{cases} 0 & \text{if } s_1 = s_2 \\ 1 & \text{if } s_1 \neq s_2 \end{cases}$$

"discrete metric"

Show (S, d_0) is a metric space

Pf

We need to verify the conditions (i)-(iv)

(i) $d_0(s_1, s_2) = 0$ or $1 \geq 0 \quad \forall s_1, s_2 \in S$

(ii) $d_0(s_1, s_2) = 0 \iff s_1 = s_2$ by def. of d_0

(iii) <u>case 1</u> $s_1 = s_2$

$\Rightarrow d_0(s_1, s_2) \overset{def}{=} 0 \overset{def}{=} d_0(s_2, s_1)$

<u>case 2</u> $s_1 \neq s_2$ $\Rightarrow$ ok.

$\Rightarrow d_0(s_1, s_2) \overset{def}{=} 1 \overset{def}{=} d_0(s_2, s_1)$

(iv) $d_0(s_1, s_3) \overset{?}{\leq} d_0(s_1, s_2) + d_0(s_2, s_3)$

<u>case 1</u>: $s_1 = s_3$

$\Rightarrow d(s_1, s_3) = 0$

$$\underbrace{d_0(s_1, s_2)}_{\overset{(i)}{\geq 0}} + \underbrace{d_0(s_2, s_3)}_{\overset{(i)}{\geq 0}} \geq 0 = d_0(s_1, s_3)$$

<u>case 2</u> : $s_1 \neq s_3$

$\Rightarrow d_0(s_1, s_3) = 1$

Since $s_1 \neq s_3$, we have $s_2 \neq s_1$ or $s_2 \neq s_3$

<u>case 2-1</u> : $s_2 \neq s_1$

$$\Rightarrow d_0(s_1, s_3) = 1 \leq \underbrace{d_0(s_1, s_2)}_{=1} + \underbrace{d(s_2, s_3)}_{\geq 0}$$

<u>case 2-2</u>: $\underline{s_2 \neq s_3}$

$$\Rightarrow d_0(s_1, s_3) = 1 \leq \underbrace{d_0(s_1, s_2)}_{\geq 0} + \underbrace{d(s_2, s_3)}_{=1}$$

So

$$d_0(s_1, s_3) \leq d_0(s_1, s_2) + d_0(s_2, s_3)$$

$\forall s_1, s_2, s_3 \in S$

By (i) - (iv), (S, d_0) is a metric space $\#$

Remark

① On $S = \mathbb{R}^2$, we have different metrics:

$$d_e((x_1, x_2), (y_1, y_2)) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$$

$$d_0((x_1, x_2), (y_1, y_2)) = \begin{cases} 0 & \text{if } (x_1, x_2) = (y_1, y_2) \\ 1 & \text{if } (x_1, x_2) \neq (y_1, y_2) \end{cases}$$

Consider $x_m = \left(\frac{1}{m}, 0\right) \in \mathbb{R}^2$.
$m = 1, 2, \cdots$

Then

$$d_e\left(\left(\frac{1}{m}, 0\right), (0,0)\right) = \frac{1}{m} \to 0 \text{ as } m \to \infty$$

(i.e. $\lim_{m \to \infty} \left(\frac{1}{m}, 0\right) = (0,0)$ in $(\mathbb{R}^2, d_e)$)

However,

$$d_0\left(\left(\frac{1}{m}, 0\right), (0,0)\right) \overset{\forall m}{=} 1 \not\to 0$$

(i.e. $\lim_{m \to \infty} \left(\frac{1}{m}, 0\right) \neq (0,0)$ in $(\mathbb{R}^2, d_0)$)

This shows that limit and convergence heavily depend on the choice of metric.

② Let d and d' be metrics on M

If $\exists\ \alpha, \beta > 0$ s.t. indep of x, y

$$\alpha\, d(x,y) \leq d'(x,y) \leq \beta\, d(x,y)$$

$\forall\ x,y \in M$, then

$$\lim_{n\to\infty} x_n = x_* \text{ in } (M,d)$$

$$\Longleftrightarrow \lim_{n\to\infty} x_n = x_* \text{ in } (M,d')$$

Such metrics d and d' are called equivalent metrics.

Example

On $\mathbb{R}^n$, we can define another metric:

for $x=(x_1,\dots,x_n)$ $y=(y_1,\dots,y_n)\in\mathbb{R}^n$,

"1-metric" "1-norm metric"

$$d_1(x,y) = |x_1-y_1| + |x_2-y_2| + \dots + |x_n-y_n|$$

Show that $(\mathbb{R}^n, d_1)$ is a metric sp.

pf

(i) $d_1(x,y) \geq 0$ $\forall x,y\in\mathbb{R}^n$ since $|a-b|\geq 0$ $\forall a,b\in\mathbb{R}$

(ii) $d_1(x,y)=0 \Longleftrightarrow |x_1-y_1| = |x_2-y_2| = \dots = |x_n-y_n| = 0$

$\Longleftrightarrow x=y$

(iii) $d_1(x,y) = \sum_{i=1}^{n} |x_i - y_i| = \sum_{i=1}^{n} |y_i - x_i|$

$= d_1(y,x) \qquad \forall x,y \in \mathbb{R}^n$

(iv) $d_1(x,z) = \sum_{i=1}^{n} |x_i - z_i|$

$$\leq \sum_{i=1}^{n} \left(|x_i - y_i| + |y_i - z_i| \right)$$

$= d_1(x,y) + d_1(y,z)$ #

Remark $(d_2 = d_0)$

On $\mathbb{R}^n$, we have

$$d_2(x,y) \overset{①}{\leq} d_1(x,y) \overset{②}{\leq} \sqrt{n}\, d_2(x,y)$$

$\forall x,y \in \mathbb{R}^n$

Pf

① $(d_2(x,y))^2 = \sum_{i=1}^{n} (x_i - y_i)^2$

$\wedge\!\!|$

$(d_1(x,y))^2 = \left(\sum_{i=1}^{n} |x_i - y_i| \right)^2$

$= \sum_{i=1}^{n} |x_i - y_i|^2 + \sum_{i,j} 2|x_i - y_i| \cdot |x_j - y_j| \ \cdots \quad \geq 0$

$|a_1 b_1 + \cdots + a_n b_n|^2 \leq (\sum a_i^2)(\sum b_i^2)$

$|\langle \vec{v}, \vec{w} \rangle|^2 \leq \|\vec{v}\|^2 \cdot \|\vec{w}\|^2$

② By Cauchy-Schwarz ineq.,

$$\left(\sum_{i=1}^{n} \underbrace{1}_{a_i} \cdot \underbrace{|x_i - y_i|}_{b_i} \right)^2 \leq \left(\sum_{i=1}^{n} 1^2 \right) \left(\sum_{i=1}^{n} |x_i - y_i|^2 \right)$$

$$(d_1(x,y))^2 \leq n \cdot (d_2(x,y))^2$$

$$\Rightarrow d_1(x,y) \leq \sqrt{n}\, d_2(x,y). \quad \#$$

This remark shows that for $x_m, x_* \in \mathbb{R}^n$

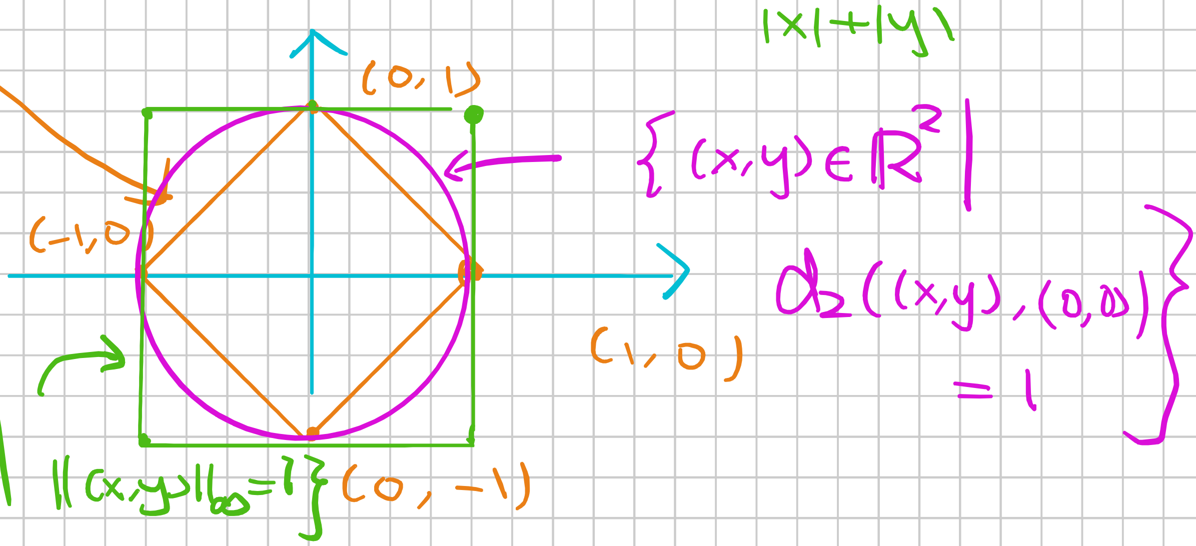
$$\lim_{m\to\infty} x_m = x_* \text{ in } (\mathbb{R}^n, d_1)$$

$$\Leftrightarrow \lim_{m\to\infty} x_m = x_* \text{ in } (\mathbb{R}^n, d_2)$$

<u>Remark</u>

Consider "unit circle in $(\mathbb{R}^2, d_1)$"

$$\{(x,y)\in\mathbb{R}^2 \mid \underbrace{d_1((x,y),(0,0))}_{=\,|x|+|y|} = 1\}$$



<u>Example</u>

On $\mathbb{R}^2$, we define sup norm of

$$x = (x_1, x_2)$$

to be
$$\|x\|_\infty := \max\{|x_1|, |x_2|\}$$
and
$$d_\infty(x,y) := \|x-y\|_\infty$$
$$= \max\{|x_1-y_1|, |x_2-y_2|\}$$
Then $(\mathbb{R}^2, d_\infty)$ is a metric space because

(i) $d_\infty(x,y) = \max\{\overset{|y_1-x_1|}{\overset{=}{|x_1-y_1|}}, \overset{|y_2-x_2|}{\overset{=}{|x_2-y_2|}}\} \geq 0$

(ii) $d_\infty(x,y) = 0 \iff 0 \geq |x_i-y_i| \geq 0$, $i=1,2$

$\iff x_i = y_i$, $i=1,2$ $\iff x=y$

(iii) $d_\infty(x,y) = d_\infty(y,x)$

(iv) NOTE:
$$\|x+y\|_\infty = \max\{\overset{|x_1|+|y_1|}{\overset{\geq}{|x_1+y_1|}}, \overset{|x_2|+|y_2|}{\overset{\geq}{|x_2+y_2|}}\}$$
$$< \max\{|x_1|+|y_1|, |x_2|+|y_2|\}$$
$$= |x_1|+|y_1| \text{ or } |x_2|+|y_2|$$
$$\leq \max\{|x_1|,|x_2|\} + \max\{|y_1|,|y_2|\}$$
$$= \|x\|_\infty + \|y\|_\infty$$

$$\Rightarrow \quad d_\infty(x,z) = \|x-z\|_\infty$$
$$= \|(x-y)+(y-z)\|_\infty$$
$$\leq \|x-y\|_\infty + \|y-z\|_\infty$$
$$= d_\infty(x,y) + d_\infty(y,z) \qquad \#$$

§ Normed vector spaces

Def

Let V be a vector space over $\mathbb{R}$.

A function

$$\|\ \| : V \longrightarrow \mathbb{R}, \quad x \longmapsto \|x\|$$

is called a norm if it satisfies:

(i) $\|x\| \geq 0 \quad \forall x \in V$,

(ii) $\|x\| = 0 \iff x = 0$,

(iii) $\|x+y\| \leq \|x\| + \|y\| \quad \forall x, y \in V$

☆ (iv) $\|\lambda \cdot x\| = |\lambda| \cdot \|x\| \quad \forall \lambda \in \mathbb{R},\ x \in V$

The pair $(V, \|\ \|)$ is called a normed vector space

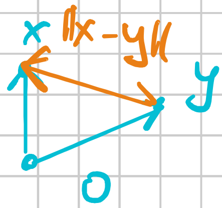
e.g.

$$V = \mathbb{R}^2, \quad \|(x,y)\| = \sqrt{x^2+y^2}$$

Prop

Let $(V, \| \|)$ be a normed vector space.

Define $d : V \times V \to \mathbb{R}$ by



$$d(x,y) := \|x-y\|$$

Then d is a metric on V,

called the induced metric of $\| \|$,

and (V, d) is a metric space

pf: exercise.

For $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$ (assume $V = \mathbb{R}^n$),

define the p-norm of x to be

$$\|x\|_p := \sqrt[p]{|x_1|^p + |x_2|^p + \cdots + |x_n|^p}$$

$p \in \mathbb{R}$, $p \neq \infty$

$(p > 1)$. e.g. $\|x\|_2 = \sqrt[2]{x_1^2 + x_2^2 + \cdots + x_n^2}$

Claim $\| \|_p$ is a norm on $\mathbb{R}^n$ $\forall p > 1$.

(i), (ii), (iv) can be verified easily.

To show (iii), we need a few results:

Lemma

Suppose $x, y \geq 0$, and $p, p' > 1$ s.t.

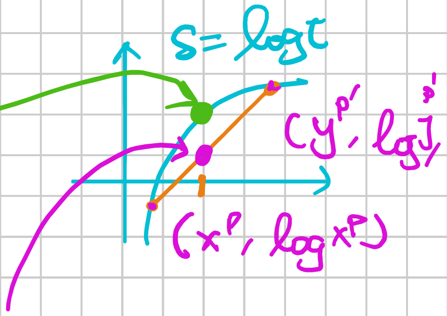
$$\frac{1}{p} + \frac{1}{p'} = 1$$

Then $$xy \leq \frac{x^p}{p} + \frac{y^{p'}}{p'}$$

Sketch of pf

Step 1: Show $g(t) = \log t$ is concave

Step 2: "Concave" implies



$a = \frac{1}{p},\ b = \frac{1}{p'}$

$a + b = 1$

$a, b \geq 0$

$ax + by$

$x \qquad y$

$$\log\left(\frac{x^p}{p} + \frac{y^{p'}}{p'}\right)$$

$$\geq \frac{1}{p}\log(x^p) + \frac{1}{p'}\log(y^{p'})$$

$$= \log(xy) \quad \Rightarrow \text{ ok} \qquad \#$$

Lemma (Hölder's inequality)

Let $p > 1$, $\frac{1}{p} + \frac{1}{p'} = 1$. Suppose $a_1, \cdots, a_n, b_1, \cdots, b_n \geq 0$.

Then

$$(*) \quad \sum_{j=1}^{n} a_j b_j \leq \left(\sum_{j=1}^{n} a_j^p\right)^{\frac{1}{p}} \cdot \left(\sum_{j=1}^{n} b_j^{p'}\right)^{\frac{1}{p'}}$$

Remark:

$p = 2 \Rightarrow p' = 2$

$\Rightarrow$ Cauchy ineq.

pf

case 1: $\sum_{j=1}^{n} a_j^p = 0$ or $\sum_{j=1}^{n} b_j^{p'} = 0$

$\Rightarrow a_1 = \cdots = a_n = 0$ $\qquad \Rightarrow b_1 = \cdots = b_n = 0$

$\Rightarrow \sum a_j b_j = 0$

$\Rightarrow (*)$ holds.

case 2: $\sum_{j=1}^{n} a_j^p \neq 0$ and $\sum_{j=1}^{n} b_j^{p'} \neq 0$

Consider

$$x_j := \frac{a_j}{\left(\sum_{j=1}^{n} a_j^p\right)^{1/p}} \quad \text{and} \quad y_j := \frac{b_j}{\left(\sum_{j=1}^{n} b_j^{p'}\right)^{1/p'}}$$

By the previous lemma,

$$\underbrace{\sum_{j=1}^{n} x_j y_j}\leq \underbrace{\sum_{j=1}^{n} \frac{x_j^p}{p} + \sum_{j=1}^{n} \frac{y_j^{p'}}{p'}}$$

$$= \frac{1}{p}\underbrace{\sum_{j=1}^{n} \frac{a_j^p}{\sum_{j=1}^{n} a_j^p}}_{=1} + \frac{1}{p'}\underbrace{\sum_{j=1}^{n} \frac{b_j^{p'}}{\sum_{j=1}^{n} b_j^{p'}}}_{=1}$$

$$= \frac{1}{p} + \frac{1}{p'} = \underbrace{1}$$

So

$$\sum_{j=1}^{n} \frac{a_j b_j}{\left(\sum_{j=1}^{n} a_j^p\right)^{1/p} \cdot \left(\sum_{j=1}^{n} b_j^{p'}\right)^{1/p'}} \leq 1$$

$$\Rightarrow \sum_{j=1}^{n} a_j b_j \leq \left(\sum_{j=1}^{n} a_j^p\right)^{1/p} \cdot \left(\sum_{j=1}^{n} b_j^{p'}\right)^{1/p'}$$

#

Lemma

If $\frac{1}{p} + \frac{1}{p'} = 1$, $p > 1$, then

$$\|x\|_p = \max_{\|y\|_{p'} = 1} \underbrace{\left|\sum_{j=1}^{n} x_j y_j\right|}$$

pf

① By Hölder's ineq,

"$\geq$"

$$\underbrace{\left|\sum x_j y_j\right|} \leq \sum |x_j y_j| \leq \|x\|_p \cdot \|y\|_{p'}$$

if $\|y\|_{p'} = 1$

$= \underline{\|x\|_p}$

$$\Rightarrow \max_{\|y\|_{p'}=1} \left| \sum_{j=1}^{n} x_j y_j \right| \le \|x\|_p$$

② " $\ge$ "

(2-1) $x=(0,\cdots,0) \Rightarrow \|x\|_p = 0 \Rightarrow$ " $\ge$ " holds

(2-2) $x \neq 0 \Rightarrow c := \left(\sum_{j=1}^{n} |x_j|^p \right)^{1/p'} \neq 0$

Choose

$$\check{y} = (\check{y}_1, \cdots, \check{y}_n) \text{ with}$$

$$\check{y}_j = \frac{|x_j|^{p-1} \cdot \operatorname{sign}(x_j)}{c}$$

$\because \frac{1}{p} + \frac{1}{p'} = 1$

$\Rightarrow \frac{1}{p'} = 1 - \frac{1}{p}$

$= \frac{p-1}{p}$

$$\Rightarrow \sum_{j=1}^{n} |\check{y}_j|^{p'} = \sum_{j=1}^{n} \frac{\overbrace{|x_j|^{(p-1)p'}}^{=|x_j|^p}}{\underbrace{\sum_{j=1}^{n} |x_j|^p}_{=c^{p'}}}$$

$p' = \frac{p}{p-1}$

$$= 1$$

and

$$\sum_{j=1}^{n} x_j \check{y}_j = \sum \frac{x_j |x_j|^{p-1} \cdot \operatorname{sign}(x_j)}{c}$$

$= |x_j|$

$$= \frac{\sum |x_j|^p}{\left(\sum |x_j|^p \right)^{\frac{1}{p'}}} = \|x\|_p$$

$\frac{1}{p'} = 1 - \frac{1}{p}$

$= \left(\sum |x_j|^p \right)^{1-(1-\frac{1}{p})}$

Conclusion: $\exists \check{y}$ s.t. $\|\check{y}\|_{p'} = 1$ and $\left| \sum x_j \check{y}_j \right| = \|x\|_p$

$$\Rightarrow \max_{\|y\|_{p'}=1} \left| \sum x_j y_j \right| \ge \sum x_j \check{y}_j = \|x\|_p$$

#

Thm (Minkowski's ineq)

$$\|x+y\|_p \le \|x\|_p + \|y\|_p$$

pf

$(\langle x,y\rangle = \sum x_j y_j)$

$$\|x+y\|_p = \max_{\|u\|_{p'}=1} \Big| \underbrace{\sum (x_j+y_j) u_j}_{= \langle x+y, u\rangle} \Big|$$

$$= \max_{\|u\|_{p'}=1} |\langle x,u\rangle + \langle y,u\rangle|$$

$$\le \max_{\|u\|_{p'}=1} (|\langle x,u\rangle| + |\langle y,u\rangle|) \quad (1<p<\infty)$$

$$\le \left(\max_{\|u\|_{p'}=1} |\langle x,u\rangle|\right) + \left(\max_{\|u\|_{p'}=1} |\langle y,u\rangle|\right)$$

$$= \|x\|_p + \|y_p\| \qquad \#$$

NOTE:

For $p=1$ and $p=\infty$,

$\|x\|_1 = |x_1| + \cdots + |x_n|$ — 1-norm

$\|x\|_\infty = \max\{|x_1|, \dots, |x_n|\}$ — sup norm

are also norms on $\mathbb{R}^n$.

Cor

For $1 \le p \le \infty$, the pair $(\mathbb{R}^n, \| \|_p)$ is a normed vector space. In particular, $(\mathbb{R}^n, d_p)$ is a metric sp. where

$$d_p(x,y) = \|x-y\|_p = \left(\sum |x_j - y_j|^p\right)^{1/p}$$

Equivalent metrics

Recall

Let d, d' be metrics on a set M.
They are equivalent if $\exists\, \alpha, \beta > 0$ s.t.

$$\alpha \cdot d(x,y) \le d'(x,y) \le \beta \cdot d(x,y)$$

$x, y \in M$.

Exercise

Let d, d' be equivalent metrics on M,
$x_n, x_* \in M$. Then

$$x_n \to x_* \quad \text{in } (M, d)$$
$$\iff x_n \to x_* \quad \text{in } (M, d')$$

Remark

"Equivalence of metrics" is an equivalence realation.