Calculus (II) — Homework 7 (Fall 2026)

1. Simplify.
   (a) $(3\vec{a}\cdot\vec{b}) - (\vec{a}\cdot 2\vec{b})$.
   (b) $\vec{a}\cdot(\vec{a}-\vec{b}) + \vec{b}\cdot(\vec{b}+\vec{a})$.
   (c) $(\vec{a}-\vec{b})\cdot\vec{c} + \vec{b}\cdot(\vec{c}+\vec{a})$.
   (d) $\vec{a}\cdot(\vec{a}+2\vec{c}) + (2\vec{b}-\vec{a})\cdot(\vec{a}+2\vec{c}) - 2\vec{b}\cdot(\vec{a}+2\vec{c})$.

2. Consider vectors in $\mathbb{R}^3$.
   (a) Find $\vec{a},\vec{b},\vec{c}\neq\vec{0}$ such that $\vec{a}\cdot\vec{b} = \vec{a}\cdot\vec{c}$ but $\vec{b}\neq\vec{c}$.
   (b) Show that if $\vec{u}\cdot\vec{b} = \vec{u}\cdot\vec{c}$ for all *unit vectors* $\vec{u}$ (i.e. $\left\|\vec{u}\right\| = 1$), then $\vec{b} = \vec{c}$.

3. Let $\vec{a},\vec{b}\in\mathbb{R}^3$.
   (a) Show that for all vectors $\vec{a}$ and $\vec{b}$
   $$4(\vec{a}\cdot\vec{b}) = \left\|\vec{a}+\vec{b}\right\|^2 - \left\|\vec{a}-\vec{b}\right\|^2.$$
   (b) Show that $\vec{a}\perp\vec{b}$ iff $\left\|\vec{a}+\vec{b}\right\| = \left\|\vec{a}-\vec{b}\right\|$.
   (c) Show that, if $\vec{a}$ and $\vec{b}$ are nonzero vectors such that
   $$(\vec{a}+\vec{b})\perp(\vec{a}-\vec{b}) \qquad \text{and} \qquad \left\|\vec{a}+\vec{b}\right\| = \left\|\vec{a}-\vec{b}\right\|,$$
   then the parallelogram generated by $\vec{a}$ and $\vec{b}$ is a square.

4. Show that
   $$|\vec{a}\cdot\vec{b}| \leq \left\|\vec{a}\right\|\left\|\vec{b}\right\|,$$
   and the equality holds iff $\vec{a}$ and $\vec{b}$ are *parallel*, i.e. there exists λ such that $\vec{a} = \lambda\vec{b}$ or $\vec{b} = \lambda\vec{a}$.

5. Prove the *parallelogram law*:
   $$\left\|\vec{a}+\vec{b}\right\|^2 + \left\|\vec{a}-\vec{b}\right\|^2 = 2\left\|\vec{a}\right\|^2 + 2\left\|\vec{b}\right\|^2.$$

6. Calculate the $\vec{f}'(t)$ and $\vec{f}''(t)$.
   (a) $\vec{f}(t) = (1+2t)\,\vec{\imath} + (3-t)\,\vec{\jmath} + \cos t\vec{k}$.
   (b) $\vec{f}(t) = e^t(\vec{\imath}-\vec{\jmath}) + e^{-2t}(\vec{\jmath}-\vec{k})$.
   (c) $\vec{f}(t) = ((t^2\,\vec{\imath}-\vec{\jmath})\cdot(\vec{\imath}-t^2\,\vec{\jmath}))\,\vec{\imath}$.

7. Find $\lim_{t\to 0}\vec{f}(t)$ if it exists. Explain why if the limit does not exist.
   (a) $\vec{f}(t) = (1+2t)\,\vec{\imath} + (3-t)\,\vec{\jmath} + \dfrac{t}{|t|}\vec{k}$.
   (b) $\vec{f}(t) = e^t(\vec{\imath}-\vec{\jmath}) + e^{-2t}(\vec{\jmath}-\vec{k})$.
   (c) $\vec{f}(t) = \dfrac{\sin t}{2t}\,\vec{\imath} + e^{2t}\,\vec{\jmath} + \dfrac{t^2}{e^t}\,\vec{k}$.
   (d) $\vec{f}(t) = t^2\,\vec{\imath} + \dfrac{1-\cos t}{3t}\,\vec{\jmath} + \dfrac{t}{t+1}\,\vec{k}$.

8. Let $\vec{f}$ be a differentiable vector-valued function. Show that if $\left\|\vec{f}(t)\right\| \neq 0$, then
   $$\frac{d}{dt}\left(\left\|\vec{f}(t)\right\|\right) = \frac{\vec{f}(t)\cdot\vec{f}'(t)}{\left\|\vec{f}(t)\right\|}.$$

9. Let $\vec{\gamma}$ be a differentiable curve on an interval I. Show that $\left\|\vec{\gamma}(t)\right\|$ is constant iff $\vec{\gamma}(t)\cdot\vec{\gamma}'(t) = 0$ for all $t\in I$.