Calculus (II) — Homework 2 (Fall 2026)

1. State whether the sequence converges and, if it does, find the limit.

   (a) $a_n = 2^{2/n}$.

   (b) $a_n = \left(\frac{2}{n}\right)^n$.

   (c) $a_n = \dfrac{\ln(n+1)}{n}$.

   (d) $a_n = \dfrac{4^{100n}}{n!}$.

   (e) $a_n = \displaystyle\int_{-n}^{0} e^{2x}\,dx$.

   (f) $a_n = n^2 \sin\dfrac{\pi}{n}$.

   (g) $a_n = \left(\dfrac{n-1}{n}\right)^n$.

   (h) $a_n = \displaystyle\int_0^{1/n} \cos e^x\,dx$.

   (i) $a_n = \left(\dfrac{1}{2} + \dfrac{3}{n}\right)^{3n}$.

2. Evaluate the following limits as $x \to \infty$.

   (a) $\displaystyle\lim_{x\to\infty} \frac{\ln(x^2+1)}{x}$

   (b) $\displaystyle\lim_{x\to\infty} x^2 e^{-x}$

   (c) $\displaystyle\lim_{x\to\infty} \left(1 + \frac{3}{x}\right)^{2x}$

   (d) $\displaystyle\lim_{x\to\infty} \frac{x + \sin x}{x}$ (*Warning:* L'Hôpital's Rule applies only when the limit of the ratio of derivatives exists or equals $\pm\infty$.)

3. Show that if two sequences a_n and b_n grow at the same rate, then $a_n = O(b_n)$ and $b_n = O(a_n)$.

4. Prove the following.

   (a) $e^n = o(e^{e^n})$.

   (b) $\ln(\ln n) = o(\ln n)$.

   (c) $3n^5 - 100n^2 + 5n + 1 = O(n^5)$.

   (d) $2^n = O(2^{n^2})$.

Before attempting the improper integrals, review the following core techniques:

- ***u*-substitution:** $\displaystyle\int f(g(x))g'(x)\,dx = \int f(u)\,du$, where $u = g(x)$.
- **Integration by parts:** $\displaystyle\int u\,dv = uv - \int v\,du$.
- **Partial fraction decomposition:** Decompose rational functions $\frac{P(x)}{Q(x)}$ into simpler fractions after factoring $Q(x)$.
- **Standard formulas:**

$$\int \frac{dx}{x^2 + a^2} = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C, \qquad \int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C.$$

5. Evaluate the integrals.

   (a) $\displaystyle\int_0^\infty \frac{dx}{x^2+1}$.

   (b) $\displaystyle\int_{-\infty}^0 xe^x\,dx$.

   (c) $\displaystyle\int_{-1}^\infty \frac{dx}{x^2+5x+6}$.

   (d) $\displaystyle\int_{-\infty}^\infty \frac{1}{e^x + e^{-x}}\,dx$.

6. This problem shows that $\displaystyle\int_{-\infty}^\infty f(x)\,dx$ and $\displaystyle\lim_{b\to\infty}\int_{-b}^{b} f(x)\,dx$ are different.

   (a) Show that $\displaystyle\int_0^\infty \frac{2x}{x^2+1}\,dx$ diverges and hence that $\displaystyle\int_{-\infty}^\infty \frac{2x}{x^2+1}\,dx$ diverges.

   (b) Show that

$$\lim_{b\to\infty}\int_{-b}^{b} \frac{2x}{x^2+1}\,dx = 0.$$