Calculus, week 9, Fall 2026

Recall

Consider a vector-valued function

$$\vec{f}(t) = (f_1(t), f_2(t), f_3(t))$$
$$= f_1(t)\,\vec{i} + f_2(t)\,\vec{j} + f_3(t)\,\vec{k}$$

Such a function $\vec{f}(t)$ is differentiable
$\Leftrightarrow$ $f_1(t), f_2(t), f_3(t)$ are all differentiable.

In this case,

$$\vec{f}'(t) = (f_1'(t), f_2'(t), f_3'(t))$$
$$= f_1'(t)\,\vec{i} + f_2'(t)\,\vec{j} + f_3'(t)\,\vec{k}$$

For example, if

$$\vec{f}(t) = (t^2, \sin t, e^{t^2})$$

then

$$\vec{f}'(t) = ((t^2)', (\sin t)', (e^{t^2})')$$
$$= (2t, \cos t, e^{t^2}\cdot 2t)$$

Remark

① Recall (from Calculus I):

If $f(t)$ is differentiable at $t=c$, then $f(t)$ is continuous at $t=c$.

② Recall (from last week):

$\vec{f}(t) = (f_1(t), f_2(t), f_3(t))$ is continuous at $t=c$

$\Leftrightarrow$ $f_1(t)$, $f_2(t)$, $f_3(t)$ are all continuous at $t=c$.

③ Therefore, if $\vec{f}(t)$ is differentiable at $t=c$,

$\Rightarrow$ $f_1(t)$, $f_2(t)$, $f_3(t)$ are all differentiable at $t=c$

① $\Rightarrow$ $f_1(t)$, $f_2(t)$, $f_3(t)$ are all continuous at $t=c$

② $\Rightarrow$ $\vec{f}(t)$ is continuous at $t=c$

"Differentiable $\Rightarrow$ Continuous"

Thm (§14.2)

Let $\vec{f}(t)$, $\vec{g}(t)$ be differentiable vector-valued functions, $\alpha, \beta \in \mathbb{R}$. Then

(i) $(\alpha \vec{f}(t) + \beta \vec{g}(t))' = \alpha \vec{f}'(t) + \beta \vec{g}'(t)$

If $\vec{f}(t) = (f_1(t), f_2(t), f_3(t))$

$\vec{g}(t) = (g_1(t), g_2(t), g_3(t))$

then $\alpha \vec{f}(t) + \beta \vec{g}(t) = (\alpha f_1(t) + \beta g_1(t), \alpha f_2(t) + \beta g_2(t), \alpha f_3(t) + \beta g_3(t))$

LHS: $(\alpha \vec{f}(t) + \beta \vec{g}(t))'$

$= ((\alpha f_1(t) + \beta g_1(t))', (\alpha f_2(t) + \beta g_2(t))', (\alpha f_3(t) + \beta g_3(t))')$

Calculus I ↓

$= (\alpha f_1'(t) + \beta g_1'(t), \alpha f_2'(t) + \beta g_2'(t), \alpha f_3'(t) + \beta g_3'(t))$

$= \alpha (f_1'(t), f_2'(t), f_3'(t)) + \beta (g_1'(t), g_2'(t), g_3'(t))$

$= \alpha \vec{f}'(t) + \beta \vec{g}'(t) = \text{RHS}$

(ii) If $u(t)$ is a real-valued differentiable function, then

$(u(t) \vec{f}(t))' = u'(t) \vec{f}(t) + u(t) \vec{f}'(t)$

(iii) $(\vec{f}(t) \cdot \vec{g}(t))' = \vec{f}'(t) \cdot \vec{g}(t) + \vec{f}(t) \cdot \vec{g}'(t)$

(iv) $(\vec{f}(u(t)))' = u'(t) \vec{f}'(u(t))$

↑ "Chain rule"

Example

Let $\vec{f}(t) = (t, t^2, \cos t)$

$\vec{g}(t) = (t, -1, e^t)$

① $\vec{f}'(t) = ((t)', (t^2)', (\cos t)')$

$= (1, 2t, -\sin t)$

$\vec{g}'(t) = ((t)', (-1)', (e^t)')$

$= (1, 0, e^t)$

② $(\vec{f}(t) - \vec{g}(t))' = \vec{f}'(t) - \vec{g}'(t)$

$= (1, 2t, -\sin t) - (1, 0, e^t)$

$= (0, 2t, -\sin t - e^t)$

③ $(t^2 \vec{f}(t) + \vec{g}(t^3))'$

$\overset{(i)}{=} (t^2 \vec{f}(t))' + (\vec{g}(t^3))'$

$\overset{(ii)}{=} (t^2)' \vec{f}(t) + t^2 \vec{f}'(t) + (\vec{g}(t^3))'$

$\overset{(iv)}{=} 2t \vec{f}(t) + t^2 \vec{f}'(t) + \underbrace{(t^3)'}_{3t^2} \vec{g}'(t^3)$

$$= 2t(t, t^2, \cos t) + t^2(1, 2t, -\sin t) + 3t^2(1, 0, e^{t^3})$$

$$= (2t^2 + t^2 + 3t^2,\ 2t^3 + 2t^3 + 0,\ 2t\cos t - t^2\sin t + 3t^2 e^{t^3})$$

$$= (6t^2,\ 4t^3,\ 2t\cos t - t^2\sin t + 3t^2 e^{t^3})$$

#

Geometry of Curve

A differentiable vector-valued function

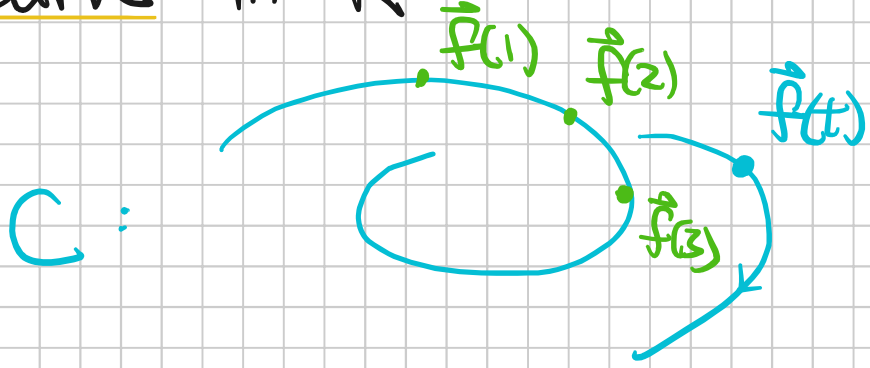
$$\vec{f}(t) = (f_1(t), f_2(t), f_3(t))$$

$$= f_1(t)\vec{i} + f_2(t)\vec{j} + f_3(t)\vec{k}$$

$(t \in (a, b))$

traces out a directed curve in $\mathbb{R}^3$.

It is called a differentiable parametrized curve in $\mathbb{R}^3$

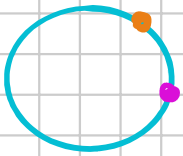


We consider $\vec{f}(t)$ as a parametrization 參數化 of the curve $C = \text{im}(\vec{f})$

$$C = \{ \vec{f}(t) \in \mathbb{R}^3 \mid t \in (a,b) \}$$

For example. Consider

$$C = \{(x,y,0) \in \mathbb{R}^3 \mid x^2+y^2=1 \}$$



The functions

$$\vec{f}(t) = (\cos t, \sin t, 0), \quad t \in \mathbb{R}$$

$$\vec{g}(t) = (\cos(2\pi t), \sin(2\pi t), 0), \quad t \in \mathbb{R}$$

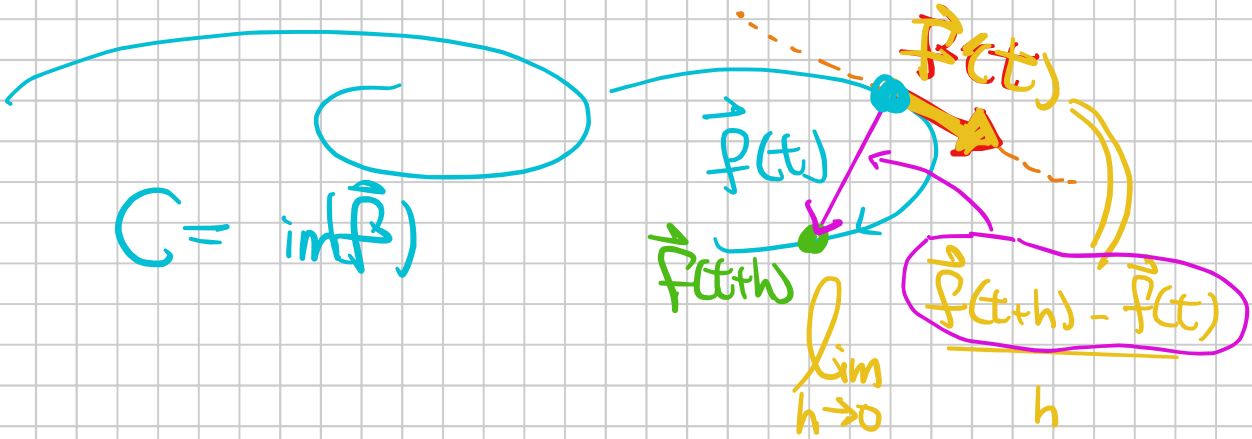
are two parametrizations of the circle C.

Def (Def 14.3.1)

The vector

$$\vec{f}'(t) = (f_1'(t), f_2'(t), f_3'(t))$$

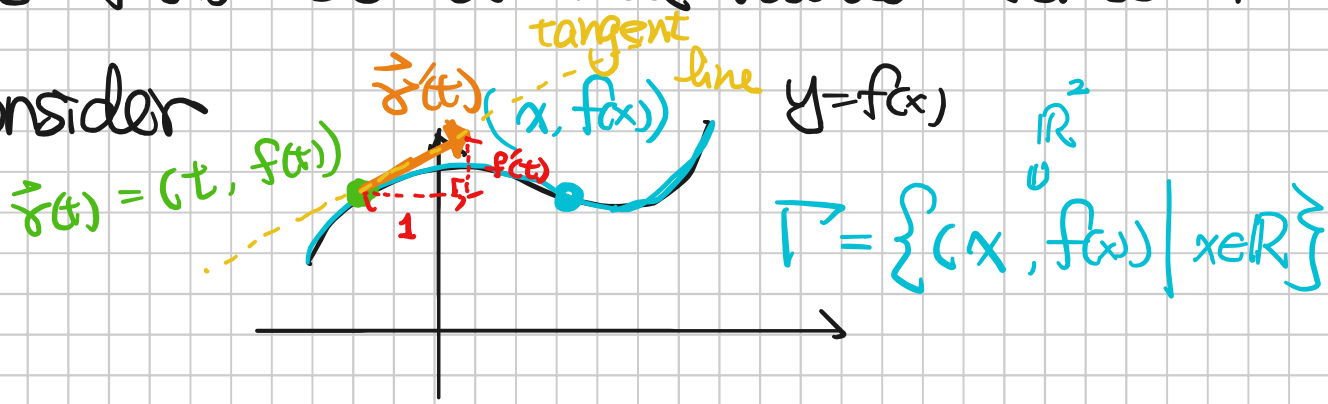
is called a tangent vector of the curve $\text{im}(\vec{f})$ at the point $\vec{f}(t)$.



Remark

Let $f(x)$ be a differentiable real-valued function

Consider



The graph Γ is a curve, which can be parametrized by

$$\vec{\gamma}(t) = (t, f(t))$$

Then

$$\vec{\gamma}'(t) = (1, f'(t))$$

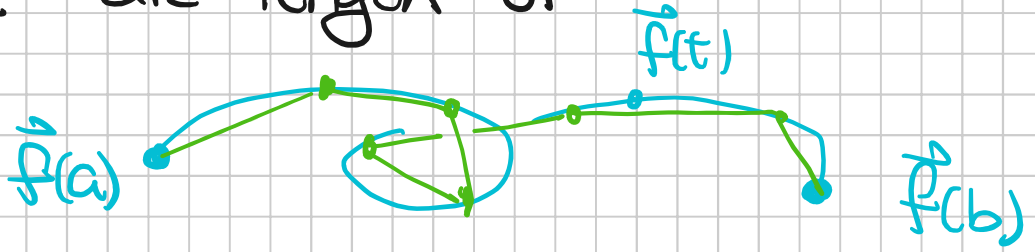
is a tangent vector of Γ at $\vec{\gamma}(t)$ $= (t, f(t))$

Arc length of a curve

Thm (Thm 14.4.2)

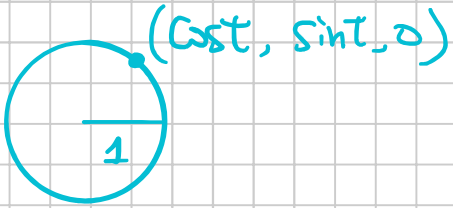
Let $\vec{f}(t)$, $t \in [a,b]$, be a continuously differentiable parametrized curve in $\mathbb{R}^3$

The arc length of



is

$$\int_a^b \| \vec{f}'(t) \| \, dt \qquad (*)$$



Example

① $C = \{ (x,y,0) \in \mathbb{R}^3 \mid x^2+y^2=1 \}$

We know its arc length is 2π.

Let us try to use $(*)$ to get 2π:

Step1: Choose a parametrization of C:

$$\vec{f}(t) = (\cos t, \sin t, 0)$$

$0 \le t \le 2\pi$. 逆時針轉一圈

Step2: Apply the formula:

$$\text{arc length} = \int_0^{2\pi} \| \vec{f}'(t) \| \, dt$$

where

$$\vec{f}'(t) = \left((\cos t)', (\sin t)', (0)'\right)$$

$$= (-\sin t, \cos t, 0)$$

$$\Rightarrow \|\vec{f}'(t)\| = \sqrt{(-\sin t)^2 + (\cos t)^2 + 0^2}$$

$$= \sqrt{\sin^2 t + \cos^2 t} = 1$$

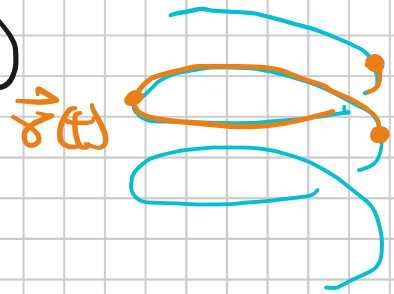
So

$$\text{arc length} = \int_0^{2\pi} 1 \, dt = 2\pi \quad \#$$

② Find the arc length of the parametrized curve:

$$\vec{\gamma}(t) = (\cos t, \sin t, t)$$

$t \in [0, 2\pi]$.



sol

$$\vec{\gamma}'(t) = \left((\cos t)', (\sin t)', (t)'\right)$$

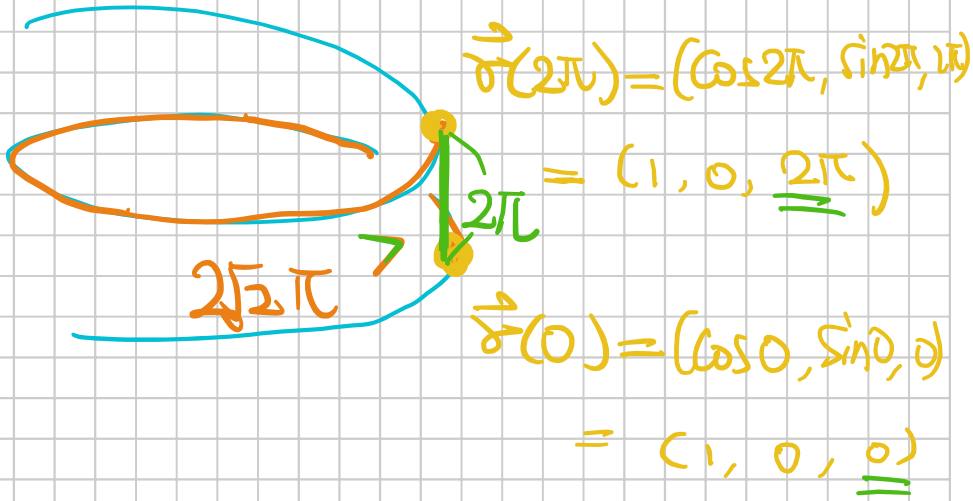
$$= (-\sin t, \cos t, 1)$$

$$\|\vec{\gamma}'(t)\| = \sqrt{\underbrace{(-\sin t)^2 + (\cos t)^2}_{=1} + 1^2}$$

$$= \sqrt{2}$$

So the arc length is

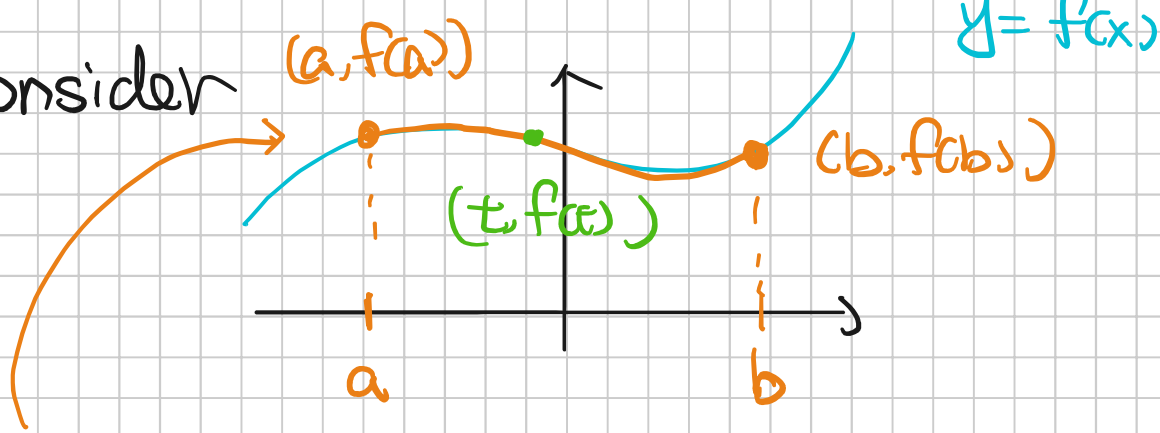
$$\int_0^{2\pi} \sqrt{2}\, dt = \underline{\underline{2\sqrt{2}\,\pi}} \quad \#$$



Remark

Let $f(x)$ be a continuously differentiable real-valued function.

Consider



Recall that this part can be parametrized by

$$\vec{r}(t) = (t, f(t)). \qquad a \le t \le b$$

Then the arc length of this part is given by

$$\int_a^b \|\vec{\gamma}'(t)\|\,dt = \int_a^b \sqrt{1+(f'(x))^2}\,dx$$

$$\vec{\gamma}'(t) = (c(t)', f'(t)) = (1, f'(t))$$

$$\|\vec{\gamma}'(t)\| = \sqrt{1^2+(f'(t))^2}$$

§ Functions of serval variables

From now on, we consider (real-valued) functions of several variables, such as

$$f(x,y) = x^2+y^2$$

$$g(x,y,z) = x^2+y^2+z^2$$

We will introduce derivatives of such fuctions in this chapter.

Def (Def 15.4.1)

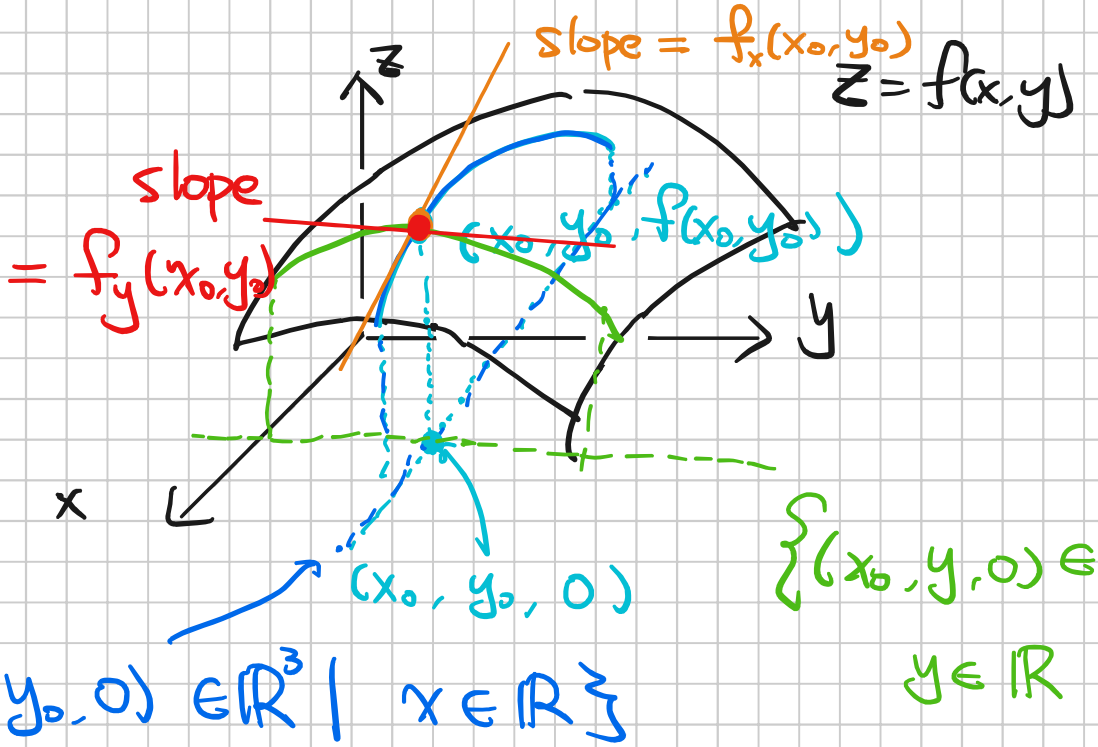
Let $f(x,y)$ be a function of 2 variables. The partial (偏微分) derivative of f with respect to x (with respect to y) are

$$f_x(x,y) = \frac{\partial f}{\partial x} = \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h}$$

Recall: $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

$$\left(f_y(x,y) = \frac{\partial f}{\partial y} = \lim_{h\to 0} \frac{f(x,y+h)-f(x,y)}{h} \right)$$

provided there limits exist.



$\{(x_0, y, 0) \in \mathbb{R}^3 \mid y \in \mathbb{R}\}$

$\{(x, y_0, 0) \in \mathbb{R}^3 \mid x \in \mathbb{R}\}$

Example

Calculate f_x and f_y:

① $f(x,y) = x^2 + y^2$

把 y 當成一固定的數字

$\Rightarrow f_x = \frac{\partial x^2}{\partial x} + \frac{\partial y^2}{\partial x} = 2x + 0$

$= 2x$

把 x 當成一固定數字

$f_y = \frac{\partial x^2}{\partial y} + \frac{\partial y^2}{\partial y} = 0 + 2y = 2y$ #

② $f(x,y) = x \cdot \tan^{-1}(xy)$

$\Rightarrow f_x = \frac{\partial f}{\partial x} = \frac{\partial x}{\partial x} \cdot \tan^{-1}(xy) + x \cdot \frac{\partial \tan^{-1}(xy)}{\partial x}$ ($\frac{\partial x}{\partial x} = 1$)

product rule: $(fg)' = f'g + fg'$

$= \tan^{-1}(xy) + x \cdot \left(\frac{1}{1+(xy)^2} \cdot \frac{\partial (xy)}{\partial x} \right)$ ($\frac{\partial (xy)}{\partial x} = y$)

- $(\tan^{-1} t)' = \frac{1}{1+t^2}$ $(= f'(t))$
- $(f(g(x)))' = f'(g(x)) \cdot g'(x)$ (Chain rule)

$f(t) = \tan^{-1}(t)$, $g(x) = xy$

$f(g(x)) = f(xy) = \tan^{-1}(xy)$

$= \tan^{-1}(xy) + \frac{xy}{1+(xy)^2}$

$f_y = \frac{\partial (x \tan^{-1}(xy))}{\partial y}$ product rule $= \frac{\partial x}{\partial y} \tan^{-1}(xy) + x \cdot \frac{\partial \tan^{-1}(xy)}{\partial y}$ ($\frac{\partial x}{\partial y} = 0$)

$= x \cdot \left(\frac{1}{1+(xy)^2} \cdot \frac{\partial (xy)}{\partial y} \right)$ ($\frac{\partial (xy)}{\partial y} = x$)

$= \frac{x^2}{1+(xy)^2}$

#

③ $f(x,y) = e^{xy} + \ln(x^2+y)$

$\Rightarrow f_x \overset{\text{chain rule}}{=} e^{xy} \cdot \frac{\partial(xy)}{\partial x} + \frac{1}{x^2+y} \cdot \frac{\partial(x^2+y)}{\partial x}$ ($\frac{\partial(xy)}{\partial x} = y$, $\frac{\partial(x^2+y)}{\partial x} = 2x$)

$= ye^{xy} + \frac{2x}{x^2+y}$

$f_y \overset{\text{chain rule}}{=} e^{xy} \cdot \frac{\partial(xy)}{\partial y} + \frac{1}{x^2+y} \cdot \frac{\partial(x^2+y)}{\partial y}$ ($\frac{\partial(xy)}{\partial y} = x$, $\frac{\partial(x^2+y)}{\partial y} = 1$)

$= xe^{xy} + \frac{1}{x^2+y}$ #

Def (Def 15.4.2)

Let $f = f(x,y,z)$. The partial derivatives of f are

$$f_x = \frac{\partial f}{\partial x} = \lim_{h\to 0} \frac{f(x+h,y,z) - f(x,y,z)}{h}$$

$$f_y = \frac{\partial f}{\partial y} = \lim_{h\to 0} \frac{f(x,y+h,z) - f(x,y,z)}{h}$$

$$f_z = \frac{\partial f}{\partial z} = \lim_{h\to 0} \frac{f(x,y,z+h) - f(x,y,z)}{h}$$

provided the limits exist.

Example

Calculate $f_x(0,1,2)$, $f_y(0,1,2)$, $f_z(0,1,2)$

① $f(x,y,z) = xy^2z^3$

$\Rightarrow f_x = y^2z^3$

$(ax)' = a$, $a = y^2z^3$

$f_y = 2xyz^3$ $\quad (by^2)' = 2by$, $b = xz^3$

$f_z = 3xy^2z^2$ $\quad (cz^3)' = 3cz^2$, $c = xy^2$

$\Rightarrow f_x(0,1,2) = 1^2 \cdot 2^3 = 8$

$f_y(0,1,2) = 2 \cdot 0 \cdot 1 \cdot 2^3 = 0$

$f_z(0,1,2) = 3 \cdot 0 \cdot 1^2 \cdot 2^2 = 0$ #

② $f(x,y,z) = x^2 e^{\frac{xy}{z}}$, $z \neq 0$

$\Rightarrow f_x = \frac{\partial x^2}{\partial x} \cdot e^{\frac{xy}{z}} + x^2 \cdot \frac{\partial e^{\frac{xy}{z}}}{\partial x} = 0$

$= 2x e^{\frac{xy}{z}}$ $\Rightarrow f_x(0,1,2) = 0$

$$f_y = x^2 \cdot \frac{\partial e^{y/z}}{\partial y} = x^2 \cdot e^{y/z} \cdot \frac{\partial \frac{y}{z}}{\partial y}$$

$$= x^2 \cdot e^{y/z} \cdot \frac{1}{z} \Rightarrow f_y(0,1,2) = 0$$

$$f_z = x^2 \frac{\partial e^{y/z}}{\partial z} = x^2 \cdot e^{\frac{y}{z}} \frac{\partial \frac{y}{z}}{\partial z}$$

$$= x^2 \cdot e^{y/z} \left(-\frac{y}{z^2}\right) \Rightarrow f_z(0,1,2) = 0$$

*

Remark

If $f = f(x_1, x_2, x_3, \cdots, x_n)$, then

$$f_{x_k} = \frac{\partial f}{\partial x_k}$$

$$= \lim_{h \to 0} \frac{f(x_1, \cdots x_{k-1}, x_k + h, x_{k+1}, \cdots, x_n) - f(x_1, \cdots, x_n)}{h}$$

e.g. $f(x,y,z,t) = xy + z^2 t$

$$f_t = \frac{\partial (xy)}{\partial t}^{=0} + \frac{\partial (z^2 t)}{\partial t} = z^2$$

Def (Higher order partial derivative, p.782)

A second order partial derivative

of f is a partial derivative of a partial derivative of f

More generally, a k-th order partial derivative of f is a partial derivative of a $(k-1)$-th order partial derivative of f

Example

$$f(x,y,z) = x y^2 z^3$$

$$\Rightarrow f_x = y^2 z^3, \quad f_y = 2xy z^3, \quad f_z = 3xy^2 z^2$$

2n order partial derivatives:

$$(f_x)_x = f_{xx} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial y^2 z^3}{\partial x} = 0$$

$$f_{xy} = (f_x)_y = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial (y^2 z^3)}{\partial y} = 2yz^3$$

$$f_{xz} = (f_x)_z = \frac{\partial^2 f}{\partial z \partial x} = \frac{\partial (y^2 z^3)}{\partial z} = 3y^2 z^2$$

$$f_{yx} = (f_y)_x = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial (2xyz^3)}{\partial x} = 2yz^3$$

$$f_{yy} = \frac{\partial^2 f}{\partial y^2} = 2xz^3$$

$$f_{yz} = \frac{\partial^2 f}{\partial z \partial y} = 6xyz^2$$

$$f_{zx} = (f_z)_x = \frac{\partial^2 f}{\partial x \partial z} = 3y^2z^2$$

$$f_{zy} = \frac{\partial^2 f}{\partial y \partial z} = 6xyz^2$$

$$f_{zz} = \frac{\partial^2 f}{\partial z^2} = 6xy^2z$$

3rd order : 27 , e.g. $f_{xy^2} = \frac{\partial^3 (xy^2z^3)}{\partial y^2 \partial x}$

$$= \frac{\partial^2 (y^2z^3)}{\partial y^2} = 2z^3$$

Q: Do we have

$$(f_{x_i})_{x_j} = (f_{x_j})_{x_i}$$

in general ?

A: We need an assumption.

(continuity)

Recall $f = f(x)$ is continuous at $x = c$

$$\Leftrightarrow \lim_{x \to c} f(x) = f(c)$$

Def (§15.6)

Let $f(\vec{x}) = f(x_1, x_2)$ $\quad (\vec{x} = (x_1, x_2))$

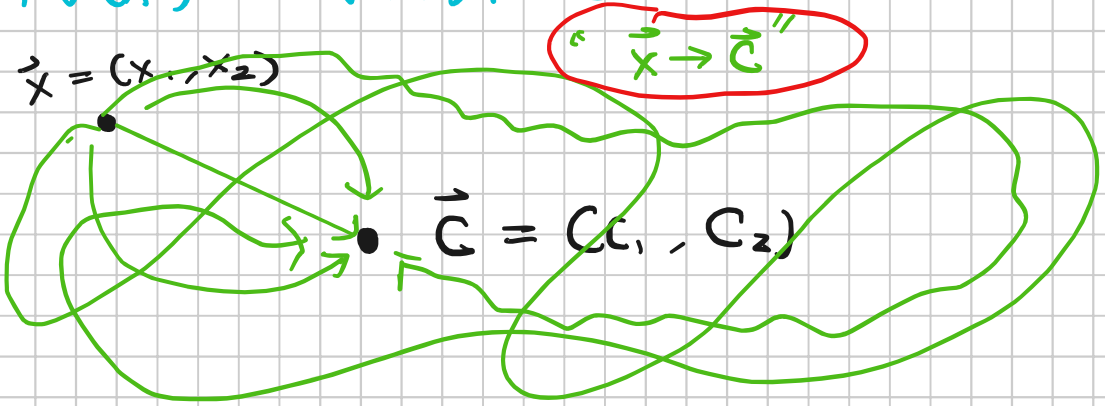
f is <u>continuous</u> at $\vec{x} = \vec{c}$ if

$$\lim_{\vec{x} \to \vec{c}} f(\vec{x}) = f(\vec{c})$$

?? $(x_1, x_2) \longrightarrow (c_1, c_2)$??

The definition of limits of this type is difficult: $\forall \varepsilon > 0 \ \exists \delta > 0$ s.t.

$|f(\vec{x}) - f(\vec{c})| < \varepsilon$ when $(0 <) \|\vec{x} - \vec{c}\| < \delta$



Example

The following functions are continuous on $\mathbb{R}^2$ (or $\mathbb{R}^3$)

① Constant functions (e.g. $f(x,y) = 1 \quad \forall x, y \in \mathbb{R}$)

② Polynomials: e.g. x, y, $x+y$, xy, $x+y^2+xy^2+x^3y^2+1$, etc.

③ If $P(x,y)$ is continuous, then

$e^{P(x,y)}$, $\sin P(x,y)$, $\cos P(x,y)$

and combination of them are continuous.

e.g. $e^{x^2+y} \cdot \cos(x^4+y^2-1+xy^3) + e^{\cos(\sin(e^{x^2+y}))}$

Thm (P. 783)

Let $f = f(x,y)$. Suppose

f, f_x, f_y, f_{xy}, f_{yx}

are continuous on $\mathbb{R}^2$. Then

$$f_{xy} = f_{yx} \quad \text{on } \mathbb{R}^2$$

Remark

If f is smooth, i.e. its partial derivatives of any order exist and are all continuous,

then $f_{xyy} = (f_{xy})_y \overset{\text{Thm}}{=} (f_{yx})_y = f_{yxy}$

$$= (f_y)_{xy} \overset{\text{Thm}}{=} (f_y)_{yx} = f_{y^2x}$$

$$f_{xyz} = (f_{xy})_z \overset{\text{Thm}}{=} (f_{yx})_z = f_{yxz}$$

$$= (f_y)_{xz} \overset{\text{Thm}}{=} (f_y)_{zx} = f_{yzx}$$

$$= \cdots$$

Gradient

For convenience, unless otherwise stated, we assume the functions are smooth.

"partial derivative"

$\updownarrow$

"total derivative"

$\parallel$

"collecting all the first order partial derivatives"

Def (Def 16.1.2, 16.1.3)

Let $f = f(\vec{x}) = f(x, y, z)$.

The gradient of f at $\vec{x} \in \mathbb{R}^3$ is the vector

$$\nabla f(\vec{x}) = \left(f_x(\vec{x}),\ f_y(\vec{x}),\ f_z(\vec{x}) \right) \in \mathbb{R}^3$$

$$= \frac{\partial f}{\partial x}(\vec{x})\,\vec{i} + \frac{\partial f}{\partial y}(\vec{x})\,\vec{j} + \frac{\partial f}{\partial z}(\vec{x})\,\vec{k}$$

If $f = f(x,y)$, then

$$\nabla f(\vec{x}) = \left(f_x(\vec{x}),\ f_y(\vec{x}) \right) \in \mathbb{R}^2$$

Example

$$f(x,y,z) = x y^2 z^3$$

$$f_x = y^2 z^3, \quad f_y = 2xyz^3, \quad f_z = 3xy^2 z^3$$

$$\Rightarrow \nabla f = \left(y^2 z^3,\ 2xyz^3,\ 3xy^2 z^3 \right)_{\in \mathbb{R}^3}$$

$$\nabla f(0,1,2) = (8,\ 0,\ 0) \in \mathbb{R}^3$$

$$= 8\vec{i}$$

#