Calculus, week 8, Fall 2026

We have considered real-valued functions of one variable.

$$f: \mathbb{R} \to \mathbb{R}$$

e.g. $f(x) = \cos e^x$, $f(x) = x^2 + x - 1$

From now on, we will consider more general functions $f: \mathbb{R}^n \to \mathbb{R}^m$.

In particular, we will consider

- $f: \mathbb{R} \to \mathbb{R}^3$ (or $\mathbb{R}^2$) —
  curves in $\mathbb{R}^3$ or vector-valued functions
- $f: \mathbb{R}^3$ (or $\mathbb{R}^2$) $\to \mathbb{R}$ —
  multivariable real-valued functions
  e.g. $f(x,y) = \cos x \cdot e^y + 6xy^2$

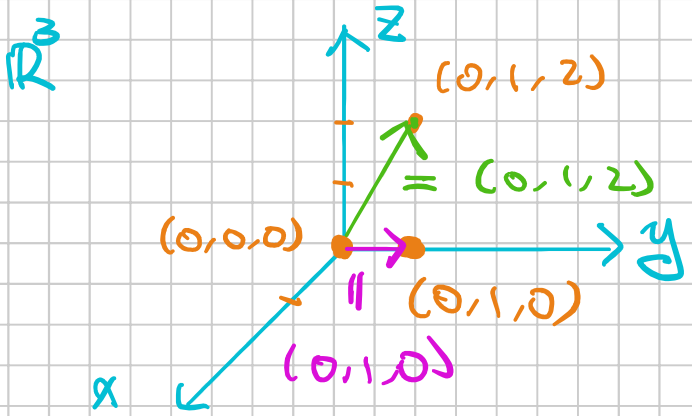
We start with a study of $\mathbb{R}^3$.

§ Vectors in a 3-dimensional space

A point in $\mathbb{R}^3$ is usually represented by a triple (a_1, a_2, a_3) of real numbers

e.g. $(0,0,0)$, $(0,1,0)$, $(0,1,2) \in \mathbb{R}^3$

Such a triple also can be considered as the vector from $(0,0,0)$ to (a_1, a_2, a_3)



Def (§13.2)

A vector $\vec{a}$ in $\mathbb{R}^3$ is an ordered triple of real numbers:

$$\vec{a} = (a_1, a_2, a_3) \in \mathbb{R}^3$$

Let $\vec{a} = (a_1, a_2, a_3)$, $\vec{b} = (b_1, b_2, b_3) \in \mathbb{R}^3$

$r \in \mathbb{R}$. (r is a real number)

(i) Equal:

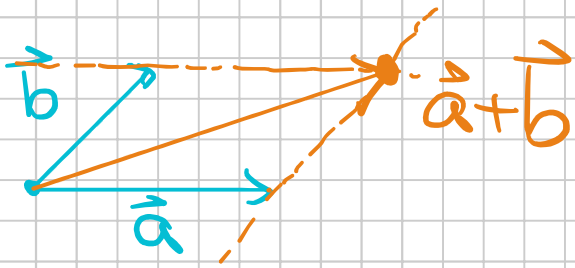
$$\vec{a} = \vec{b} \iff \begin{cases} a_1 = b_1 \\ a_2 = b_2 \\ a_3 = b_3 \end{cases}$$

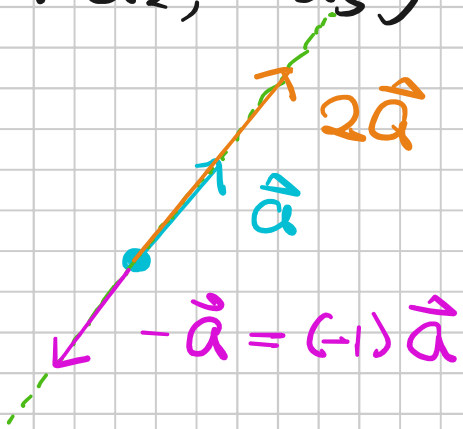
(ii) Addition:

$$\vec{a} + \vec{b} = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$$

(iii) Scalar product:

$$r \cdot \vec{a} = (ra_1, ra_2, ra_3)$$





Common notations:

$$-\vec{a} = (-1)\vec{a} = (-a_1, -a_2, -a_3)$$

$$\vec{a} - \vec{b} = \vec{a} + (-1)\vec{b}$$

$$= (a_1 - b_1,\ a_2 - b_2,\ a_3 - b_3)$$

$$\vec{0} = (0, 0, 0)$$

Thm

Let $\vec{a} = (a_1, a_2, a_3)$, $\vec{b} = (b_1, b_2, b_3)$, $\vec{c} = (c_1, c_2, c_3) \in \mathbb{R}^3$

and $\alpha, \beta \in \mathbb{R}$

Then

(a) $\vec{a} + \vec{b} = \vec{b} + \vec{a}$

$(a_1+b_1, a_2+b_2, a_3+b_3) = (b_1+a_1, b_2+a_2, b_3+a_3)$

means:
$$\begin{cases} (a_1+b_1)+c_1 = a_1+(b_1+c_1) \\ (a_2+b_2)+c_2 = a_2+(b_2+c_2) \\ (a_3+b_3)+c_3 = a_3+(b_3+c_3) \end{cases}$$

means:
$$\begin{cases} a_1+b_1 = b_1+a_1 \\ a_2+b_2 = b_2+a_2 \\ a_3+b_3 = b_3+a_3 \end{cases}$$

(ii) $(\vec{a}+\vec{b})+\vec{c} = \vec{a}+(\vec{b}+\vec{c})$

(iii) $\vec{a}+\vec{0} = \vec{a} = \vec{0}+\vec{a}$

(iv) $\vec{a}+(-\vec{a}) = \vec{0}$

(v) $1\cdot\vec{a} = \vec{a}$

(vi) $(\alpha\beta)\cdot\vec{a} = \alpha(\beta\vec{a})$

(vii) $\alpha(\vec{a}+\vec{b}) = \alpha\vec{a}+\alpha\vec{b}$

(viii) $(\alpha+\beta)\vec{a} = \alpha\vec{a}+\beta\vec{a} = \begin{pmatrix}\alpha a_1+\beta a_1,\\ \alpha a_2+\beta a_2,\\ \alpha a_3+\beta a_3\end{pmatrix}$

$= (\alpha+\beta)a_1, (\alpha+\beta)a_2, (\alpha+\beta)a_3)$

means: $\begin{cases}(\alpha+\beta)a_1 = \alpha a_1+\beta a_1\\ (\alpha+\beta)a_2 = \alpha a_2+\beta a_2\\ (\alpha+\beta)a_3 = \alpha a_3+\beta a_3\end{cases}$

Example

Let $\vec{a} = (1,-1,2)$, $\vec{b} = (2,3,-1)$

$\vec{c} = (0,1,0)$

① $\vec{a}-\vec{b} = (1-2, -1-3, 2-(-1))$

$= (-1, -4, 3)$ #

② $2\vec{a}+3\vec{b}-4\vec{c} = 3\vec{b}+2(\vec{a}-2\vec{c})$

$a-2\vec{c} = (1-2\cdot0, -1-2\cdot1, 2-2\cdot0)$

$= (1, -3, 2)$

$$\Rightarrow 2\vec{a}+3\vec{b}-4\vec{c} = 3\vec{b}+2(\vec{a}-2\vec{c})$$

$$= 3\cdot(2,3,-1)+2\cdot(1,-3,2)$$

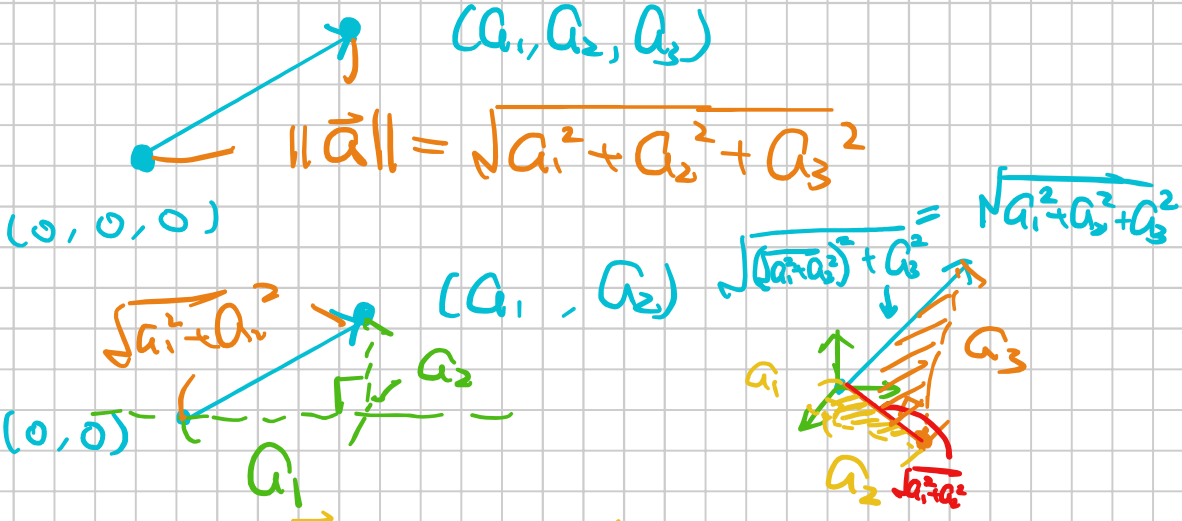
$$= (3\cdot 2+2\cdot 1,\ 3\cdot 3+2\cdot(-3),\ 3\cdot(-1)+2\cdot 2)$$

$$= (8,\ 3,\ 1) \quad \#$$

Def

The norm (or magnitude, length) of the vector $\vec{a}=(a_1,a_2,a_3)$ is

$$\|\vec{a}\| = \sqrt{a_1^2+a_2^2+a_3^2}$$



Thm means: $\vec{a}, \vec{b}$ are vectors in $\mathbb{R}^3$

Let $\vec{a}, \vec{b} \in \mathbb{R}^3$, $\alpha \in \mathbb{R}$

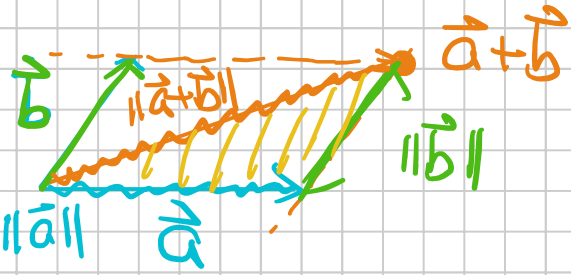
(i) $\|\vec{a}\| \geq 0$ $\quad \sqrt{a_1^2+a_2^2+a_3^2} = 0 \Leftrightarrow a_1^2+a_2^2+a_3^2 = 0$

(ii) $\|\vec{a}\| = 0 \iff \vec{a} = \vec{0}$ $\quad \Leftrightarrow a_1=a_2=a_3=0$

(iii) $\|\alpha \cdot \vec{a}\| = |\alpha| \, \|\vec{a}\|$

$\|\alpha \cdot \vec{a}\| = \sqrt{(\alpha a_1)^2 + (\alpha a_2)^2 + (\alpha a_3)^2} = |\alpha| \sqrt{a_1^2 + a_2^2 + a_3^2}$

(iv) $\|\vec{a} + \vec{b}\| \leq \|\vec{a}\| + \|\vec{b}\|$ (Triangle inequality 三角不等式)



Example

Let $\vec{a} = (1, -1, 2)$, $\vec{b} = (2, 3, -1)$

① $\|\vec{a}\| = \sqrt{1^2 + (-1)^2 + 2^2} = \sqrt{6}$

② $\|\vec{b}\| = \sqrt{2^2 + 3^2 + (-1)^2} = \sqrt{14}$

③ $\|\vec{a} + \vec{b}\| = \|(1+2, -1+3, 2+(-1))\|$

$= \|(3, 2, 1)\| = \sqrt{3^2 + 2^2 + 1^2} = \sqrt{14}$

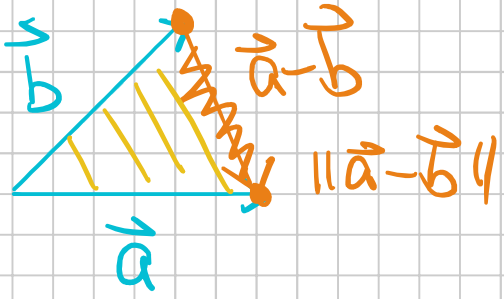
NOTE: In this example,

$\|\vec{a} + \vec{b}\| = \sqrt{14} < \sqrt{6} + \sqrt{14} = \|\vec{a}\| + \|\vec{b}\|$

④ $\|\vec{a} - \vec{b}\| = \|(1-2, -1-3, 2-(-1))\|$

$= \|(-1, -4, 3)\| = \sqrt{1 + 16 + 9}$

$= \sqrt{26} > \|\vec{a}\| - \|\vec{b}\| = \sqrt{6} - \sqrt{14}$



$\|\vec{a}-\vec{b}\| \geq \|\vec{a}\| - \|\vec{b}\|$

$\Leftrightarrow \|\vec{a}-\vec{b}\| + \|\vec{b}\| \geq \|\vec{a}\|$

Def (Def 13.3.1)

For $\vec{a}=(a_1,a_2,a_3)$, $\vec{b}=(b_1,b_2,b_3) \in \mathbb{R}^3$, we define the dot product $\vec{a}\cdot\vec{b}$ by

$$\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$$

e.g.

$$(1,0,1)\cdot(1,1,0) = 1\cdot1+0\cdot1+1\cdot0 = 1$$

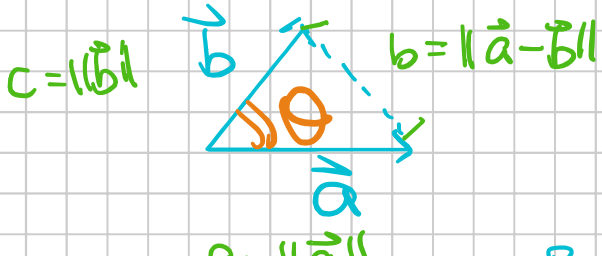
The geometric meaning of $\vec{a}\cdot\vec{b}$ is:

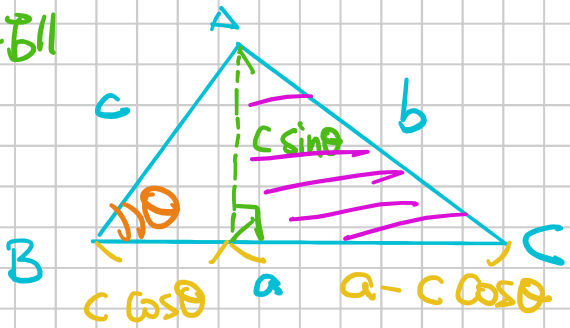
Thm

Let θ be the angle from $\vec{a}$ to $\vec{b}$.

Then

$$\vec{a}\cdot\vec{b} = \|\vec{a}\|\cdot\|\vec{b}\|\cdot\cos\theta$$





pf

Recall: the law of cosine

$$b^2 = a^2+c^2-2ac\cos\theta$$

$$b^2 = (c\sin\theta)^2 + (a-c\cos\theta)^2$$

$$= c^2\sin^2\theta + a^2 + c^2\cos^2\theta - 2ac\cos\theta$$

$$\Rightarrow \|\vec{a}-\vec{b}\|^2 = \|\vec{a}\|^2 + \|\vec{b}\|^2 - 2\|\vec{a}\|\|\vec{b}\|\cos\theta$$

$$\Rightarrow \|\vec{a}\|\|\vec{b}\|\cos\theta = \frac{1}{2}\left(\|\vec{a}\|^2+\|\vec{b}\|^2-\|\vec{a}-\vec{b}\|^2\right)$$

$$= \frac{1}{2}\Big((a_1^2+a_2^2+a_3^2) + (b_1^2+b_2^2+b_3^2) - ((a_1-b_2)^2 + (a_2-b_2)^2 + (a_3-b_3)^2)\Big)$$

$$= \frac{1}{2}\Big(a_1^2+a_2^2+a_3^2+b_1^2+b_2^2+b_3^2 - a_1^2+2a_1b_1-b_1^2 - a_2^2+2a_2b_2-b_2^2 - a_3^2+2a_3b_3-b_3^2\Big)$$

$$= a_1b_1 + a_2b_2 + a_3b_3 = \vec{a}\cdot\vec{b}$$

So $\vec{a}\cdot\vec{b} = \|\vec{a}\|\|\vec{b}\|\cos\theta$ #

In particular,

$\vec{a} \perp \vec{b}$ (meaning $\theta = 90^\circ$ or 270°)

$\Leftrightarrow \vec{a}\cdot\vec{b} = 0$

Example

① The vectors $(2,1,1)$ and $(1,-1,-3)$ are perpendicular because

垂直的

$$(2,1,1)\cdot(1,1,-3)$$

$$= 2\cdot 1 + 1\cdot 1 + 1\cdot(-3) = 2+1-3 = 0$$

② Let θ be the angle between the vectors $(1,1,0)$ and $(0,1,1)$.

$\theta = ?$

Sol

$$(1,1,0)\cdot(0,1,1) = 1\cdot 0 + 1\cdot 1 + 0\cdot 1 = 1$$

Thm $= \|(1,1,0)\|\,\|(0,1,1)\|\cos\theta$

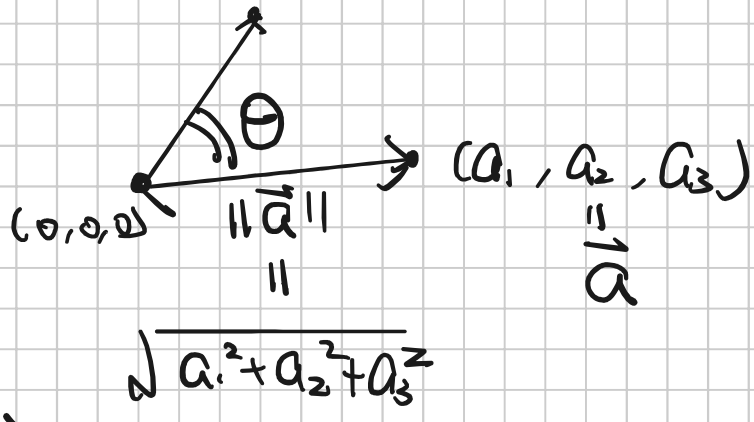
$$= \sqrt{1^2+1^2+0^2}\cdot\sqrt{0^2+1^2+1^2}\cdot\cos\theta$$

$$\Rightarrow \sqrt{2}\cdot\sqrt{2}\cdot\cos\theta = 1 \Rightarrow \cos\theta = \frac{1}{2}$$

$$\Rightarrow \theta = \frac{\pi}{3} \qquad \#$$

Recall

$\vec{b} = (b_1, b_2, b_3)$



$$\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2 + a_3b_3 = \|\vec{a}\|\,\|\vec{b}\|\cos\theta$$

Thm (§13.3)

Let $\vec{a}, \vec{b}, \vec{c} \in \mathbb{R}^3$, $\alpha, \beta \in \mathbb{R}$. Then

(i) $\vec{a} \cdot \vec{a} = \|\vec{a}\|^2$

$= a_1 a_1 + a_2 a_2 + a_3 a_3$

(ii) $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$

(iii) $\vec{a} \cdot (\alpha \vec{b} + \beta \vec{c}) = \vec{a} \cdot (\alpha \vec{b}) + \vec{a} \cdot (\beta \vec{c})$

$= \alpha(\vec{a} \cdot \vec{b}) + \beta(\vec{a} \cdot \vec{c})$

NOTE: $(\alpha \vec{b} + \beta \vec{c}) \cdot \vec{a} \overset{(ii)}{=} \vec{a} \cdot (\alpha \vec{b} + \beta \vec{c})$

$\overset{(iii)}{=} \alpha(\vec{a} \cdot \vec{b}) + \beta(\vec{a} \cdot \vec{c}) \overset{(ii)}{=} \alpha(\vec{b} \cdot \vec{a}) + \beta(\vec{c} \cdot \vec{a})$

Example

Suppose

$\|\vec{a}\| = 1$, $\|\vec{b}\| = 3$, $\|\vec{c}\| = 4$

$\vec{a} \cdot \vec{b} = 0$, $\vec{a} \cdot \vec{c} = 1$, $\vec{b} \cdot \vec{c} = -2$

Find

① $3\vec{a} \cdot (\vec{b} + 4\vec{c}) = ?$

② $(\vec{a} - \vec{b}) \cdot (2\vec{a} + \vec{b}) = ?$

③ $((\vec{b} \cdot \vec{c})\vec{a} - (\vec{a} \cdot \vec{c})\vec{b}) \cdot \vec{c} = ?$

Sol

① $3\vec{a} \cdot (\vec{b} + 4\vec{c}) \overset{(iii)}{=} 3\vec{a} \cdot \vec{b} + 3\vec{a} \cdot (4\vec{c})$

$\overset{(iii)}{=} 3\,\vec{a} \cdot \vec{b} + (3 \cdot 4)\,\vec{a} \cdot \vec{c}$

$= 3 \times 0 + 12 \times 1 = 12$ #

② $(\vec{a} - \vec{b}) \cdot (2\vec{a} + \vec{b})$

$$= \vec{a}\cdot(2\vec{a}) + \vec{a}\cdot\vec{b} - \vec{b}\cdot(2\vec{a}) - \vec{b}\cdot\vec{b} \quad \text{(iii)}$$

$$= 2\|\vec{a}\|^2 + \vec{a}\cdot\vec{b} - 2\vec{b}\cdot\vec{a} - \|\vec{b}\|^2 \quad \text{(iii) + (i)}$$

$\|\vec{a}\|^2 = 1^2$, $\vec{a}\cdot\vec{b} = 0$, $\vec{b}\cdot\vec{a} = \vec{a}\cdot\vec{b} = 0$ (iii), $\|\vec{b}\|^2 = 3^2$

$$= 2 - 9 = -7 \quad \#$$

③ $((\vec{b}\cdot\vec{c})\vec{a} - (\vec{a}\cdot\vec{c})\vec{b})\cdot\vec{c}$

$\vec{b}\cdot\vec{c} = -2$, $\vec{a}\cdot\vec{c} = 1$

$$= (-2\vec{a} - \vec{b})\cdot\vec{c}$$

$$= -2\vec{a}\cdot\vec{c} - \vec{b}\cdot\vec{c} \quad (\vec{a}\cdot\vec{c} = 1,\ \vec{b}\cdot\vec{c} = -2)$$

$$= -2 - (-2) = 0 \quad \#$$

Remark

All the definitions and properties about vectors can be established for vectors in $\mathbb{R}^2$, and even in $\mathbb{R}^n$.

e.g. $\vec{c} = (c_1, \cdots, c_{10})$

$\vec{a} = (a_1, a_2, a_3, \cdots, a_{10}) \in \mathbb{R}^{10}$

$\vec{b} = (b_1, b_2, \cdots, b_{10}) \in \mathbb{R}^{10}$

$\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2 + \cdots + a_{10}b_{10}$

$\|\vec{a}\| = \sqrt{\vec{a}\cdot\vec{a}} \Rightarrow \|\vec{a}+\vec{b}\| \le \|\vec{a}\| + \|\vec{b}\|$

$\vec{a}\cdot(\vec{b}+\vec{c}) = \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c}$

Remark

In books, some people use "i, j, k" for vectors in $\mathbb{R}^3$:

$$i = \vec{i} = (1,0,0)$$
$$j = \vec{j} = (0,1,0)$$
$$k = \vec{k} = (0,0,1)$$

$$\Rightarrow \vec{a} = (a_1, a_2, a_3)$$
$$= (a_1, 0, 0) + (0, a_2, 0) + (0, 0, a_3)$$
$$= a_1(1,0,0) + a_2(0,1,0) + a_3(0,0,1)$$
$$= a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$$

$\vec{a} \cdot \vec{b}$

$\vec{i}\cdot\vec{j} = 0,\ \vec{i}\cdot\vec{k} = 0,\ \vec{i}\cdot\vec{k} = 0$
$\vec{i}\cdot\vec{i} = 1,\ \vec{j}\cdot\vec{j} = 1,\ \vec{k}\cdot\vec{k} = 1$

$$= (a_1\vec{i} + a_2\vec{j} + a_3\vec{k}) \cdot (b_1\vec{i} + b_2\vec{j} + b_3\vec{k})$$
$$= a_1b_1\vec{i}\cdot\vec{i} + a_1b_2\vec{i}\cdot\vec{j} + a_1b_3\vec{i}\cdot\vec{k}$$
$$+ a_2b_1\vec{j}\cdot\vec{i} + a_2b_2\vec{j}\cdot\vec{j} + a_2b_3\vec{j}\cdot\vec{k}$$
$$+ a_3b_1\vec{k}\cdot\vec{i} + a_3b_2\vec{k}\cdot\vec{j} + a_3b_3\vec{k}\cdot\vec{k}$$
$$= a_1b_1 + a_2b_2 + a_3b_3$$

Limit and vector derivative

Consider $\vec{f}: \mathbb{R} \to \mathbb{R}^3$. Such a function can be expressed as

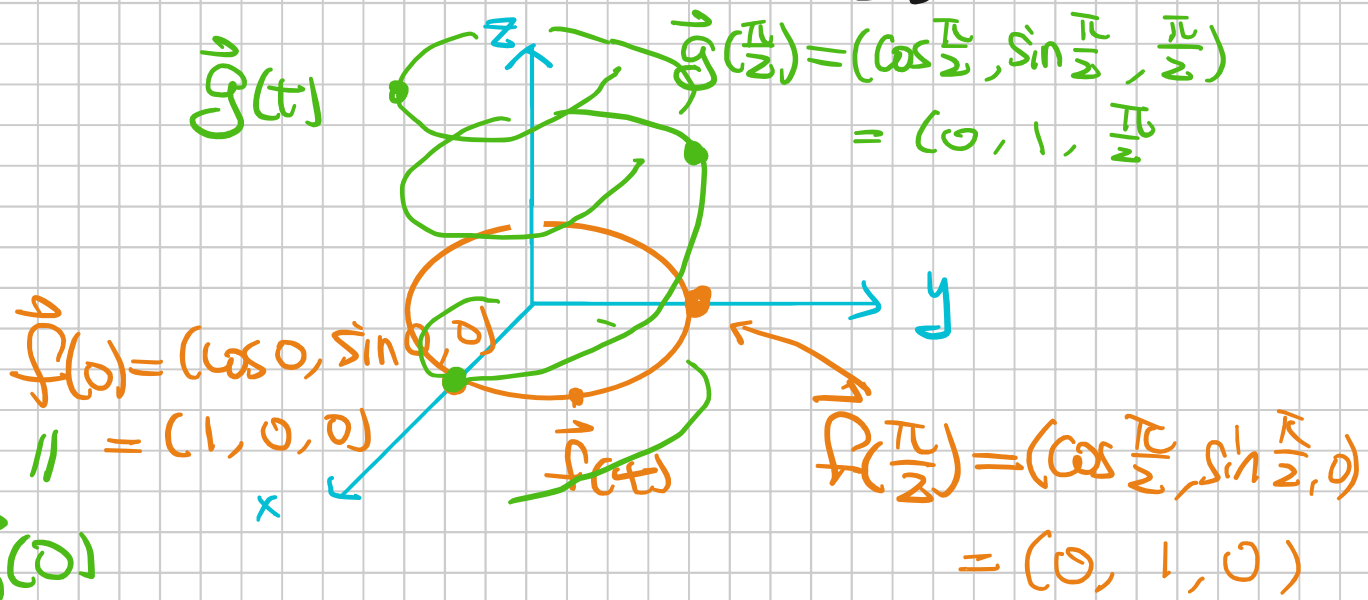
$$\vec{f}(t) = (f_1(t), f_2(t), f_3(t))$$
$$= f_1(t)\,\vec{i} + f_2(t)\,\vec{j} + f_3(t)\,\vec{k}$$

where $f_1, f_2, f_3 : \mathbb{R} \to \mathbb{R}$.

Example

① $\vec{f}(t) = \cos t\,\vec{i} + \sin t\,\vec{j}$

$$= (\cos t, \sin t, 0)$$



② $\vec{g}(t) = (\cos t, \sin t, t)$

$$= \cos t\,\vec{i} + \sin t\,\vec{j} + t\,\vec{k}$$

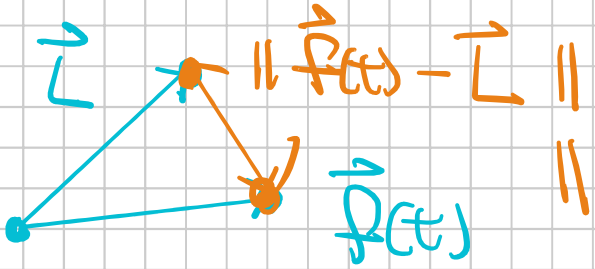
Def (Def 14.1.1)

Let $\vec{f}(t)$ be a function valued in $\mathbb{R}^3$. and $t_0 \in \mathbb{R}$. We say the limit

$\lim_{t \to t_0} \vec{f}(t) \left(= \vec{L}\right)$ exists if

there exists $\vec{L} \in \mathbb{R}^3$ (L_1, L_2, L_3) s.t.

$$\lim_{t \to t_0} \|\vec{f}(t) - \vec{L}\| = 0$$



$$\sqrt{(f_1(t)-L_1)^2 + (f_2(t)-L_2)^2 + (f_3(t)-L_3)^2}$$

In this case, we write

$$\lim_{t \to t_0} \vec{f}(t) = \vec{L}$$

Thm (14.1.4) ⊛

Let $\vec{L} = (L_1, L_2, L_3) \in \mathbb{R}^3$ and

$\vec{f}(t) = (f_1(t), f_2(t), f_3(t))$, $t_0 \in \mathbb{R}$

Then $\lim_{t\to t_0} \vec{f}(t) = \vec{L}$

$\Leftrightarrow$

$$\begin{cases} \lim_{t\to t_0} f_1(t) = L_1 \\ \lim_{t\to t_0} f_2(t) = L_2 \\ \lim_{t\to t_0} f_3(t) = L_3 \end{cases}$$

pf

$$\lim_{t\to t_0} \vec{f}(t) = \vec{L}$$

$$\Leftrightarrow \lim_{t\to t_0} \| \vec{f}(t) - \vec{L} \| = 0$$

$$= \lim_{t\to t_0} \sqrt{(f_1(t)-L_1)^2 + (f_2(t)-L_2)^2 + (f_3(t)-L_3)^2} = 0$$

Since

$$\sqrt{(f_1(t)-L_1)^2 + (f_2(t)-L_2)^2 + (f_3(t)-L_3)^2} \to 0$$

$$\| \quad \|$$

$$\sqrt{(f_1(t)-L_1)^2 + 0^2 + 0^2}$$

$$0 \le |f_1(t) - L_1|$$

we have

$$\lim_{t \to t_0} |f_1(t) - L_1| = 0$$

$$\Rightarrow \lim_{t \to t_0} f_1(t) = L_1$$

Similarly, we also have

$$\lim_{t \to t_0} f_2(t) = L_2$$

$$\lim_{t \to t_0} f_3(t) = L_3$$

The converse direction follows from the properties of limits for real-valued functions. #

Example

Let

$$\vec{f}(t) = \cos(t+\pi)\,\vec{i} + \sin(t+\pi)\,\vec{j} + e^{-t^2}\vec{k}$$

$$= \left(\underbrace{\cos(t+\pi)}_{f_1(t)},\ \underbrace{\sin(t+\pi)}_{f_2(t)},\ \underbrace{e^{-t^2}}_{f_3(t)}\right)$$

Then

$$\lim_{t\to 0} \vec{f}(t) \overset{\text{Thm}}{=} \left(\lim_{t\to 0} f_1(t),\ \lim_{t\to 0} f_2(t),\ \lim_{t\to 0} f_3(t)\right)$$

$$= \left(\lim_{t\to 0} \cos(\pi+t),\ \lim_{t\to 0} \sin(\pi,t),\ \lim_{t\to 0} e^{-t^2}\right)$$

NOTE:

$\because \cos x$ is continuous

$$\lim_{t\to 0} \cos(\pi+t) \overset{\downarrow}{=} \cos(\pi+0)$$

$$= \cos\pi = -1$$

$\because \sin x$ is continuous

$$\lim_{t\to 0} \sin(\pi+t) \overset{\downarrow}{=} \sin(\pi+0) = 0$$

$\because e^x$ is continuous

$$\lim_{t\to 0} e^{-t^2} \overset{\downarrow}{=} e^{\lim_{t\to 0}(-t^2)} = e^0 = 1$$

So $\lim_{t\to 0} \vec{f}(t) = (-1, 0, 1)$ ✻

Thm (Thm 14.1.3)

Let $\vec{f}(t)$, $\vec{g}(t)$ be vector-valued functions, and u a real-valued function

Suppoe that $\alpha, \beta \in \mathbb{R}$

$$\lim_{t\to t_0} \vec{f}(t) = \vec{L}, \quad \lim_{t\to t_0} \vec{g}(t) = \vec{M}$$

$$\lim_{t\to t_0} u(t) = A$$

Then

$$\lim_{t\to t_0} (\alpha \vec{f}(t) + \beta \vec{g}(t)) = \alpha \vec{L} + \beta \vec{M}$$

$$\left(= \alpha \lim_{t\to t_0} \vec{f}(t) + \beta \lim_{t\to t_0} \vec{g}(t)\right)$$

$$\lim_{t\to t_0} \left(\alpha f_1(t) + \beta g_1(t),\ \alpha f_2(t) + \beta g_2(t),\ \alpha f_3(t) + \beta g_3(t)\right)$$

$$\overset{\text{thm}}{=} \Big(\lim_{t\to t_0} (\alpha f_1(t) + \beta g_1(t)),$$

$$\lim_{t\to t_0} (\alpha f_2(t) + \beta g_2(t)),$$

$$\lim_{t\to t_0} (\alpha f_3(t) + \beta g_3(t)) \Big)$$

$$\overset{\text{By Calculus I}}{=} \Big(\alpha \underbrace{\lim_{t\to t_0} f_1(t)}_{=L_1} + \beta \underbrace{\lim_{t\to t_0} g_1(t)}_{M_1},$$

$$\alpha \underbrace{\lim_{t\to t_0} f_2(t)}_{L_2} + \beta \underbrace{\lim_{t\to t_0} g_2(t)}_{M_2},$$

$$\alpha \underbrace{\lim_{t\to t_0} f_3(t)}_{L_3} + \beta \underbrace{\lim_{t\to t_0} g_3(t)}_{M_3}$$

$$= \alpha(L_1, L_2, L_3) + \beta(M_1, M_2, M_3)$$

$$= \alpha \vec{L} + \beta \vec{M}$$

$$\lim_{t\to t_0} \left(u(t)\, \vec{f}(t) \right) = A \vec{L}$$

$$= \left(\lim_{t\to t_0} u(t) \right) \left(\lim_{t\to t_0} \vec{f}(t) \right)$$

$$\lim_{t\to t_0} \| \vec{f}(t) \| = \| \vec{L} \|$$

$$= \| \lim_{t\to t_0} \vec{f}(t) \|$$

$$\lim_{t\to t_0} \vec{f}(t) \cdot \vec{g}(t) = \vec{L} \cdot \vec{M}$$

$$= \left(\lim_{t\to t_0} \vec{f}(t) \right) \cdot \left(\lim_{t\to t_0} \vec{g}(t) \right)$$

Def

A vector-valued function $\vec{f}$ is continuous at c if

$$\lim_{t \to c} \vec{f}(t) = \vec{f}(c)$$

Remark

By Thm ⊛, a function

$$\vec{f}(t) = (f_1(t), f_2(t), f_3(t))$$

is continuous at c

$\Leftrightarrow \lim_{t \to c} \vec{f}(t) = \vec{f}(c)$

Thm ⊛ //

$$(f_1(c), f_2(c), f_3(c))$$

$$\left(\lim_{t \to t_0} f_1(t), \lim_{t \to t_0} f_2(t), \lim_{t \to t_0} f_3(t)\right)$$

$\Leftrightarrow$ f_1, f_2, f_3 are continuous at c.

Recall

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$ (from Calculus I)

Def (Def 14.1.5)

A vector-valued function is differentiable if

$$\lim_{h\to 0} \frac{\vec{f}(t+h) - \vec{f}(t)}{h}$$

exists. If the limit exists, it is called the derivative of $\vec{f}$ at t and is denoted by $\vec{f}'(t)$ or $\frac{d\vec{f}}{dt}$

Thm (Page 697)

Let $\vec{f}(t) = (f_1(t), f_2(t), f_3(t))$.

Then $\vec{f}$ is differentiable at t

$\Leftrightarrow$ f_1, f_2, f_3 are differentiable at t

In this case,

$$\vec{f}'(t) = (f_1'(t), f_2'(t), f_3'(t))$$

pf

By Thm ⊛,

$$\vec{f}'(t) = \lim_{h\to 0} \frac{\vec{f}(t+h) - \vec{f}(t)}{h} \quad \text{(Def)}$$

$$= \lim_{h\to 0}\left(\frac{f_1(t+h)-f_1(t)}{h}, \frac{f_2(t+h)-f_2(t)}{h}, \frac{f_3(t+h)-f_3(t)}{h}\right)$$

Thm ⊛

$$= \left(\lim_{h\to 0}\frac{f_1(t+h)-f_1(t)}{h}, \lim_{h\to 0}\frac{f_2(t+h)-f_2(t)}{h}, \lim_{h\to 0}\frac{f_3(t+h)-f_3(t)}{h}\right)$$

$$= (f_1'(t), f_2'(t), f_3'(t)) \quad \#$$

Example

① $\vec{f}(t) = (1, 2, 3) \quad \forall t$

$\Rightarrow \vec{f}'(t) = (1', 2', 3') = (0, 0, 0)$

② $\vec{g}(t) = (t, t^2, -e^t)$

$\Rightarrow \vec{g}'(t) = ((t)', (t^2)', (-e^t)')$

$(t)' = 1 \quad (t^2)' = 2t \quad (-e^t)' = -e^t$

$$= (1, 2t, -e^t) \quad \#$$

National Tsing Hua University

Calculus II – Exam 1 – Solutions

Instructor: Hsuan-Yi Liao

October 13, 2026

Common mistakes:
2(c) $\int \cos(e^x)\,dx \neq \sin(e^x) + C$
5(a) 未確認積分函數是否有不連續點

Name: _____

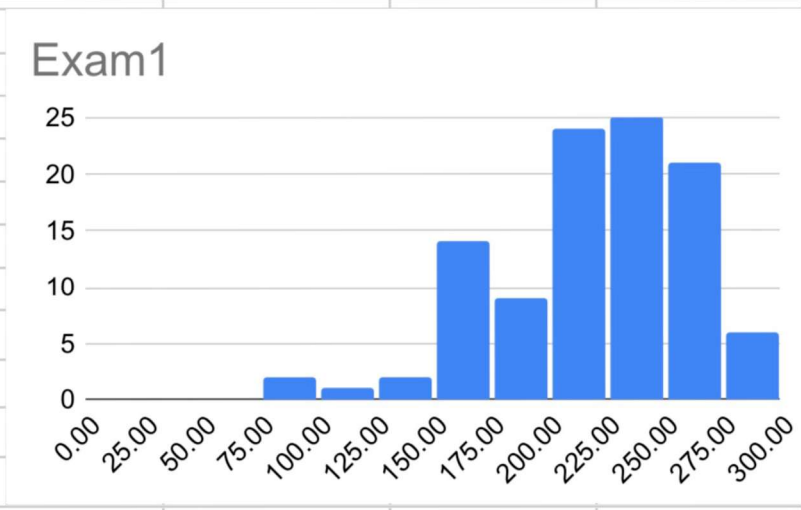
Student ID: _____

- This exam contains 11 pages (including this cover page) and 7 questions.
- The total number of points is 300.
- Time limit: **100 minutes** (10:10–11:50).
- Write down your computations or arguments in detail unless otherwise stated.
- The use of a calculator, cell phone, or any other electronic device is **NOT** permitted.
- The use of books or notes of any kind is **NOT** permitted.

Distribution of Marks

Question	Points	Score
1	24	
2	48	

Average	220.0761905
Average (except zero)	220.0761905
Quartile 0	77
Quartile 1	197
Quartile 2	225
Quartile 3	251
Quartile 4	300
標準差	44.29787615
非零數	105
不到60%人數	20
滿分人數	1



Throughout the exam, n denotes a positive integer.

1. Determine the boundedness and monotonicity (i.e. increasing or decreasing or neither) of the sequence with a_n, as indicated.

   (a) (8 points) $a_n = \dfrac{n + (-1)^n}{n}$.

   Solution: If n is odd, then $a_n = 1 - 1/n$, while if n is even, then $a_n = 1 + 1/n$. Thus

   $$0 \le a_n \le \frac{3}{2} \qquad \text{for all } n \ge 1,$$

   so the sequence is **bounded**. Since

   $$a_1 = 0 < a_2 = \frac{3}{2} > a_3 = \frac{2}{3},$$

   it is **neither increasing nor decreasing**.

   (b) (8 points) $a_n = \dfrac{2^n}{4^n + 1}$.

   Solution: The terms are positive, and

   $$\frac{a_{n+1}}{a_n} = \frac{2(4^n + 1)}{4^{n+1} + 1} < 1 \qquad (n \ge 1),$$

   because $4^{n+1} + 1 - 2(4^n + 1) = 2 \cdot 4^n - 1 > 0$. Therefore the sequence is **strictly decreasing**. Moreover,

   $$0 < a_n \le a_1 = \frac{2}{5},$$

   so it is **bounded**.

   (c) (8 points) $a_n = \sqrt{n^2 + 1}$.

   Solution: Since $(n+1)^2 + 1 > n^2 + 1$, we have $a_{n+1} > a_n$. Thus the sequence is **strictly increasing**. Also,

   $$a_n = \sqrt{n^2 + 1} > n \longrightarrow \infty,$$

   so it is **unbounded**.

2. State whether the sequence converges and, if it does, find the limit.

(a) (12 points) $a_n = \dfrac{4^n}{\sqrt{n^2+1}}$.

Solution: Let $b_n = \dfrac{1}{a_n} = \dfrac{\sqrt{n^2+1}}{4^n}$. Since $\sqrt{n^2+1} \le n+1$ and $n+1 \le 2^n$ for every $n \ge 1$,

$$0 < b_n \le \frac{2^n}{4^n} = \frac{1}{2^n} \longrightarrow 0.$$

Given any $M > 0$, for all sufficiently large n we have $0 < b_n < 1/M$, and hence $a_n = 1/b_n > M$. Thus $a_n \to +\infty$, so the sequence **diverges**.

(b) (12 points) $a_n = \dfrac{4^{100n}}{n!}$.

Solution: Put $A = 4^{100}$, which is a fixed positive constant. Then $a_n = A^n/n!$, and

$$\frac{a_{n+1}}{a_n} = \frac{A}{n+1} \longrightarrow 0.$$

Choose an integer N such that $A/(N+1) \le 1/2$. For all integers $m \ge 0$,

$$0 < a_{N+m} \le a_N \left(\frac{1}{2}\right)^m \longrightarrow 0.$$

Thus the sequence **converges to** 0. Alternatively, consider the series

$$\sum_{n=1}^{\infty} \frac{A^n}{n!}.$$

The ratio of consecutive terms is

$$\frac{A^{n+1}/(n+1)!}{A^n/n!} = \frac{A}{n+1} \longrightarrow 0,$$

so this series converges by the ratio test. By the n-th term test, its terms tend to 0; hence $A^n/n! \longrightarrow 0$, as required.

(c) (12 points) $a_n = \displaystyle\int_0^{1/n} \cos e^x \, dx$.

Solution: Since $|\cos(e^x)| \le 1$,

$$|a_n| \le \int_0^{1/n} |\cos(e^x)| \, dx \le \int_0^{1/n} 1 \, dx = \frac{1}{n} \longrightarrow 0.$$

The sequence therefore **converges to** 0 by the pinching theorem.

(d) (12 points) $a_n = n \sin \dfrac{2}{n}$.

Solution: Using $\lim_{t \to 0} \sin t/t = 1$, we obtain

$$a_n = 2 \frac{\sin(2/n)}{2/n} \longrightarrow 2.$$

The sequence **converges to** 2.

3. Evaluate the following limits as $x \to \infty$.

(a) (12 points) $\displaystyle\lim_{x\to\infty} \frac{\ln(x^2+1)}{x}$

Solution: The quotient has the form ∞/∞. By L'Hôpital's rule,

$$\lim_{x\to\infty} \frac{\ln(x^2+1)}{x} = \lim_{x\to\infty} \frac{2x}{x^2+1} = \boxed{0}.$$

(b) (12 points) $\displaystyle\lim_{x\to\infty} \left(1+\frac{3}{x}\right)^{2x}$

Solution: Let $y(x) = (1+3/x)^{2x}$ for $x > 0$. Taking logarithms and putting $t = 3/x$, we have

$$\lim_{x\to\infty} \ln y(x) = \lim_{x\to\infty} 2x \ln(1+3/x) = 6 \lim_{t\to 0^+} \frac{\ln(1+t)}{t} = 6.$$

Here the last quotient tends to 1, for example by L'Hôpital's rule. Since

$$\lim_{x\to\infty} y(x) = \lim_{x\to\infty} e^{\ln y(x)} = e^{\lim\limits_{x\to\infty} \ln y(x)} = e^6,$$

the required limit is $\boxed{e^6}$.

(c) (12 points) $\displaystyle\lim_{x\to\infty} \frac{x+\cos x}{2x}$

Solution: We have

$$\frac{x+\cos x}{2x} = \frac{1}{2} + \frac{\cos x}{2x}, \qquad \left|\frac{\cos x}{2x}\right| \le \frac{1}{2x} \longrightarrow 0 \quad (x \to \infty).$$

Therefore the required limit is $\boxed{1/2}$.

4. Evaluate the integrals.

   (a) (20 points) $\displaystyle\int_{-1}^{\infty} \frac{dx}{x^2+5x+6}$.

   Solution: Factor the denominator and use partial fractions. To find the coefficients, write

$$\frac{1}{(x+2)(x+3)} = \frac{A}{x+2} + \frac{B}{x+3}.$$

   Then $1 = A(x+3) + B(x+2)$, so $A = 1$ and $B = -1$. Hence

$$\frac{1}{x^2+5x+6} = \frac{1}{(x+2)(x+3)} = \frac{1}{x+2} - \frac{1}{x+3}.$$

   Consequently,

$$\begin{aligned}
\int_{-1}^{\infty} \frac{dx}{x^2+5x+6} &= \lim_{b\to\infty} \int_{-1}^{b} \left(\frac{1}{x+2} - \frac{1}{x+3}\right) dx \\
&= \lim_{b\to\infty} \left[\ln(x+2) - \ln(x+3)\right]_{-1}^{b} \\
&= \lim_{b\to\infty} \left(\ln(b+2) - \ln(b+3) + \ln 2\right) \\
&= \lim_{b\to\infty} \left(\ln\frac{b+2}{b+3} + \ln 2\right) = \boxed{\ln 2},
\end{aligned}$$

   because $(b+2)/(b+3) \to 1$ and $\ln 1 = 0$.

Question 4 (continued). Evaluate the integrals.

(b) (20 points) $\displaystyle\int_0^1 x \ln x \, dx$

Solution: Recall the integration by parts formula

$$\int u \, dv = uv - \int v \, du.$$

Take $u = \ln x$ and $dv = x \, dx$. Then $du = dx/x$ and $v = x^2/2$, so

$$\int x \ln x \, dx = \frac{x^2}{2} \ln x - \int \frac{x^2}{2} \frac{dx}{x} = \frac{x^2}{2} \ln x - \frac{1}{2} \int x \, dx = \frac{x^2}{2} \ln x - \frac{x^2}{4} + C.$$

The integral is improper at 0, so evaluate it as

$$\int_0^1 x \ln x \, dx = \lim_{a \to 0^+} \int_a^1 x \ln x \, dx.$$

Also, $\lim_{x \to 0^+} x^2 \ln x = 0$; for instance, L'Hôpital's rule gives

$$\lim_{x \to 0^+} \frac{\ln x}{1/x^2} = \lim_{x \to 0^+} \frac{1/x}{-2/x^3} = 0.$$

Therefore

$$\begin{aligned} \int_0^1 x \ln x \, dx &= \lim_{a \to 0^+} \left[\frac{x^2}{2} \ln x - \frac{x^2}{4} \right]_a^1 \\ &= -\frac{1}{4} - \left(\frac{a^2}{2} \ln a - \frac{a^2}{4} \right) \\ &= -\frac{1}{4} - 0 = \boxed{-\frac{1}{4}}. \end{aligned}$$

Question 4 (continued). Evaluate the integrals.

(c) (20 points) $\displaystyle\int_{-\infty}^{\infty} \frac{1}{e^{2x}+e^{-2x}}\,dx.$

Solution: Recall the substitution formula: if $u = g(x)$, then

$$\int F(g(x))g'(x)\,dx = \int F(u)\,du,$$

with the endpoints changed to their corresponding u-values. Use $u = e^{2x}$, so $du = 2e^{2x}\,dx = 2u\,dx$ and $dx = du/(2u)$. Since $e^{-2x} = 1/u$, the integrand becomes

$$\frac{dx}{e^{2x}+e^{-2x}} = \frac{du/(2u)}{u+1/u} = \frac{du}{2(1+u^2)}.$$

The substitution sends $x = -\infty, 0, \infty$ to $u = 0, 1, \infty$, respectively. We check the two improper integrals separately:

$$\begin{aligned}
\int_{-\infty}^{0} \frac{dx}{e^{2x}+e^{-2x}} &= \lim_{a\to-\infty}\int_{e^{2a}}^{1} \frac{du}{2(1+u^2)} = \lim_{a\to-\infty}\frac{1}{2}\left[\arctan u\right]_{e^{2a}}^{1} \\
&= \frac{1}{2}\left(\frac{\pi}{4}-0\right) = \frac{\pi}{8}, \\
\int_{0}^{\infty} \frac{dx}{e^{2x}+e^{-2x}} &= \lim_{b\to\infty}\int_{1}^{e^{2b}} \frac{du}{2(1+u^2)} = \lim_{b\to\infty}\frac{1}{2}\left[\arctan u\right]_{1}^{e^{2b}} \\
&= \frac{1}{2}\left(\frac{\pi}{2}-\frac{\pi}{4}\right) = \frac{\pi}{8}.
\end{aligned}$$

Both converge, so the required integral is

$$\frac{\pi}{8}+\frac{\pi}{8} = \boxed{\frac{\pi}{4}}.$$

5. Determine whether each improper integral converges or diverges.

(a) (18 points) $\displaystyle\int_{1/2}^{2} \frac{dx}{x \ln x}$

Solution: The integrand is undefined at $x = 1$, so the integral must be split there. For $x > 0$, $x \neq 1$, an antiderivative is $\ln|\ln x|$. Thus

$$\int_{1/2}^{1} \frac{dx}{x \ln x} = \lim_{b \to 1^-} (\ln|\ln b| - \ln(\ln 2)) = -\infty,$$

$$\int_{1}^{2} \frac{dx}{x \ln x} = \lim_{a \to 1^+} (\ln(\ln 2) - \ln(\ln a)) = +\infty.$$

Neither one-sided improper integral converges. Therefore the given integral **diverges**. The two infinite quantities cannot be cancelled.

(b) (18 points) $\displaystyle\int_{1}^{\infty} \frac{\sqrt{x+1}}{x^2}\, dx$

Solution: For $x \geq 1$, we have $x + 1 \leq 2x$, and therefore

$$0 \leq \frac{\sqrt{x+1}}{x^2} \leq \frac{\sqrt{2x}}{x^2} = \frac{\sqrt{2}}{x^{3/2}}.$$

Since

$$\int_{1}^{\infty} \frac{\sqrt{2}}{x^{3/2}}\, dx = 2\sqrt{2} < \infty,$$

the given integral **converges** by comparison.

6. Does the series converge absolutely, converge conditionally, or diverge? Explain your reasoning.

(a) (12 points) $\displaystyle\sum_{k=1}^{\infty} \frac{3+\sin k}{\sqrt{k+2}}$.

Solution: For every $k \geq 1$, we have $3+\sin k \geq 2$ and $k+2 \leq 3k$. Thus

$$\frac{3+\sin k}{\sqrt{k+2}} \geq \frac{2}{\sqrt{k+2}} \geq \frac{2}{\sqrt{3}}\,\frac{1}{\sqrt{k}}.$$

Since $\displaystyle\sum_{k=1}^{\infty} 1/\sqrt{k}$ is a divergent p-series with $p = 1/2$, the given series **diverges** by comparison.

(b) (12 points) $\displaystyle\sum_{k=1}^{\infty} \frac{5^k}{k!}$.

Solution: Put $b_k = 5^k/k! > 0$. Then

$$\lim_{k\to\infty} \frac{b_{k+1}}{b_k} = \lim_{k\to\infty} \frac{5}{k+1} = 0 < 1.$$

The ratio test shows that $\sum b_k$ converges. As all terms are positive, the series **converges absolutely**.

(c) (12 points) $\displaystyle\sum_{k=2}^{\infty} \frac{(-1)^k}{k\ln k}$.

Solution: Let $b_k = 1/(k\ln k)$ for $k \geq 2$. The terms b_k are positive, tend to 0, and decrease, since

$$\frac{d}{dx}\left(\frac{1}{x\ln x}\right) = -\frac{\ln x + 1}{x^2(\ln x)^2} < 0 \qquad (x \geq 2).$$

The alternating-series test therefore gives convergence. However,

$$\int_2^{\infty} \frac{dx}{x\ln x} = \lim_{b\to\infty} \left(\ln(\ln b) - \ln(\ln 2)\right) = +\infty.$$

By the integral test, $\displaystyle\sum_{k=2}^{\infty} 1/(k\ln k)$ diverges. Hence the given series **converges conditionally**.

Question 6 (continued). Does the series converge absolutely, converge conditionally, or diverge? Explain your reasoning.

(d) (12 points) $\displaystyle\sum_{k=2}^{\infty}(-1)^k\frac{\ln k}{k}$.

Solution: Let $b_k = (\ln k)/k$. Then $b_k > 0$ for $k \geq 2$, and $b_k \to 0$. Furthermore,

$$\frac{d}{dx}\left(\frac{\ln x}{x}\right) = \frac{1-\ln x}{x^2} < 0 \qquad (x > e),$$

so b_k decreases for $k \geq 3$. The alternating-series test applies to the tail; the finitely many initial terms do not affect convergence. On the other hand,

$$\int_3^{\infty} \frac{\ln x}{x}\,dx = \lim_{b\to\infty} \frac{(\ln b)^2 - (\ln 3)^2}{2} = +\infty.$$

Thus the series of absolute values diverges by the integral test. The given series **converges conditionally**.

(e) (12 points) $\displaystyle\sum_{k=1}^{\infty}(-1)^k k\sin(1/k)$.

Solution: The magnitudes of the terms satisfy

$$\left|(-1)^k k\sin(1/k)\right| = k\sin(1/k) = \frac{\sin(1/k)}{1/k} \longrightarrow 1.$$

The terms therefore do not tend to 0, so the series **diverges** by the necessary condition for convergence.

(f) (12 points) $\displaystyle\sum_{k=1}^{\infty}\frac{\sin(\pi k/2)}{k\sqrt{k}}$.

Solution: For every $k \geq 1$,

$$\left|\frac{\sin(\pi k/2)}{k\sqrt{k}}\right| \leq \frac{1}{k^{3/2}}.$$

The p-series $\displaystyle\sum_{k=1}^{\infty} 1/k^{3/2}$ converges because $3/2 > 1$. By comparison, the given series **converges absolutely**.

7. Let $f(x) = \sin x$.

(a) (8 points) Write the Taylor expansion of $f(x)$ at 0 up to the term in x^3, with remainder denoted by $R_3(x)$. Show your computation.

Solution: We have

$$f'(x) = \cos x, \qquad f''(x) = -\sin x, \qquad f^{(3)}(x) = -\cos x,$$

and hence

$$f(0) = 0, \qquad f'(0) = 1, \qquad f''(0) = 0, \qquad f^{(3)}(0) = -1.$$

Taylor's theorem gives

$$\begin{aligned}\sin x &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + R_3(x) \\ &= x - \frac{x^3}{6} + R_3(x).\end{aligned}$$

(b) (8 points) Use Taylor's theorem to show that

$$|R_3(x)| \le \frac{|x|^4}{24}$$

for every real number x.

Solution: By Taylor's theorem, for each real number x there is a number c between 0 and x such that

$$R_3(x) = \frac{f^{(4)}(c)}{4!}x^4 = \frac{\sin c}{24}x^4.$$

Since $|\sin c| \le 1$,

$$|R_3(x)| = \frac{|\sin c|}{24}|x|^4 \le \frac{|x|^4}{24}.$$

(c) (8 points) Use the expansion in part (a), with the remainder term omitted, to approximate $\sin(1/10)$. Express your approximation as a fraction in lowest terms, and show that its absolute error is less than $\dfrac{1}{100000}$.

Solution: Taking $x = 1/10$ in part (a), we obtain

$$\sin(1/10) \approx \frac{1}{10} - \frac{1}{6}\left(\frac{1}{10}\right)^3 = \frac{599}{6000}.$$

By part (b), the absolute error satisfies

$$\begin{aligned}\left|\sin(1/10) - \frac{599}{6000}\right| &= |R_3(1/10)| \\ &\le \frac{1}{24}\left(\frac{1}{10}\right)^4 = \frac{1}{240000} < \frac{1}{100000}.\end{aligned}$$