Calculus, week 7, Fall 2026

Q: $\sum_{k=0}^{\infty} a_k x^k = ?$

Thm (Thm 12.9.1, 12.9.2)

If

$$f(x) = \sum_{k=0}^{\infty} a_k x^k \quad \text{on } (-c, c)$$

then f is differentiable on $(-c, c)$

and

$$f'(x) = \sum_{k=0}^{\infty} (a_k x^k)'$$

$$= \sum_{k=1}^{\infty} k a_k x^{k-1}$$

$$= a_1 + 2a_2 x + 3a_3 x^2 + \cdots$$

Example

① $\sum_{k=1}^{\infty} \frac{x^k}{k} = ?$ on $(-1, 1)$

sol

Recall

$$\sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$$

on $(-1, 1)$.

$\Rightarrow \left(\sum_{k=0}^{\infty} x^k\right)' = \left(\frac{1}{1-x}\right)'$

Thm II: $\sum_{k=1}^{\infty} k \cdot x^{k-1} = \frac{-1 \cdot (-1)}{(1-x)^2} = \frac{1}{(1-x)^3}$

Recall: $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$

Let $f(x) = \sum_{k=1}^{\infty} \frac{x^k}{k}$

Thm $\Rightarrow f'(x) = \sum_{k=1}^{\infty} \left(\frac{x^k}{k}\right)' = \sum_{k=1}^{\infty} k \cdot \frac{x^{k-1}}{k}$

$= \sum_{k=1}^{\infty} x^{k-1} = 1 + x + x^2 + x^3 + \cdots$

$= \frac{1}{1-x}$

Recall $(\ln x)' = \frac{1}{x}$ $\quad (\ln(1-x))' = \frac{-1}{1-x}$

$\Rightarrow (-\ln(1-x) + C)' = \frac{1}{1-x}$

$\Rightarrow f(x) = -\ln(1-x) + C$

Q: $C = ?$

$f(0) = \sum_{k=1}^{\infty} \frac{0^k}{k} = 0$

$-\ln(1-0) + C = C$

So $f(x) = \sum_{k=1}^{\infty} \frac{x^k}{k} = -\ln(1-x) = \ln \frac{1}{1-x}$ #

② $\sum_{k=1}^{\infty} k x^k = ?$

Sol

Since $\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$ on $(-1,1)$

we have

$$\left(\frac{1}{1-x}\right)' = \sum_{k=1}^{\infty} (x^k)' \quad \text{on } (-1,1)$$

$$= \sum_{k=1}^{\infty} k \cdot x^{k-1} = \frac{\sum_{k=1}^{\infty} k x^k}{x}$$

$$\Rightarrow \sum_{k=1}^{\infty} k x^k = x \cdot \left(\frac{1}{1-x}\right)' \quad \text{on } (-1,1)$$

$$= x \cdot \frac{-(1-x)'}{(1-x)^2} = \frac{x}{(1-x)^2} \quad \#$$

Remark

By $\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$, we have

$$\left(\frac{1}{1-x}\right)' = \frac{1}{(1-x)^2} = \sum_{k=1}^{\infty} k x^{k-1}$$

$$\left(\frac{1}{1-x}\right)'' = \left((1-x)^{-2}\right)' \overset{\text{chain rule}}{=} (-2)\cdot(1-x)^{-3}\cdot(-1)$$

$$= \frac{2}{(1-x)^3} = \sum_{k=1}^{\infty} (k x^{k-1})'$$

$$= \sum_{k=2}^{\infty} k(k-1) x^{k-2} \quad \text{on } (-1,1)$$

$$\left(\frac{1}{1-x}\right)''' = \left(2\cdot(1-x)^{-3}\right)' = 2\cdot(-3)\cdot(1-x)^{-3-1}\cdot(-1)$$

$$= \frac{6}{(1-x)^4} = \sum_{k=3}^{\infty}\left(k(k-1)x^{k-2}\right)'$$

$$= \sum_{k=3}^{\infty} k(k-1)(k-2)x^{k-3} \quad \text{on } (-1,1)$$

Example

$$\sum_{k=0}^{\infty} k^2 x^k = ?$$

Sol

Recall: $\frac{1}{(1-x)^2} = \sum_{k=1}^{\infty} k x^{k-1}$ on $(-1,1)$

$$\Rightarrow \frac{x}{(1-x)^2} = \sum_{k=1}^{\infty} k\cdot x^k$$

$$\Rightarrow \left(\frac{x}{(1-x)^2}\right)' = \sum_{k=1}^{\infty}\left(k\cdot x^k\right)'$$

$$= \sum_{k=1}^{\infty} k\cdot k\cdot x^{k-1} = \sum_{k=1}^{\infty} k^2\cdot x^{k-1}$$

$$\Rightarrow \sum_{k=1}^{\infty} k^2 x^k = x\cdot\left(\sum_{k=1}^{\infty} k^2\cdot x^{k-1}\right) = x\cdot\left(\frac{x}{(1-x)^2}\right)'$$

$$= \sum_{k=0}^{\infty} k^2 x^k$$

$$x\cdot\left(\frac{(x)'\cdot(1-x)^2 - x\cdot\left((1-x)^2\right)'}{\left((1-x)^2\right)^2}\right)$$

$$= x \cdot \frac{1\cdot(1-x)^2 - x\cdot 2(1-x)\cdot(-1)}{(1-x)^4} \quad (1-2x+x^2 \;+\; 2(x-x^2))$$

$$= \frac{x - x^3}{(1-x)^4} \qquad \text{on } (-1,1) \quad \#$$

Example

Find $y = y(x)$ s.t.

$$\circledast \quad y'' + y = 0$$

sol

Assume $y = \sum_{k=0}^{\infty} a_k x^k$

$$\Rightarrow \quad y' = \sum_{k=1}^{\infty} k a_k x^{k-1}$$

$$y'' = \sum_{k=2}^{\infty} k(k-1) a_k x^{k-2}$$

By $\circledast$

$$\sum_{k=2}^{\infty} k(k-1) a_k x^{k-2} + \sum_{k=0}^{\infty} a_k x^k = 0$$

$k = k'+2$ (the x^{k-2} term)

$k = k'$ (the x^k term)

$\Leftrightarrow \; k = k'+2$

$k = 2 \Leftrightarrow k' = 0$

$$= \sum_{k'=0}^{\infty} (k'+2)(k'+2-1) a_{k'+2} x^{k'} \quad (k' \leftrightarrow k)$$

$$+ \sum_{k=0}^{\infty} a_k x^k$$

$$= \sum_{k=0}^{\infty} (k+2)(k+1)\, a_{k+2}\, x^k + \sum_{k=0}^{\infty} a_k x^k$$

$$= \sum_{k=0}^{\infty} \left((k+2)(k+1)\, a_{k+2} + a_k \right) x^k = 0$$

$$\Rightarrow \quad (k+2)(k+1)\, a_{k+2} + a_k = 0 \quad \forall k$$

$$\Rightarrow \quad a_{k+2} = -\frac{a_k}{(k+2)(k+1)} \quad \forall k$$

So

(i) $a_{2m} \overset{(k=2m-2)}{=} -\frac{a_{2m-2}}{2m \cdot (2m-1)}$

$k=2m-4$

$$= -\frac{1}{2m(2m-1)} \left(-\frac{a_{2m-4}}{(2m-2)(2m-3)} \right)$$

$k=2m-6$

$$= \cdots$$

$k=0$

$$= \left(-\frac{1}{2m(2m-1)} \right) \left(-\frac{1}{(2m-2)(2m-3)} \right) \cdots$$

$$\left(-\frac{1}{2 \cdot 1} \right) \cdot a_0$$

$$a_{2m} = (-1)^m \cdot \frac{1}{(2m)!}\, a_0$$

(ii) a_{2m+1}

$a_{k+2} = -\frac{a_k}{(k+2)(k+1)}$

$k=2m-1$

$$= -\frac{a_{2m-1}}{(2m+1)\ 2m}$$

$k=2m-3$

$$= \left(-\frac{1}{(2m+1)\,2m}\right)\left(-\frac{a_{2m-3}}{(2m-1)(2m-2)}\right)$$

$k=1$

$$= \cdots = \left(-\frac{1}{(2m+1)\,2m}\right)\left(-\frac{1}{(2m-1)(2m-2)}\right)$$

$$\cdots \left(-\frac{1}{3\cdot 2}\right) a_1$$

$$= (-1)^m \frac{1}{(2m+1)!}\, a_1$$

So

$$y = \sum_{k=0}^{\infty} a_k x^k$$

$$= \sum_{m=0}^{\infty} a_{2m} x^{2m} + \sum_{m=0}^{\infty} a_{2m+1} x^{2m+1}$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m)!}\, a_0 \cdot x^{2m}$$

$$+ \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!}\, a_1 \cdot x^{2m+1}$$

$$= a_0 \left(\sum_{m=0}^{\infty} \frac{(-1)^m}{(2m)!} x^{2m}\right)$$

$$+ a_1 \left(\sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} x^{2m+1}\right)$$

Recall

$$\cos x = \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m)!} x^{2m}$$

$$\sin x = \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} x^{2m+1}$$

$y'' + y = 0$

$\Rightarrow y(x) = a_0 \cos x + a_1 \sin x$

for some numbers a_0 and a_1 #

Thm (Thm 12.9.3)

If $f(x) = \sum_{k=0}^{\infty} a_k x^k$ on $(-c, c)$,

then

$$\sum_{k=0}^{\infty} \frac{a_k}{k+1} x^{k+1} = a_0 x + \frac{a_1}{2} x^2 + \frac{a_2}{3} x^3 + \cdots$$

converges to an antiderivative of $f(x)$ on $(-c, c)$.

Equivalently,

$$\int \left(\sum_{k=0}^{\infty} a_k x^k \right) dx = \sum_{k=0}^{\infty} \frac{a_k}{k+1} x^{k+1} + C$$

Example

Show that

NOTE Why not $\ln x$?
Because $\ln 0$ is NOT defined.

$$\ln(1+x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} x^{k+1}$$

for $x \in (-1, 1)$

sol

Recall

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k \quad \text{on } (-1,1)$$

$$\Rightarrow \frac{1}{1+x} = \sum_{k=0}^{\infty} (-x)^k \quad \text{on } (-1,1)$$

$$\Rightarrow \int \frac{1}{1+x}\, dx = \ln(1+x) + C'$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} x^{k+1} + C \quad \text{on } (-1,1)$$

$$\Rightarrow \ln(1+x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} x^{k+1} + C \quad \text{on } (-1,1)$$

Q: $C = ?$

At $x=0$, we have

$$\ln(1+0) = \ln 1 = 0$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} 0^{k+1} + C = C$$

$$\Rightarrow C = 0$$

Therefore,

$$\ln(1+x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} x^{k+1}$$

for $x \in (-1, 1)$. #

Q: How about $x = \pm 1$?

At $x=-1$, $\ln(1-1) = \ln 0$ is NOT OK

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} (-1)^{k+1} = -\sum_{k=0}^{\infty} \frac{1}{k+1}$$

diverges.

At $x=1$,

$$\ln(1+1) = \ln 2$$

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} 1^{k+1} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1}$$

$$= \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$$

Q: $\ln 2 = \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1}$?

A: Yes. by the following thm:

Thm (Abel's Thm, Thm 12.9.5)

Suppose $\sum_{k=0}^{\infty} a_k x^k$ converges on $(-c, c)$

and $f(x) = \sum_{k=0}^{\infty} a_k x^k$ on $(-c, c)$.

(ii) If f is left continuous at c (i.e. $\lim_{x \to c^-} f(x) = f(c)$) and $\sum_{k=0}^{\infty} a_k c^k$ converges, then

$$f(c) = \sum_{k=0}^{\infty} a_k c^k$$

(iii) If f is right continuous at $-c$ (i.e. $\lim_{x \to -c^+} f(x) = f(-c)$) and $\sum_{k=0}^{\infty} a_k (-c)^k$ converges, then

$$f(-c) = \sum_{k=0}^{\infty} a_k (-c)^k.$$

Example

① $f(x) = \ln(1+x)$ is continuous for $x > -1$

$\Rightarrow f$ is (left) continuous at $x = 1$

We also know that

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

converges (Alternating Series)

Abel's Thm $\Rightarrow$

$$f(1) = \ln 2 = \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1}$$

② $\tan^{-1}x = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1}$

$= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$

for $x \in [-1, 1]$

pf of ②

Recall that

$(\tan^{-1}x)' = \frac{1}{1+x^2}$

$= \frac{1}{1-(-x^2)} \overset{\text{等比级数}}{=} \sum_{k=0}^{\infty} (-x^2)^k$

for $x \in (-1, 1)$

$\tan^{-1}x = \int (\tan^{-1}x)' \, dx$

$= \int \frac{1}{1+x^2} \, dx$

$= \int \sum_{k=0}^{\infty} (-x^2)^k \, dx$

$= \sum_{k=0}^{\infty} (-1)^k \int x^{2k} dx + C$

$= \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1} + C \quad x \in (-1, 1)$

Q: $C = ?$

At $x = 0$,

$$\tan^{-1} 0 = 0 = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} 0^{2k+1} + C$$

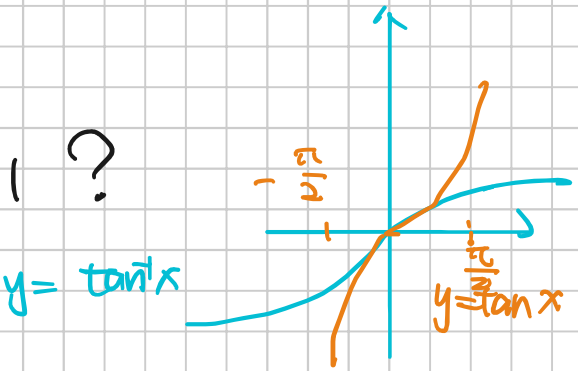
$$= C$$

$\Rightarrow C = 0$

So $$\tan^{-1} x = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1}$$

for $x \in (-1, 1)$

Q: What if $x = \pm 1$?



(i) At $x = 1$:

Since $\tan^{-1} x$ is continuous on $(-\infty, \infty) \supseteq [-1, 1]$

$\Rightarrow \tan^{-1} x$ is (left) continuous at 1

Also, $\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} 1^{2k+1}$ converges by the Alternating Series Test.

$\Rightarrow$ Abel's Thm $\tan^{-1} 1 = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} 1^{2k+1}$

(ii) At $x=-1$ is similar.

So $$\tan^{-1}x = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1}$$

for $x \in [-1, 1]$ #

Summary of Taylor series and power series

A function is called smooth if it is infinitely differentiable.

smooth function $f(x)$

Taylor series at 0 ↓

↑ Sum when $r>0$

r = radius of convergence

power series $\sum_{k=0}^{\infty} a_k x^k$

- Taylor series: Start with $f(x)$
  - Taylor series of f at 0 is $$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$$

- The Taylor series does not necessarily converge to $f(x)$, even at points where the series converges.

For example,

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

is a smooth function s.t.

$$f^{(k)}(0) = 0 \qquad \forall\, k \geq 0$$

$\Rightarrow$ Its Taylor series is the zero series which converges everywhere, but agrees with $f(x)$ only at $x=0$.

- Taylor's Thm:

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k + R_n(x)$$

where

$$R_n(x) = \frac{1}{n!} \int_0^x f^{(n+1)}(t) \cdot (x-t)^n \, dt$$

$$= \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1}$$

with a number c between 0 and x

(c depends on n and x)

- $f(x) \overset{\circledast}{=} \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$

$\Leftrightarrow \lim_{n\to\infty} R_n(x) = 0$

- The problem is: $\lim_{n\to\infty} R_n(x)$ can be difficult.

Another method for obtaining $\circledast$:

> Use known expansions + substitution, addition, differentiation, integration, ... (△)

- Known expansions:

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \cdots \quad \forall x \in (-\infty, \infty)$$

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

$\forall x \in (-\infty, \infty)$

$$\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

$\forall x \in (-\infty, \infty)$

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \cdots \quad \forall x \in (-1, 1)$$

$$\ln(1+x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} x^{k+1} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots \quad \forall x \in (-1, 1]$$

$$\tan^{-1} x = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots \quad \forall x \in [-1, 1]$$

- Power series: Start with a series

$$\sum_{k=0}^{\infty} a_k x^k = a_0 + a_1 x + a_2 x^2 + \cdots$$

- Convergence:

Step 1: Compute the radius of convergence

$$r = \frac{1}{\lim_{k\to\infty} \sqrt[k]{|a_k|}} = \frac{1}{\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right|}$$

If $r>0$, then $\sum_{k=0}^{\infty} a_k x^k$ converges for $x \in (-r, r)$

And it diverges for $|x| > r$

Step 2: Check the convergence at the endpoints separately.

- What's the sum ?

Use ◬:

For $x \in (-r, r)$,

$$\left(\sum_{k=0}^{\infty} a_k x^k\right)' = \sum_{k=1}^{\infty} k a_k x^{k-1}$$

$$\int_0^x \left(\sum_{k=0}^{\infty} a_k t^k\right) dt = \sum_{k=0}^{\infty} \frac{a_k}{k+1} x^{k+1}$$

- If $r>0$, the function

$$f(x) = \sum_{k=0}^{\infty} a_k x^k \quad \text{for } x \in (-r, r),$$

satisfies

$$f^{(k)}(0) = k!\, a_k$$

$\Rightarrow$ Taylor series of $f(x)$ at 0 is

$$\sum_{k=0}^{\infty} a_k x^k$$

Example

Find the Taylor series of the functions at 0 and check wether the series converges to the given function.

① $\cosh x = \dfrac{e^x + e^{-x}}{2} = ?$

sol

Since $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$ $\quad \forall x \in (-\infty, \infty)$

$\Rightarrow e^{-x} = \sum_{k=0}^{\infty} \frac{(-x)^k}{k!} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} x^k$

$\Rightarrow \dfrac{e^x + e^{-x}}{2} = \sum_{k=0}^{\infty} \left(\dfrac{1}{k!} + \dfrac{(-1)^k}{k!} \right) \Big/ 2 \cdot x^k$ $\quad \forall x \in (-\infty, \infty)$

$= \begin{cases} 0 & \text{if } k \text{ odd} \\ \frac{2}{k!} & \text{if } k \text{ even} \end{cases}$

$k = 2m$

$$= \sum_{m=0}^{\infty} \frac{2}{(2m)!} \Big/ 2 \cdot x^{2m}$$

$$\Rightarrow \cosh x = \sum_{m=0}^{\infty} \frac{x^{2m}}{(2m)!} \quad \forall x \in (-\infty, \infty) \quad \#$$

$$= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$$

② $x^2 \cos(x^3) = ?$

Sol

Recall

$$\cos y = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} y^{2k} \qquad \forall y \in (-\infty, \infty)$$

Plug in: $y = x^3$

$$\cos x^3 = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} (x^3)^{2k} \qquad \forall x \in (-\infty, \infty)$$

$$= x^{6k}$$

$$\Rightarrow x^2 \cdot \cos(x^3) = x^2 \cdot \left(\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{6k} \right)$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^2 \cdot x^{6k} = x^{6k+2}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{6k+2} \qquad \forall x \in (-\infty, \infty)$$

③ $\ln(1+x^2) = ?$

Sol

Recall

$$\ln(1+y) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} y^{k+1} \qquad \forall y \in (-1, 1]$$

$-1 < y \leq 1$

Plug in: $y = x^2$

$$\ln(1+x^2) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} (x^2)^{k+1} = x^{2k+2}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} x^{2k+2}$$

$-1 < x^2 \leq 1$

$-1 \leq x \leq 1$

for $x \in [-1, 1]$

④ $\tan^{-1}(2x) = ?$

Sol

Recall

$$\tan^{-1} y = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} y^{2k+1} \qquad \forall y \in [-1,1]$$

Plug in: $y = 2x$

$$\tan^{-1}(2x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} (2x)^{2k+1}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k \cdot 2^{2k+1}}{2k+1} x^{2k+1}$$

$-1 \le y \le 1$

$-1 \le 2x \le 1$

$\Leftrightarrow$

$-\frac{1}{2} \le x \le \frac{1}{2}$

for $x \in [-\frac{1}{2}, \frac{1}{2}]$

⑤

$$f(x) = \begin{cases} \frac{\sin x}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$$

Sol

Recall

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} \qquad \forall x \in (-\infty, \infty)$$

$$\Rightarrow \frac{\sin x}{x} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \frac{x^{2k+1}}{x} = x^{2k}$$

if $x \neq 0$

$$= 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots$$

check $x=0$ separately $= f(x)$

$\forall x \in (-\infty, \infty)$

(5') $f'(0) = ?$ $f''(0) = ?$ $f^{(100)}(0) = ?$

Sol

Since $f(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k}$ $\forall x \in (-\infty, \infty)$

$\sum_{m=0}^{\infty} \frac{f^{(m)}(0)}{m!} x^m$ $\quad = \quad \frac{f^{(2k)}(0)}{(2k)!}$

$\because r = \infty > 0$

we have

$$f^{(2k)}(0) = (2k)! \cdot \frac{(-1)^k}{(2k+1)!}$$

$$f^{(2k+1)}(0) = 0$$

$\Rightarrow$ $f'(0) = 0$

$$f''(0) = f^{(2\cdot 1)}(0) = 2! \cdot \frac{(-1)^1}{3!} = -\frac{1}{3}$$

$$f^{(100)}(0) = 100! \cdot \frac{(-1)^{50}}{101!} = \frac{1}{101}$$ #

(6) $g(x) = \int_0^x e^{-t^2} dt = ?$

Sol

Recall $e^y = \sum_{k=0}^{\infty} \frac{y^k}{k!}$ $\forall y \in (-\infty, \infty)$

Plug in: $y = -t^2$

$$e^{-t^2} = \sum_{k=0}^{\infty} \frac{(-t^2)^k}{k!} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} t^{2k} \quad \forall t \in (-\infty, \infty)$$

$$\Rightarrow \quad g(x) = \int_0^x e^{-t^2}\,dt = \sum_{k=0}^{\infty} \int_0^x \frac{(-1)^k}{k!} t^{2k}\,dt$$

(↑ if $|x| < r$)

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \underbrace{\int_0^x t^{2k}\,dt}_{= \frac{t^{2k+1}}{2k+1}\Big|_{t=0}^{x} = \frac{x^{2k+1}}{2k+1}}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,(2k+1)} x^{2k+1} \qquad \forall x \in (-\infty, \infty)$$

#

Remark (A special technique: using diff. eq.)

Let $\alpha \in \mathbb{R}$ (e.g. $(1+x)^{\frac{1}{2}} = \sqrt{1+x}$ or $(1+x)^{\frac{1}{3}} = \sqrt[3]{1+x}$)

$$f(x) = (1+x)^{\alpha} = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k \qquad \text{for } |x| < 1$$

where

$$\binom{\alpha}{k} = \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-k+1)}{k!}$$

pf

Step1 Find Taylor series of $(1+x)^{\alpha}$:

$$f'(x) = \alpha \cdot (1+x)^{\alpha-1}$$

$$f''(x) = \alpha(\alpha-1) \cdot (1+x)^{\alpha-2}$$

$$\vdots$$

$$f^{(k)}(x) = \alpha(\alpha-1)\cdots(\alpha-k+1) \cdot (1+x)^{\alpha-k}$$

$\Rightarrow$ Its Taylor series is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = \sum_{k=0}^{\infty} \underbrace{\binom{\alpha}{k}}_{= a_k = \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}} \cdot x^k$$

Step 2: Convergence.

$$r = \frac{1}{\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right|} = \frac{1}{\lim_{k\to\infty}\left|\frac{\frac{\alpha(\alpha-1)\cdots(\alpha-k)}{(k+1)!}}{\frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}}\right|}$$

$$= \frac{1}{\lim_{k\to\infty}\left|\frac{\alpha-k}{k+1}\right|} = 1$$

So $\sum_{k=0}^{\infty} a_k x^k$ converges for $|x|<1$

Denote

$$\varphi(x) = \sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k$$

for $|x|<1$

Q: $\varphi(x) = f(x)$?

Step 3: Derive a diff. eq. for $\varphi(x)$

$$\varphi'(x) = \sum_{k=1}^{\infty} k a_k x^{k-1} = \sum_{k=1}^{\infty} k \cdot \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!\,(k-1)!} x^{k-1}$$

$$a_{k-1} = \frac{\alpha\cdots(\alpha-k+2)}{(k-1)!}$$

$$x\varphi'(x) = \sum_{k=1}^{\infty} k a_k x^k = \sum_{k=1}^{\infty} (\alpha-k+1)\, a_{k-1} x^{k-1}$$

$$= \alpha \sum_{k=1}^{\infty} a_{k-1} x^{k-1} - \sum_{k=1}^{\infty} (k-1) a_{k-1} x^{k-1}$$

$$= \varphi(x) \qquad = x\varphi'(x)$$

$$= \alpha \varphi(x) - x\varphi'(x)$$

Conclusion:

$$\circledast \quad \begin{cases} (1+x)\varphi'(x) = \alpha\varphi(x) \\ \varphi(0) = 1 \end{cases}$$

Step 4: Solve $\circledast$

$$\Rightarrow \varphi(x) = (1+x)^{\alpha}$$

$\#$