# Calculus, week 6, Fall 2026

**Recall** (Taylor expansion)

$$f(x) \rightsquigarrow \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$$

or $$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x-a)^k$$

function $\rightsquigarrow$ series

**Today**: We approach this problem from the other direction:

Consider $\sum_{k=0}^{\infty} a_k x^k$ ← we will focus on this or $\sum_{k=0}^{\infty} a_k (x-a)^k$

without a given function $f(x)$.

**Def**

A power series $\sum_{k=0}^{\infty} a_k x^k$

(i) converges at $c$ if $\sum_{k=0}^{\infty} a_k c^k$ converges;

(ii) converges on the set $S$ if

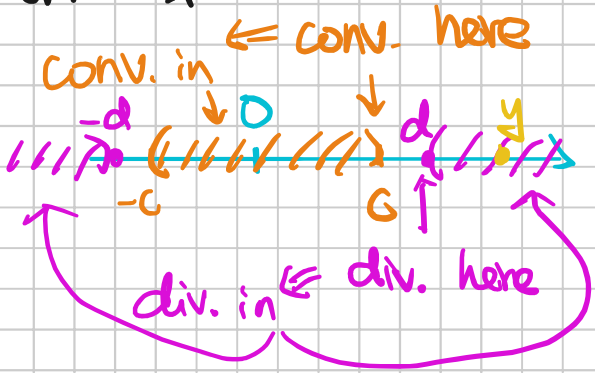
$\sum_{k=0}^{\infty} a_k x^k$ Converges at each $x \in S$.

**Thm** (Thm 12.8.2)

(ii) If $\sum_{k=0}^{\infty} a_k x^k$ converges at $c \neq 0$, then it converges absolutely at all $x$ with $|x| < |c|$.

i.e. $\sum_{k=0}^{\infty} |a_k x^k|$ conv. ($\Rightarrow \sum_{k=0}^{\infty} a_k x^k$ conv.)

(iii) If $\sum_{k=0}^{\infty} a_k x^k$ diverges at $d$, then it diverges at all $x$ with $|x| > |d|$




pf

It suffices to prove (ii):

Suppose $\sum_{k=0}^{\infty} a_k c^k$ converges

n-th Term $\Rightarrow$ Test

$$\lim_{k\to\infty} a_k c^k = 0$$

$\Rightarrow$ For $k$ sufficiently large,

$$|a_k c^k| < 1$$

$$\Rightarrow \quad |a_k x^k| = |a_k \cdot c^k| \left|\frac{x}{c}\right|^k < \left|\frac{x}{c}\right|^k$$

If $|x| < |c|$, then

$$\left|\frac{x}{c}\right| < 1$$

Since $\sum_{k=0}^{\infty} \left|\frac{x}{c}\right|^k$ Converges ← 等比级数

by the Comparison Test, we have

$$\sum_{k=0}^{\infty} |a_k x^k| \quad \text{converges}$$

$$\Rightarrow \quad \sum_{k=0}^{\infty} a_k x^k \quad \text{converges absolutely} \quad \#$$

By this theorem, we know there are exactly 3 possibilities for a power series:

Case 1:

$\sum_{k=0}^{\infty} a_k x^k$ Converges ONLY at $x=0$

e.g. $\sum_{k=1}^{\infty} k^k x^k$ :

If $x \neq 0$, since $\sqrt[k]{|k^k x^k|} = k \cdot |x| \to \infty > 1$ $\Rightarrow$ Root Test

$\sum_{k=1}^{\infty} k^k x^k$ diverges.

## case2:

$\sum_{k=0}^{\infty} a_k x^k$ Converges absolutely at all real numbers.

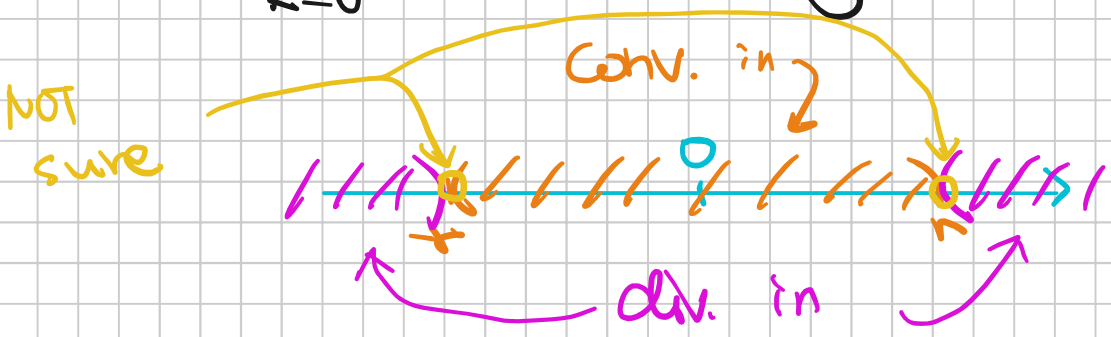
e.g. $\sum_{k=0}^{\infty} \frac{x^k}{k!}$ $(= e^x)$ Converges absolutely at all real numbers

## case3

There exists $r > 0$ s.t.

(i) $\sum_{k=0}^{\infty} a_k x^k$ Converges absolutely for $|x| < r$

(ii) $\sum_{k=0}^{\infty} a_k x^k$ diverges for $|x| > r$.



e.g. $\sum_{k=0}^{\infty} x^k$ (等比级数)

($r=1$) Converges absolutely for $|x|<1$

diverges for $|x|>1$

**Def**

The radius of convergence (收敛半径) of $\sum_{k=0}^{\infty} a_k x^k$ is

(i) 0 if it is case1

(ii) $\infty$ if it is case2

(iii) $r$ if it is case3

The interval of convergence (收敛区间) of $\sum_{k=0}^{\infty} a_k x^k$ is the maximal interval on which it converges

**Example**

① $\sum_{k=0}^{\infty} k^k x^k$:

Its radius of convergence is 0

Its interval of convergence is $\{0\}$.

② $\sum_{k=0}^{\infty} \frac{x^k}{k!}$ :

Its radius of convergence is $\infty$.

Its interval of convergence is $(-\infty, \infty)$

③ $\sum_{k=0}^{\infty} x^k$ :

Its radius of convergence is 1

Its interval of convergence is $(-1, 1)$

$(-1,1)$ or $(-1,1]$ or $[-1,1)$ or $[-1,1]$

Check:

$x=1$: $\sum_{k=0}^{\infty} 1^k = 1+1+1+\cdots$ diverges

$x=-1$: $\sum_{k=0}^{\infty} (-1)^k = 1-1+1-1+\cdots$ diverges

**Remark**

The radius of convergence $= r$ of $\sum_{k=0}^{\infty} a_k x^k$ is usually computed by the Root Test or Ratio Test:

① (Root Test)

If $\lim_{k\to\infty} \sqrt[k]{|a_k|} = L$, then

$$r = \frac{1}{L}$$

② (Ratio Test)

If $\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = L$, then

$$r = \frac{1}{L}$$

Why do we have there formulas?

① By Root Test, $\sum_{k=0}^{\infty} a_k x^k$ converges if

$$\lim_{k\to\infty} \sqrt[k]{|a_k x^k|} < 1$$

$$\left(\lim_{k\to\infty} \sqrt[k]{|a_k|}\right) \cdot |x| < 1$$

$$\Leftrightarrow \quad |x| < \frac{1}{\lim_{k\to\infty} \sqrt[k]{|a_k|}}$$

Also, by Root Test, it diverges if

$$\lim_{k\to\infty} \sqrt[k]{|a_k x^k|} > 1$$

$$\left(\lim_{k\to\infty}\sqrt[k]{|a_k|}\right)\cdot|x| > 1$$

$$\Leftrightarrow \quad |x| > \frac{1}{\lim_{k\to\infty}\sqrt[k]{|a_k|}}$$

Conclusion

$$|x| < \frac{1}{\lim_{k\to\infty}\sqrt[k]{|a_k|}} \Rightarrow \sum_{k=0}^{\infty} a_k x^k \text{ converges}$$

$$|x| > \frac{1}{\lim_{k\to\infty}\sqrt[k]{|a_k|}} \Rightarrow \sum_{k=0}^{\infty} a_k x^k \text{ diverges}$$

So its radius of convergence

$$= \frac{1}{\lim_{k\to\infty}\sqrt[k]{|a_k|}}$$

② Similarly, by Ratio Test, we check the number

$$\rho = \lim_{k\to\infty}\left|\frac{a_{k+1}x^{k+1}}{a_k x^k}\right| = \lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right||x|$$

$$\rho < 1 \Leftrightarrow |x| < \frac{1}{\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right|}$$

$\Rightarrow \sum_{k=0}^{\infty} a_k x^k$ converges

$P > 1 \iff |x| > \dfrac{1}{\lim\limits_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right|}$

$\Rightarrow \sum_{k=0}^{\infty} a_k x^k$ diverges

So its radius of convergence

$$= \frac{1}{\lim\limits_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right|}$$

Example

Find the interval of convergence:

① $\sum_{k=1}^{\infty} \frac{x^k}{k}$ $\left(a_k = \frac{1}{k}\right)$

sol

Step1 Compute the radius of conv.

$$= \frac{1}{\lim\limits_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right|} = \frac{1}{\lim\limits_{k\to\infty} \frac{\frac{1}{k+1}}{\frac{1}{k}}}$$

$$= \lim_{k\to\infty} \frac{(k+1)/k}{(k)/k} = \lim_{k\to\infty} \frac{1+\overbrace{\frac{1}{k}}^{\to 0}}{1} = 1$$

Step 2 Check the convergence at the endpoints $(x = \pm 1)$

At $x = 1$, the series is

$$\sum_{k=1}^{\infty} \frac{1^k}{k} = \sum_{k=1}^{\infty} \frac{1}{k}$$

which diverges.

At $x = -1$, the series is

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k} = -\frac{1}{1} + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} \cdots$$

which converges (Alternating Series)

So the interval of convergence is

$$[-1, 1)$$

#

② $\sum_{k=1}^{\infty} \frac{x^k}{k^2}$ $\left(a_k = \frac{1}{k^2}\right)$

Sol

Step 1 Radius of convergence is

$$\frac{1}{\lim_{k\to\infty}\sqrt[k]{\frac{1}{k^2}}} = \frac{1}{\lim_{k\to\infty}\frac{1}{(\sqrt[k]{k})^2}} = \frac{1}{\frac{1}{1^2}}$$

($\sqrt[k]{k} \to 1$)

$$= 1$$

Step 2 Check endpoints.

At $x = 1$,

$$\sum_{k=1}^{\infty} \frac{x^k}{k^2} = \sum_{k=1}^{\infty} \frac{1}{k^2}$$

which converges (p-series)

At $x = -1$,

$$\sum_{k=1}^{\infty} \frac{x^k}{k^2} = \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2}$$

which converges (Alternating Series)

So the interval of conv. is

$$[-1, 1]$$ #

③ $$\sum_{k=1}^{\infty} \frac{k}{6^k} x^k \quad \left(a_k = \frac{k}{6^k}\right)$$

Sol

Step 1 Radius of conv. is

$$\frac{1}{\lim_{k\to\infty}\sqrt[k]{\frac{k}{6^k}}} = \frac{1}{\lim_{k\to\infty}\frac{\sqrt[k]{k}}{6}} = 6$$

($\sqrt[k]{k} \to 1$)

Step2 Check endpoints:

At $x=6$,

$$\sum_{k=1}^{\infty}\frac{k}{6^k}x^k = \sum_{k=1}^{\infty}\frac{k}{6^k}\cdot 6^k = \sum_{k=1}^{\infty} k$$

which diverges (n-th Term Test)

At $x=-6$,

$$\sum_{k=1}^{\infty}\frac{k}{6^k}x^k = \sum_{k=1}^{\infty}\frac{k}{6^k}(-6)^k = \sum_{k=1}^{\infty}(-1)^k k$$

($(-1)^k$)

which diverges (n-th Term Test)

So the interval of conv. is

$$(-6, 6)$$ #

④ $$\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2 3^k}(x+2)^k$$ ($a_k = \frac{(-1)^k}{k^2 3^k}$, $a=-2$)

sol

Step1 Radius of Conv. is

$$\frac{1}{\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right|} = \frac{1}{\lim_{k\to\infty}\left|\frac{\frac{(-1)^{k+1}}{(k+1)^2 3^{k+1}}}{\frac{(-1)^k}{k^2 3^k}}\right|}$$

$$= \frac{1}{\lim_{k\to\infty}\left|\frac{k^2/k^2}{(k+1)^2/k^2}\cdot\frac{-1}{3}\right|}$$

$$= \frac{1}{\frac{1}{3}\cdot\lim_{k\to\infty}\frac{1}{\left(\frac{k}{k}+\frac{1}{k}\right)^2}} \qquad \frac{1}{k}\to 0$$

$$= \frac{1}{\frac{1}{3}\cdot 1} = 3 = r$$

Step 2 Check endpoints. $|x-a| = r$

At $x = a-r = -2-3 = -5$,

$$\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2 3^k}(x+2)^k = \sum_{k=1}^{\infty}\frac{(-1)^k}{k^2 3^k}(-3)^k = \sum_{k=1}^{\infty}\frac{1}{k^2}$$

which converges (p-series)

At $x = a+r = -2+3 = 1$,

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2 3^k}(x+2)^k = \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2 3^k} 3^k = \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2}$$

which converges (Alternating Series)

So the interval of convergence is

$$[a-r, a+r] = [-5, 1] \quad \#$$